A Note on The Functions Which Are Not Polynomial Time Computable From Their Graphs

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§1. Introduction

In classical recursion theory, it is fundamental that a function and its graph have the same degrees of unsolvability. The graph of a function is recursive in the function, and every function is computable from its graph. However, if we consider the polynomial time bounded computation, it is not always the case that every function is computable from its graph. A trivial counter example is the function $f(x)=2^x$. Clearly, f is not polynomial time computable, while its graph G_f is polynomial time computable. To avoid such triviality, we shall restrict the functions which we consider in this note to the ones with polynomial growth rate: A function $f: \mathbb{N} \rightarrow \mathbb{N}$ is said to have *polynomial growth rate* if there is a polynomial $p(t)$ such that $|f(x)| \leq p(|x|)$ for all $x \in \mathbb{N}$ where $|x|$ denotes the length of x in binary expression. We identify $\mathbb{N}$ with the set Σ^* of finite strings over $\Sigma = \{0, 1\}$ ordered as

$$\lambda < 0 < 1 < 00 < 01 < 10 < 11 < 000 < 001 < \dots,$$

where λ is the empty string. It is known that if $P=NP$ then any function with polynomial growth rate is polynomial time computable if and only if its graph is. It is, however, possible to construct a function with polynomial growth rate such that it is not polynomial time computable from its graph. In §2, we shall prove that there are $2^{\aleph_0}$ many such functions. In §3, it will be shown that there are infinitely many recursive functions which are not polynomial time computable from their graphs and also are mutually independent with respect to polynomial time Turing reduction.

Our model of oracle Turing machine has an oracle tape on which each query to the oracle and the value returned from the oracle are written. The elapsed time reflects the length of the string written on the oracle tape. Thus, in particular, when a string x is queried to an oracle f , $|x| + |f(x)|$ is counted in the computation time. Throughout this note, let $\{(M_e, P_e) | e \in \mathbb{N}\}$ be a fixed recursive enumeration of all pairs (M, p) where M is a polynomial time bounded oracle Turing machine and

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p is a polynomial which bounds the running time of M . We may assume $p_e(t) > t$ for all e and t . $M_e(f; x)$ denotes the output value of M_e with oracle f on input x . f is *polynomial time Turing reducible* (p-T reducible) to g iff there is an e such that $f = M_e(g)$. Then, $\leq_p^f$ is a reflexive and transitive relation. $f \equiv_p^f g$ iff $f \leq_p^f g$ and $g \leq_p^f f$. The equivalence class of f is called *p-T degree* of f and denoted by $\deg_p^f(f)$. The *recursive join* of f and g is defined as

$$(f \oplus g)(x) = \langle f(x), g(x) \rangle.$$

As usual, $\deg_p^f(f \oplus g)$ is the least upper bound of $\deg_p^f(f)$ and $\deg_p^f(g)$.

§2. Functions and graphs

Suppose $f : \mathbb{N} \rightarrow \mathbb{N}$ is a given function. The graph of f is the set defined by

$$G_f = \{ \langle x, y \rangle \mid f(x) = y \}.$$

In classical recursion theory, f is computable from G_f , querying $\langle x, 0 \rangle, \langle x, 1 \rangle, \dots$ to G_f until G_f returns the value 1. This simple algorithm clearly takes an exponential time of the length of input x even if $|f(x)|$ is bounded by a polynomial of $|x|$. It is natural to ask whether we can find an ingenious algorithm to compute f from G_f in polynomial time. In this section, we shall prove that such an algorithm does not exist in general. In fact, there are continuum many functions which are not computable from their graphs in polynomial time.

Theorem 1. *There are $2^{\aleph_0}$ functions $f : \Sigma^* \rightarrow \Sigma^*$ such that*

- (i) $|f(x)| = |x|$
- (ii) $f \not\leq_p^f G_f$.

Proof. We shall construct a sequence $\{l_n\}_{n \in \mathbb{N}}$ of integers and a family $\{\varphi_\sigma\}_{\sigma \in \Sigma^*}$ of finite functions by stages to satisfy the following.

- (1) $l_0 < l_1 < l_2 < \dots$
- (2) if $|\sigma| = n$, then $\text{dom}(\varphi_\sigma) = \{x \in \Sigma^* : |x| < l_n\}$.
- (3) $|\varphi_\sigma(x)| = |x|$ for all $x \in \text{dom}(\varphi_\sigma)$.
- (4) $\varphi_{\sigma 0}$ and $\varphi_{\sigma 1}$ are both extensions of φ_σ and $\varphi_{\sigma 0} \neq \varphi_{\sigma 1}$.
- (5) if $|\sigma| = e + 1$, then for any function f which extends φ_σ it holds that $f \neq M_e(G_f)$.

Suppose we complete the construction. For any $\alpha \in 2^\mathbb{N}$, let $f_\alpha = \bigcup_{n \in \mathbb{N}} \varphi_{\alpha \upharpoonright n}$. Then f_α is total by (1) and (2). By (3), $|f_\alpha(x)| = |x|$, so f_α has polynomial growth rate. By (4), if $\alpha \neq \beta$, then $f_\alpha \neq f_\beta$, and by (5) it holds that $f_\alpha \neq M_e(G_{f_\alpha})$ for all e , so that $f_\alpha \not\leq_p^f G_{f_\alpha}$.

Now we define φ_σ and l_e so that they have the desired properties.

Stage 0. We set $l_0 = 0$, and $\varphi_\lambda = \emptyset$.

Stage $e+1$. Suppose l_e and φ_σ are already defined where $|\sigma|=e$. We shall define l_{e+1} and $\varphi_{\sigma 0}, \varphi_{\sigma 1}$. Let s be a sufficiently large number such that $l_e \leq s$ and $p_e(s)+1 < 2^s$. We set $l_{e+1} = p_e(s)+1$. Note that s and l_{e+1} do not depend on σ . Define G_0 by

$$G_0 = \left\{ \langle x, y \rangle \mid \begin{array}{l} (|x| < l_e \ \& \ \varphi_\sigma(x) = y) \vee \\ (l_e \leq |x| < l_{e+1} \ \& \ x \neq 0^s \ \& \ y = 0^{|x|}) \end{array} \right\}$$

and compute $M_e(G_0; 0^s)$. Let $y_0 = M_e(G_0; 0^s)$. During the computation of $M_e(G_0; 0^s)$, at most $p_e(s)$ strings are queried to the oracle G_0 . Since the set $Y = \{\langle 0^s, y \rangle : |y| = s\}$ has 2^s elements, there are at least 2 elements of Y which are not queried to G_0 . So we pick an element $\langle 0^s, y_1 \rangle$ of Y so that $y_1 \neq y_0$ and $\langle 0^s, y_1 \rangle$ is never queried. Define $\varphi_{\sigma 0}$ as

$$\varphi_{\sigma 0}(x) = \begin{cases} \varphi_\sigma(x) & \text{if } |x| < l_e \\ y_1 & \text{if } x = 0^s \\ 0^{|x|} & \text{if } l_e \leq |x| < l_{e+1} \text{ and } x \neq 0^s. \end{cases}$$

G_0 and $G\varphi_{\sigma 0}$ differ only on $\langle 0^s, y_1 \rangle$, and in the computations of $M_e(G_0; 0^s)$ and $M_e(G\varphi_{\sigma 0}; 0^s)$, $\langle 0^s, y_1 \rangle$ is never queried to the oracles. It follows that

$$M_e(G\varphi_{\sigma 0}; 0^s) = M_e(G_0; 0^s) = y_0 \neq y_1 = \varphi_{\sigma 0}(0^s).$$

Thus, the condition (5) is satisfied for $\varphi_{\sigma 0}$.

We define $\varphi_{\sigma 1}$ in the same manner using the set G_1 defined below in place of G_0 and taking the value $1^{|x|}$ for x such that $l_e \leq |x| < l_{e+1}$ and $x \neq 0^s$.

$$G_1 = \left\{ \langle x, y \rangle \mid \begin{array}{l} (|x| < l_e \ \& \ \varphi_\sigma(x) = y) \vee \\ (l_e \leq |x| < l_{e+1} \ \& \ x \neq 0^s \ \& \ y = 1^{|x|}) \end{array} \right\}$$

We can see that the conditions (2)–(4) follow from the definition of $\varphi_{\sigma 0}$ and $\varphi_{\sigma 1}$.

We have seen that there are functions which are not polynomial time computable from their graphs. Note that there are also $2^{\aleph_0}$ many functions f that are computable from their graphs in polynomial time. For any $A \subseteq \Sigma^*$, let

$$f_A(x) = \begin{cases} 1^{|x|} & \text{if } x \in A, \\ 0^{|x|} & \text{if } x \notin A. \end{cases}$$

Then $|f_A(x)| = |x|$. f_A is computed from G_A in polynomial time as follows. Given $x \in \Sigma^*$, query $\langle x, 1^{|x|} \rangle$ to G_A . If the answer is positive, then output $1^{|x|}$, else output $0^{|x|}$.

It should also be noticed that every function f has the same p-T degree as its projection R_f where $R_f = \{\langle x, y \rangle \mid y \leq f(x)\}$, see [2]. Since the reduction of f to R_f fully uses the features of p-T reducibility, it can not be expected that every function is reducible to its projection with respect to stronger polynomial time reducibilities

than $\leq_p^*$. It is easy to construct by simple diagonalization a recursive function f such that f is not p-btt reducible to any set, see [1] for the definition of p-btt reducibility.

§3. p-T Degrees of recursive functions

In this section, we shall construct effectively an infinite collection of recursive functions which are mutually incomparable with respect to $\leq_p^*$.

Theorem 2. *There is a recursive family $\{f_i \mid i \in \mathbb{N}\}$ of recursive functions such that*

- (i) *for all i and $x \in \Sigma^*$, $|f_i(x)| = |x|$,*
- (ii) *for all i , $f_i \not\leq_p^* G_{f_i}$,*
- (iii) *for every n and for every $i \leq n$,*

$$f_i \not\leq_p^* f_0 \oplus \cdots \oplus f_{i-1} \oplus f_{i+1} \oplus \cdots \oplus f_n.$$

Proof. We construct an independent family $\{f_i\}_{i \in \mathbb{N}}$ of recursive functions with the properties (i)-(ii). The requirements for $\{f_i\}_{i \in \mathbb{N}}$ are the following.

$$(P_{ij}) \quad f_i \neq M_j(G_{f_i}),$$

$$(Q_{nij}) \quad f_i \neq M_j(\hat{f}_i),$$

where $\hat{f}_i = f_0 \oplus \cdots \oplus f_{i-1} \oplus f_{i+1} \oplus \cdots \oplus f_n$. Let $\{R_m\}_{m \in \mathbb{N}}$ be a recursive enumeration of all the requirements.

Stage 0. We set $l_0 = 1$ and $f_k(\lambda) = \lambda$ for all k .

Stage $m+1$. Suppose l_m and $f_k \upharpoonright \{x \mid |x| < l_m\}$ are already defined. First we consider the case where R_m is P_{ij} . In this case, let s be the smallest integer such that $l_m \leq s$ and $p_j(s) + 1 < 2^s$, and set $l_{m+1} = p_j(s) + 1$. Define G_i by

$$G_i = \{\langle x, y \rangle \mid (|x| < l_m \ \& \ f_i(x) = y) \vee (l_m \leq |x| < l_{m+1} \ \& \ x \neq 0^s \ \& \ y = 0^{|x|})\},$$

and compute $M_j(f_i; 0^s)$. Let $y_0 = M_j(f_i; 0^s)$. Since the computation of $M_j(G_i; 0^s)$ terminates within $p_j(s)$ steps, at most $p_j(s)$ many pairs of form $\langle 0^s, y \rangle$ with $|y| = s$ are queried to G_i . $p_j(s) + 1$ is less than 2^s . Therefore, there are at least two strings y_1 and y_2 of length s such that neither of $\langle 0^s, y_1 \rangle$ and $\langle 0^s, y_2 \rangle$ is queried to G_i . One of y_1 and y_2 , say y_1 , is different from y_0 . Then we set

$$f_i(x) = \begin{cases} f_i(x) & \text{if } |x| < l_m, \\ y_1 & \text{if } x = 0^s, \\ 0^{|x|} & \text{if } l_m \leq |x| < l_{m+1} \text{ and } x \neq 0^s. \end{cases}$$

For $k \neq i$, we simply set

$$(*) \quad f_k(x) = \begin{cases} f_k(x) & \text{if } |x| < l_m, \\ 0^{|x|} & \text{if } l_m \leq |x| < l_{m+1}, \end{cases}$$

Suppose R_m is Q_{nij} . In this case, we set $l_{m+1} = p_j(l_m) + 1$. For $k \neq i$, we define f_k on $\{x \mid |x| < l_m\}$ as $(*)$. Then, the value of $M_j(\hat{f}_i; 0^{l_m})$ is defined. Let y_1 is the least string of length l_m which is different from $M_j(\hat{f}_i; 0^{l_m})$. To meet the requirement Q_{nij} , we set

$$f_i(x) = \begin{cases} f_i(x) & \text{if } |x| < l_m, \\ y_1 & \text{if } x = 0^{l_m}, \\ 0^{|x|} & \text{if } l_m \leq |x| < l_{m+1} \text{ and } x \neq 0^{l_m}. \end{cases}$$

This ends the construction. As in the proof of Theorem 1, it is easy to see that the family $\{f_i \mid i \in \mathbb{N}\}$ so constructed has the required properties. $\square$

References

- [1] K. Ambos-Spies, *On the structure of the polynomial time degrees of recursive sets* (Habilitationsschrift), Lehrstuhl für Informatik, Universität Dortmund, Germany (1984).
- [2] G. Miller, *Riemann's hypothesis and tests of primality*, J. Computer and System Science **13** (1976), 300-317.