IS-A Relation, the Principle of Comprehension  
and the Doctrine of Limitation of Size  

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1. Introduction

The main aim of this paper is to give a justification of what is called the principle of comprehension. A troublesome fact concerning this principle is that the simple formulation of it leads us to a contradiction. In order to avoid such circumstances, the doctrine of size limitation was introduced. Thus the so-called operation of separation today enjoys the status of axiom which the principle of comprehension enjoyed at the time of the birth of set theory. Since we think that such logical inconveniences arouse because of the incorrect formulation of this principle on the logical system which lacks the ability to express the logical structure of natural language in a proper way, we shall show what the correct formulation of the principle of comprehension must be with respect to the logical system which expresses the logical structure of natural language, without taking any recourse to the limitation of size doctrine. We shall be concerned with a logical system in which the correct formulation of the principle of comprehension is a provable formula. We shall show that the axiom of separation and that of replacement are derivable in that system.

For this purpose we shall put forward a logical system which captures the logical features of natural language by extending appropriately the usual first-order predicate calculus which is today on the market (we shall hereafter refer to it as LI) with some appropriate logico-linguistic devices.

We shall show that the axiom of separation and that of replacement are equivalent to each other in the system in question. They are tightly related to each other due to the comprehension principle correctly formulated. This might be a queer fact to those who are working on the standard set theory (cum LI), for it is considered usually that the axiom of replacement is stronger than that of separation. This phenomenon in set theory is essentially related the axiom of power-set which is by nature of generative character. Our system has no operations of generative character, and this is the reason why the two axioms are equivalent in the system in question. There is one more formula which is equivalent to these axioms, i.e. a weak version of the correctly formulated principle of comprehension. As for the

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2 IS-A relation and the principle of comprehension

We need at first to put forward a logical system in which IS-A relation is expressible, since LI lacks the ability to express a notion of great importance, i.e. what is now called in the field of artificial intelligence IS-A relation, which has been known widely under the name of *copula* in the history of logic.

Now our first step is to examine the principle of *comprehension* which is usually considered to be fundamental to the notion of *set*. The most intuitive form of comprehension is:

\[
ACS \quad [a](a \subseteq cl(\phi) \equiv \phi(a))
\]

or in such a system as ZF, it is stated in a weaker and modest form in which set/class-formation operation does not appear:

\[
ACS1 \quad [\exists b][a](a \subseteq b. \equiv \phi(a))
\]

Such formulations of the principle of comprehension are of set-theoretical character.

We now turn to its everyday logical version. Let us consider the following formula:

\[
[ACO] \quad [a](a \text{ is-a that-which } \phi \text{ s iff } a \phi \text{ s})
\]

Comparing [ACO] with ACS, we notice that [that-which] corresponds to set/class formation operator \([cl(.)]\), and [is-a] to \([\varepsilon]\), respectively. And it is not difficult to realize that this also well captures our intuitive principle of *comprehension*.

It should be noticed and borne in mind that ACS(1) is stated in terms of set-theoretical notions, while [ACO] is just a linguistic transformation with which we are well aquainted in everyday reasoning. We use very often [that which]-operator in everyday language and reasoning, while \(cl\)-operator in ACS is foreign to our usual everyday reasoning and intellectual life management. As for [ACO] we only need for its formulation IS-A relation and the *linguistic* operation [that-which] of the category \(n/(s/n)\). Thus [ACO] does not contain any set-theoretical notions. Now the question is this: which one does reflect our intuitive notion of *comprehension* in a natural way? It is without doubt [ACO], at least insofar as we are concerned with the logic by means of which we manage our everyday intellectual.

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1 The system we are going to put forward is the system which is proposed in Waragai [1990].
2 On IS-A relation, cf. e.g. Brachman [1983].
3 Another latin name was 'esse tertium adjacens'.
4 This is usually written \([\{\cdot | \phi(.)\}]\).
matters in life.

Since we have \([ACO]\), we can conclude that it is not necessarily (or by no
means) required that the way of the formulation of the principle of comprehension
be set-theoretical. We can well formulate the principle in everyday logico-linguistic
apparatus. This suggests that the principle of comprehension is primarily of nature
of everyday logic and not of set theoretical character.

Now that it is also clear that \([ACO]\) leads us to an contradiction, we shall turn
to the correct formulation of \([ACO]\) which is free of contradiction.

3 Comprehension, the doctrine of limitation of size and hidden semantics

In the usual explanation of the axiom of separation, the notion of the limitation
of size in made use of. Why was it necessary to introduce the notion of size
limitation of sets? It is of course to eschew anitionomies such as Russell's. Our
strategy is essentially different from this standard one. The reason for such antin-
omies should be looked for in the unnoticed mixture of syntax and semantics of LI
on which set theory is based. LI has a hidden semantics concerning the expression
of the expressions of the semantic category of names. So we shall put forward
another way of solution concerning the problematic points as to the principle of
comprehension. An usual explanations for the need of the doctrine of limitation of
size runs as follows:

The axiom of comprehension turned out to be iconsistent and therefore
cannot be used as an axiom of set theory. However, since this axiom is so
close to our intiuitive concept of set we shall try to retain a considerable
number of instance of this axiom...

Our guiding principle, for the system ZF, will be to admit only those
instances of the axiom schema of comprehension which assert the existence of
the sets which are not too "big" compared to sets which we already have.

We shall call this principle as the limitation of size doctrine.\(^5\)

Usually the expressions of the category of names are supposed to designate individual objects, i.e. a name stands for an objects and the domain of quantifiers constitutes of individuals. But this is a matter which belongs to semantics, and semantics and syntax are, from a purely theoretical point of view, two completely different notions. As a consequence, it should be strongly maintained that LI which is essentially a purely syntactic system, i.e. a purely mechanical symbol manipulation system presupposes some specific semantics as to names. LI is indeed a sytem in which to be is to be the value of a variable\(^6\) is the case. As a sequal to this hidden presupposed semantics, we have no logical device to express that something is an

\(^5\) Fraenkel, Bar-Hillel, Levy [1973], 32.
\(^6\) Cf. Quine [1948], Gochet [1984].
object within a system.

Taking into consideration the fact that syntax and semantics are two different matters, a natural question arises; namely what happens if we get free of this hidden semantic presupposition? That which happens is this: what satisfies \( \phi(.) \) usually need not be an individual name. For example, \( \langle \text{is-a-man}\rangle(.) \) is satisfied by \[ \text{John} \] as well as \[ \text{man} \], i.e. both by an individual name and a general one. \( \langle \text{is-a-man}\rangle(\text{John}) \) and \( \langle \text{is-a-man}\rangle(\text{man}) \) become syntactically on a par. From this on, we shall accept general names as logical units, running against Quinian Thesis.

Now let us return to the problem of correct formulation of the principle of comprehension. The most crucial point with respect to the principle of comprehension is the following:

**Principium Comprehensionis (PC):** We collect *only* individuals in doing comprehension.

Therefore we have to have some device which enables us to select only individuals in doing comprehension. Now let us suppose that we have, unlike LI, a system in which the predicate \( \langle . \rangle \text{is-an-object} \) is expressible. We write it \( \text{ob}(a) \), reading it \( \langle a \text{ is-an-object}\rangle \). In such a system, following PC, a natural and correct formulation (in terms of set theory) of the principle of comprehension is this:

\[
\text{ACN} \quad [a] (a \in \text{cl}(\phi) \equiv \text{ob}(a) \land \phi(a))
\]

because in comprehending, we repeat, we collect *only* objects. Notice that from ACN we can no longer deduce the well-known Russellian antinomy. What we get is that Russellian property is one which does not construct an object. We can avoid in this way Russellian paradox. According to our view, the Russellian paradox occurred due to the fact that a specific semantics was smuggled into syntax. That is, the need for the size limitation doctrine comes from the confusion of syntax and semantics as well as the lack of the ability of the usual predicate calculus to express the predicate \( \text{ob}(.) \).

Now it should be noticed that we have in set theory the predicate \( \text{ob}(.) \), for in set theory, to be an object is to be a set (or an Urelement), and any member of a set is an object. So we have the *axiom of separation* is a modified form of ACN or an instance scheme of ACN. Thus the principle of comprehension implies the axiom of separation. In the usual treatment of the principle of comprehension, the predicate \( \text{ob}(.) \) is already present in a hidden way in ACN, whence ACN reduces to simple ANS, and a contradiction arises. Thus seen, we have no special, logically a priori reason to accept the doctrine of the limitation of size. Rather it

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7 Even \([\text{every man}]\) or \([\text{some man}]\), i.e. quantified names will do. This problem will be discussed in another paper.
seems that we should try to examine carefully the presupposition of LI, i.e. the logical system we usually and unconsciously make use of.

Now let us turn to the everyday version of ACN. It looks as follows:

\[ [AC] \quad [a](a \text{ is-a that which } \phi\text{'s iff } ob(a) \text{ and } \phi(a)) \]

Notice that this formulation is totally free of set theoretical notions. All we here need are [IS-A] relation, [that-which] operator, and the predicate [ob(.)]. They are all independent notions from set-theoretical ones. Evidently, this formulation of the principle of comprehension prohibits the Russellian paradox, too.

A few words as to ACN are not out of place to end this section. As cited in the previous chapter, the reason for introducing the axiom of separation is to retain a considerable number of instance of this axiom (of comprehension), and since the axiom of separation (scheme) is an instance of ACN, what should have been done at the very beginning stage of investigation into the cause of such a paradox as Russell's was not to take recourse to limitation of size, but to examine the semantical presupposition of LI. That LI presupposes something of specific semantical character should have been an easy matter to notice, for Aristotelian syllogistic system is not based on the semantical presupposition in question.

Now a natural question is this: which is more natural ACN or [AC]? If we respect the richness of expressive power of everyday logic and naturalness, it seems that we should choose [AC]. What is more, we have a very nice fact that [AC] is already inherent to the logic of natural language. This will be presented in a later section, and this is a fact that makes clear that the notion of comprehension is a thesis which already holds at the level of logic governing natural language.

\section{A logical system in which IS-A relation is expessible}

Now let us present an axiomatic system in which IS-A relation is expressible. We obtain it as natural extension of LI. Hereafter let us refer to the intended system as LI*.

The first extension is concerned with the logical status of general names. We accept them, following the tradition of syllogism, as logically independent units. An individual name is a specific kind of general name; i.e. \([a]\) is an individual name iff \([a]\) has just one extension. Empty names like 'Pegasus', 'the round square' are also regarded to be general names.

\subsection{Primitive symbols and their intended meaning}

As the primitive symbols LI* has the following four:

<table>
<thead>
<tr>
<th>symbol</th>
<th>intended meaning</th>
<th>semantic category</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(.) is-identical-with(.)</td>
<td>s/n, n</td>
</tr>
</tbody>
</table>
4.2 Naive semantics of the sentence of the form \([a \varepsilon b]\)

We give here a naive semantics for the formula \([a \varepsilon b]\). The truth condition of this formula is the following:

\[
\text{Tr } [a \varepsilon b] \text{ is true iff } a \text{ is an object and } a \text{ is a } b
\]

On the basis of Tr, we can express \([a \text{ is an object}]\) by \([a \varepsilon a]\), so our system has the predicate \([ob(.)]\), or even the name \([\text{object}]\), which only metaphorically corresponds to \([V]\) of set theory.\(^8\) And \([a \text{ is identical with } b]\), i.e. \([a = b]\) can be expressed by \([a \varepsilon b \land b \varepsilon a]\). That this is in fact so is provable in LI\(^+\).

4.3 Axioms of LI\(^+\)

We state the axioms of LI\(^+\). They are the following:

\[
\begin{align*}
\text{AI1} & \quad [ab](a = b \supset b = a) \\
\text{AI2} & \quad [abc](a = b \land b = c \supset a = c) \\
\text{A2} & \quad [xy](x = y \supset \phi(x) = \phi(y)) \\
\text{A3} & \quad [a](\phi(a) = \phi(\text{trm} \langle \varepsilon^*(a) \rangle)) \\
\text{A4} & \quad [xa](x \varepsilon a = \varepsilon^*(a)(x)) \\
\text{A5} & \quad [a](a = a = \varepsilon^*a) \\
\text{A6} & \quad \text{trm} \langle \phi \rangle \subseteq \text{trm} \langle \psi \rangle = . \quad [\exists x](x = x \land \phi(x)) \land \\
& \quad [xy](x = x \land y = y \land \phi(x) \land \phi(y) \supset x = y) \land \\
& \quad [x](x = x \land \phi(x) \supset \psi(x))
\end{align*}
\]

About the details of this system, the reader is cordially asked to refer to Waragai [1990].\(^9\)

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\(^8\) For the sake of exactitude, it should be mentioned that the most general name \([\text{object}]\) does not by any means substitutes for \([V]\), i.e. \([\text{the class of all sets}]\). The correspondence is just an apparent one.

\(^9\) LI\(^+\) is inferentially equivalent to the first-order part of Leśniewski’s Ontology. For a proof, cf. Waragai [1990]. About Ontology itself, cf. Lejewski [1958].
4.4 Some basic theorems of LI⁺

From these axioms, we have the following theorems:

\[
\begin{align*}
T1 & \vdash (a \vDash b \land b \vDash c \supset a \vDash c) \\
T2 & \vdash (a \vDash b \supset a \vDash a) \\
T3 & \vdash (a = b \supset a \vDash b \vDash b \vDash a)
\end{align*}
\]

T1 shows that the notion of identity and singular predicative IS-A are not independent. The former is reducible to the latter.\(^{10}\) We here quote a passage from Russell:

\begin{quote}
The is of "Socrates is human" expresses the relation of subject and predicate; the is of "Socrates is a man" expresses identity. It is a disgrace to human race that it has chosen to employ the same word for these two entirely different ideas—a disgrace which a symbolic logical language of course remedies.\(^{11}\)
\end{quote}

This view runs totally against T1. We maintain that Russell was wrong in this point.

\[
T4 \vdash (a \vDash \text{term}<\phi> \supset a \vDash a \land \phi(a))
\]

This expresses precisely and correctly what the principle of comprehension intends to express. We shall refer to T4 as comprehension lemma. That T4 holds in LI⁺ shows that the notion of comprehension is adequately captured in LI⁺. A point which should be noticed is that LI⁺ has nothing to do with set-theoretical notions. Hence the principle of comprehension has also nothing to do with set theory. Thus, the principle of comprehension is not a principle proper to set theory, but is a principle inherent to the logic of natural language. This is a very important result.

\[
T5 \vdash (a \vDash b \equiv (\exists x)(x \vDash a) \land (xy)(x \vDash a \land y \vDash a \supset x \vDash y) \land (x)(x \vDash a \supset x \vDash b))
\]

For the proofs of the above theses, the reader is asked to refer to Waragai [1990]. T5 shows that the logical behaviour of [\(\varepsilon\)] is completely determined, while set-theoretical membership relation does not enjoy such a fact. Now we define [IS-A] relation of generic/generic type.

\[
D1 \vdash (a \subset b \equiv (x)(x \vDash a \supset x \vDash b))
\]

We obtain:

\[
T6 \vdash (a \vDash b \supset a \subset b) \quad [D1, T1]
\]

\(^{10}\) Cf. Sommers [1982]. Hintikka [1983].

\(^{11}\) Russell [1919], 172, my bold face.
T7 \[(ab)(ae b.\iff ae a \land a \subseteq b)\] \[T2, T6/D1\]
T7 states that \([e]\) is a special case of \([IS-A]\) relation of generic/generic type.

5 On the axiom of separation

5.1 The axiom of separation is deducible from comprehension lemma

We show in this section that a formula which is completely parallel to the axiom of separation of axiomatic set-theories is a formula provable in LI+. Indeed it is deducible from comprehension lemma. For this purpose we put:

D2 \[a](\phi(a).\iff ae b \land \psi(a))\]

Applying this formula to comprehension lemma, i.e. T2, we obtain:

T8 \[a](ae trm<\phi>.\iff ae a \land ae b \land \psi(a))\] \[T4\]

but, since we have T1 as a thesis in LI+, we have

T9 \[ab](ae trm<\phi>.\iff ae b \land \psi(a))\] \[T1\]

From this we immediately obtain as a thesis of LI+:

T10 \[b][\exists c](\forall a)(ae c.\iff ae b \land \psi(a)))\] \[T8\]

which is completely parallel to the axiom of separation in axiomatic set theory. Since LI+ has nothing to do with set-theoretical notions, the axiom of separation is a thesis in everyday logic and not proper to set theory. Let us record it as a theorem:

Theorem 1: In LI+, the comprehension lemma implies the axiom of separation.

We shall refer to T9 as separation lemma hereafter.

5.2 Weak comprehension lemma and its equivalence to the axiom of separation

We shall refer to the following thesis as weak comprehension lemma.

T11 \[\exists b](\forall a)(ae b.\iff ae a \land \phi(a))\] \[T4 or T10; b/a\]

Thus we have the following theorem:

Theorem 2: both the comprehension lemma and the separation lemma imply the weak comprehension lemma.

It is almost evident that the weak comprehension lemma implies the separation lemma. For proof, take \([ae c \land \phi(a)]\) for \([\phi(a)]\) in T11.
Theorem 3: the weak comprehension lemma implies the axiom of separation.

Combining them we have:

Theorem 4: In LI+, the weak comprehension lemma is equivalent to the separation lemma.

5.3 On the naturalness of the axiom of separation and the logical legitimacy of the doctrine of size limitation

At this point, a natural, fundamental and essential question arises from the result we obtained in the previous section; namely why does the axiom of separation look natural? Historically, this axiom is related with the notion of limitation of size of sets. The usual explanation why such an axiom is needed and natural is related to the notion of size of sets. Such a paradox like Russellian one arises because we are concerned with sets whose size is all too big. Therefore, so goes on the usual explanation, we impose a restriction on the bare axiom of comprehension. If a property is given, then we are allowed to construct a set from an already existing set by selecting the elements that satisfy the given property.

\[ \text{ASS} \left[ b \left( \exists c \right) \left[ a \left( a \in c \land a \in b \Rightarrow a \in \phi \left( a \right) \right) \right] \right] \]

But is the notion of size limitation really needed to explain the nature of comprehension operation, or in other words, to explain human ability of comprehension? ASS and T10 are completely isomorphic in form, and T10 is obtained from T4. Now let us consider what we had to do to obtain T4? All we needed was to write up the hidden semantic presuppositional requirement that the arguments of predicates must be individual, by accepting general names and IS-A relation as logical units. By extending the usual predicate calculus on LI, we obtain easily and in quite a natural way T6, the formula which expresses our intuitive notion of comprehension. Here appears no notion of limitation of size. This suggests us that the doctrine of limitation of size is something of Deus ex machina character used without any sufficiently deep philosophical reflection on the nature of comprehension and on the hidden semantics smuggled into a syntactic mechanism which should be free of semantical constraint. Thus seen, the notion of limitation of size seems to be unnatural.

Despite the fact above mentioned, the axiom of separation looks natural. Why so? The reason we can give is that this axiom is present already in the logical mechanism of natural language on which we do our intellectual jobs. Therefore we put the following as a claim:

Claim 1: 1) The axiom of separation looks natural because it is already acceptable in the logical mechanism of natural language.
2) The axiom of separation is not an axiom proper to set theory.
3) The notion of limitation of size is indifferent to the naturalness of this axiom.

To express this in a shaper form, it could be expressed as follows:

**Claim 2:** the doctrine of limitation of size does not justify the axiom of separation.

To try to justify the principle of comprehension and the axiom of separation on the basis of the doctrine of limitation of size looks out of point.

6 On the axiom of replacement in LI+

6.1 The axiom of replacement in LI+ implies the separation lemma

What we show is that LI+-version of the axiom of replacement implies as is in set theory the case LI+-version of the axiom of separation.

Before stating the axiom of replacement in set theory, let us define functional relation:

\[ \text{DSFnc} \quad \text{Fnc}\{R\} \equiv \{xyz\}(R(x, y) \land R(x, z) \supset y = z) \]

And the axiom of replacement is the following:

\[ \text{ARS} \quad [a](\text{Fnc}\{R\} \supset [\exists b][y](y \epsilon b \equiv [x](x \epsilon a \land R(x, y)))) \]

It should be noticed that, according to our standpoint, DSFnc still suffers the lack of explicit formulation of the hidden semantical presupposition. According to our standpoint, the correct formular corresponding to DSFnc is the following:

\[ \text{DOFnc} \quad \text{Fnc}\{R\} \equiv \{xyz\}(R(x, y) \land R(x, z) \land x \epsilon x \supset y \epsilon z) \]

Notice that the clause expressing \([x \ in-an \ object]\) is stated explicitly. The clause is in DSFnc unwritten, supposed as a semantical presupposition implicitly. Now the LI+-version of the axiom of replacement becomes:

\[ \text{ARO} \quad [a](\text{Fnc}\{R\} \supset [\exists b][y](y \epsilon b \equiv [\exists x](x \epsilon a \land R(x, y)))) \]

We show that in LI+-version of the axiom of replacement implies LI+-version of the axiom of separation. To show this we define a functional relation \(W\) in the following way:

\[ D3 \quad [xy]W(x, y) \equiv \phi(x) \land x = y \]

Now it is easy to check that \(W\) is a functional relation:

\[ T12 \quad [xyz](W(x, y) \land W(x, z) \land x \epsilon x \supset y = z) \quad [D3, D3, A1] \]
Thus we have the following theorem:

**Theorem 5**: In LI⁺, the axiom of replacement implies the separation lemma.

6.2 The (weak) comprehension lemma in LI⁺ implies the axiom of replacement

We show in this section that the LI⁺-version of the axiom of replacement is a thesis of LI⁺.

Now take a step further to the proof intended. Given a general name ‘a’ and functional relation R, we can think of the predicate \[(.\) is-an-R-image-of-a\], which is satisfied by an object which is an R-image of an object that is a. We introduce the predicate \[\text{Im} [R, a] (.\)\] of the category \((s/n)/(s/n,n),n)\):

\[D4 \quad [a]([y](\text{Im}[R, a](y).\beta.[\exists x](x,y)))\]

Now the following holds:

\[T19 \quad [a](\text{Fnc}[R, \exists y](x\in a \land R(x,y)).\beta.y=y))\]
\[T20 \quad [a](\text{Fnc}[R, \exists y]((\exists x)((y\in y \land [\exists x](x\in a \land R(x,y))))
\quad .\beta.x\in a \land R(x,y))))\]
\[T21 \quad [a](\text{Fnc}[R, \exists y](y\in y \land \text{Im}[R, a](y)).\beta.\text{Im}[R, a](y)))\]

Using the comprehension lemma, we have:

\[T22 \quad [a](\text{Fnc}[R, \exists y]([y](\forall x \in \text{Im}[R, a] >.\beta.\text{Im}[R, a](y)))\]
\[T23 \quad [a](\text{Fnc}[R, \exists y]([y](\forall x \in \text{Im}[R, a] >.\beta.[\exists x](x\in a \land R(x,y))))\]

Thus we finally reached:

\[T24 \quad [a](\text{Fnc}[R, \exists b]([y](y\notin b.\beta.[\exists x](x\in a \land R(x,y))))))\]

Thus we established the following theorem:
Theorem 6: LI⁺-version of the (weak) comprehension lemma implies the axiom of replacement.

That this result was obtained within LI⁺ shows the axiom of replacement is not proper to set theory, either. It is by nature of everyday logical character. Its naturalness is implicitly already, though hidden, present in our everyday logical apparatus.

Now a little careful examination of the proof steps T20-T24 makes us realize that to get T24, it is sufficient to use the weak comprehension lemma instead of the comprehension lemma. From this fact we obtain the following theorem:

Theorem 7: LI⁺-version of the axiom of replacement is deducible from the weak comprehension lemma.

Combining the theorems, we get an interesting fact:

Theorem 8: The weak comprehension lemma, the separation lemma and the axiom of replacement are equivalent in LI⁺.

Proof: Evident from Theorems 4, 5, 7.

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