Design and finite element analysis for helical gears with pinion circular arc teeth and gear parabolic curve teeth

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Abstract
A novel point contact gear drive with pinion circular arc teeth and gear parabolic curve teeth is proposed based on the application of two mismatched rack cutters. The rack cutter for generating a pinion is of normal circular arc profile, and the rack cutter for generating a gear is of normal parabolic curve profile which is tangent to the circular arc of the pinion. The number of contact points can be one or two by changing the design parameter of the parabolic curve. Equations of tooth surfaces are derived based on coordinate transformation and gear geometry theory. Computer program is developed to solve the tooth surface equations and establish the solid models. To find out the advantages and disadvantages of the proposed gear drives, the gear stress distribution and the contact deformation are investigated by using the finite element analysis. Three examples of the proposed gears and circular-arc gears are present to demonstrate the influence of the profile design parameters and the contact positions on the stress distribution. The result shows that the proposed gear drives can reduce 23.8%-35.3% contact stress or 23.2%-29.5% von Mises contact stress.

Key words: Gear geometry, Parabolic curve, Finite element analysis, Point contact

1. Introduction

Gear tooth geometry has a great effect upon the performance, strength and life of gearing systems. Helical gears with involute teeth are widely used in the industry due to they are easy to be produced with high precision and the change of gear centre distance does not cause transmission errors. However, the load capacity of the involute tooth surface is low because of the convex-to-convex contact pattern and the large specific sliding. To get the higher performance gear transmission, many gear researchers and engineers under took efforts to implement the concave-to-convex contact tooth surfaces. Wildhaber proposed circular arc tooth shapes of the gears which run on parallel axes to provide helical gearing with improved tooth contact (Wildhaber 1926). Novikov proposed circular arc tooth surface helical gears to achieve higher contact strength (Novikov 1956). The difference of the two inventions is that the Wildhaber gears are generated by a rack cutter, and Novikov gears are generated by two mismatched rack cutters. So the Wildhaber gear tooth surfaces are in line contact and the Novikov gear tooth surfaces are in point contact. Kudrjavtsev(Kudrjavtsev 1966) and Winter et al. (Winter and Looman 1961) proposed the manufacturing method of W.-N. gears by application of two mating hobs based on the idea of two mating imaginary rack cutters. By using two kinds of fly tools with circular arc profiles, Ishibashi et al. investigated the running performance of the Novikov gear pairs with the pinion of three teeth (ISHIBASHI and YOSHINO 1984). And there are many researchers in China, Russia, Germany, and Japan make great contributions to the circular-arc gears(Ariga and Nagata 1985; Litvin and Tsay 1985; Litvin and Lu 1993; F.L.Litvin and Lu 1995; Lu, Litvin et al. 1995; Wang and Chen 2001; Chen, Duan et al. 2006; Zhang, Hua et al. 2010; Qu, Peng et al. 2012).

The circular-arc gear drive is only a particular case that can transform rotation with constant gear ratio and have a point contact at every instant. Due to the bearing contact can only be improved by running the gears in the gearbox, the circular-arc gear drive has been applied for low-speed transmissions, and only soft materials has been applied. To improve the performance of circular-arc gears, Litvin et al. (L.Litvin, Fuentes et al. 2002) proposed a new type of W.-N. helical gears based on application of a double-crowned pinion tooth surface. The tooth surfaces are generated by
applying two mismatched rack cutters with the parabolic curves that are internal tangency as the normal profiles. And they also proposed a helical gear drive with the pinion of involute and the gear of parabolic curve (Litvin, Fuentes et al. 2003). However, the running in process to expend the contact area is time consuming, and the contact area cannot spread to the whole working tooth surface.

To decrease the running in time and make full use of the convex-to-concave contact pattern, a novel helical gear drive is investigated and the finite stress analysis are carried out in this paper. The organization of the rest paper is as follows. Section 2 describes the geometry of two mismatched rack cutters profiles in the normal section. Section 3 establishes the mathematical models of the proposed gears. The stress analysis is carried out in Section 4 and the paper is concluded in Section 5.

2. Geometry of generating tools

Generation of the pinion and gear tooth surfaces that are in point contact requires application of two rack cutters with mismatched surfaces that separately generate a pinion and a gear. In this work, the normal profile of the rack cutter for generating the pinion is a circular arc, and the normal profile of the rack cutter for generating the gear is a parabolic curve that is tangency to the circular arc at one or two points. The geometry of rack cutters present in this section is the basis for generating the proposed point contact gears. The normal profiles of the rack cutters for the pinion and gear are designated as $\Sigma_a$ and $\Sigma_f$ respectively.

2.1 Geometry of circular arc rack cutter

The proposed normal profile $\Sigma_a$ of the rack cutter for generating the pinion with convex tooth surface is shown in Fig. 1. Cutting edge I of the tip fillet is an arc of radius $\rho_{\min}$, which generates the root fillet of the pinion, and cutting edge II of the profile is a circular arc of radius $\rho_s$ that generates the working region of the pinion. $h_i$ and $h_j$ are the addendum and the dedendum of the rack cutter respectively. $h_l$ is the tooth depth of the rack cutter. $S_a$ is the space width of the rack cutter and the tooth thickness of the pinion. Here, we define coordinate system $S_a(0; x_a, y_a, z_a)$ fixed to the rack cutter. The origin $O_a$ is at the centre of the tooth spacing of the rack cutter.

![Pitch line](image)

Fig. 1 Normal section profile $\Sigma_a$

According to the geometry of the rack cutter, the vector $r^{(I)}(\theta_a)$ of the edge I in the coordinate system $S_a$ can be expressed as

$$r^{(I)}(\theta_a) = \begin{bmatrix} -\rho_{\min} \cos \theta_a + (\rho_s + \rho_{\min}) \sin \alpha_{\min} \\ \pm [\rho_{\min} \sin \theta_a - (\rho_s + \rho_{\min}) \cos \alpha_{\min} + l_o] \\ 0 \\ 1 \end{bmatrix}$$  \hspace{1cm} (1)

Here $\theta_a$ is the design parameter that determines the location of points on the edge I; $l_o$ represents the distance from the centre point $Q$ to the coordinate axis $x_a$; $\alpha_{\min}$ is the minimum value of design parameter $\alpha$ that determines the location of points on the working edge II.

Similarly, the vector $r^{(II)}(\alpha_a)$ of the working edge II in the coordinate system $S_a$ can be expressed as
Here $\alpha_{\text{max}}$ is the maximum angle of design parameter $\alpha_e$.

In the Eqs. (1) and (2), the upper sign indicates the left-side normal section of the rack cutter, and the lower sign indicates the right-side normal section of the rack cutter.

The transverse section of the rack cutter is determined as a section by a plane that has the orientation of $T-T$, as shown in Fig. 2. Coordinate system $S_p$ is a fixed coordinate system on the transverse section profile. $u_i$ ($i=a,f$) is the distance between the origins $O_p$ and $O_n$.

![Coordinate systems of the normal and transverse sections](image)

The rack cutter surface $\Sigma_{ap}$ is represented in coordinate system $S_p$ wherein the normal profile performs translational motion along $z_n$. Then we obtain that surface $\Sigma_{ap}$ is determined by vector function

$$r_{ap}^{(n)}(\alpha_n,u_2)=M_{ap}(u_2)r_{ap}^{(a)}(\alpha_n)$$

(3)

$$r_{ap}^{(a)}(\alpha_n,u_2)=M_{ap}(u_2)r_{ap}^{(a)}(\alpha_n)$$

(4)

Here $M_{ap}(u_2)$ is the transformation matrix from system $S_a$ to $S_p$, and it can be expressed as

$$M_{ap}(u_2)=
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & \cos \beta & \sin \beta & u_2 \sin \beta \\
0 & -\sin \beta & \cos \beta & u_2 \cos \beta \\
0 & 0 & 0 & 1
\end{bmatrix}
$$

(5)

2.2 Geometry of parabolic curve rack cutter

Fig. 3 shows the normal profile $\Sigma_f$ of the rack cutter that generates the gear with concave tooth surface. The design consists of three profiles, namely, circular arc cutting edge I, parabolic curve cutting edge II, and chamfer cutting edge III. $h_1$ is the tooth depth of the rack cutter. $h_{a2}$ is addendum of the rack cutter. $h_{f2}$ is dedendum of the rack cutter. $K-K$ is the symmetrical line of the profile. The coordinate values of the contact points in the coordinate system $S_n$ are: $P_1(\rho_n \sin \alpha_{f1}, -\rho_n \cos \alpha_{f1} + l_f + j f/2,0)$, and $P_2(\rho_n \sin \alpha_{f2}, -\rho_n \cos \alpha_{f2} + l_f + j f/2,0)$. And the pressure angles of the two points are $\alpha_{f1}$ and $\alpha_{f2}$ respectively.

![Normal section profile $\Sigma_f$](image)
The vector $\mathbf{r}_m^{(0)}(\theta_r)$ of edge I in the coordinate system $S_n$ can be expressed as

$$
\mathbf{r}_m^{(0)}(\theta_r) = \begin{bmatrix}
\rho_{\theta_r} (\cos \theta_r - 1 + h_1) \\
\mp \rho_{\theta_r} \sin \theta_r + j/2 \\
0 \\
1
\end{bmatrix}
$$

(6)

Here $\rho_{\theta_r}$ is the circular arc radius; $\theta_r$ is the design parameter that determines the location of points on edge I; $j$ is the tooth backlash.

The vector $\mathbf{r}_m^{(1)}(t)$ of edge II in the coordinate system $S_n$ can be expressed as

$$
\mathbf{r}_m^{(1)}(t) = \begin{bmatrix}
t \cos \alpha - \frac{t^2}{2p} \sin \alpha + L \sin \alpha \\
\pm \left( t \sin \alpha + \frac{t^2}{2p} \cos \alpha - L \cos \alpha + l_f \right) + \frac{j}{2} \\
0 \\
1
\end{bmatrix}
$$

(7)

Where $t$ is the independent variable of the parabolic curve, $t \in [t_1, t_2]$; $\alpha$ is the angle between the normal of the point $O_1$ and the pitch line, $\alpha = \left( \alpha_{11} + \alpha_{12} \right)/2$; $p$ is the coefficient of the parabolic curve, $p = \rho_{sa} \cos \Theta$; $\Theta$ is the half difference of the two pressure angles, $\Theta = \left( \alpha_{12} - \alpha_{11} \right)/2$; $L$ is the distance between the parabolic vertex point $O_1$ and the origin point $Q$, $L = \rho_{sa} \sin^2 \Theta / (2 \cos \Theta) + \rho_{sa} \cos \Theta$; $l_f$ represents the distance from the centre point $Q$ to the coordinate axis $x_n$, $l_f = l_n - j/2$. Derivation of equation (7) is represented in Appendix A.

The cutting edge III produces chamfer angles in the hobbing process, which reduces production cost compared to that of the traditional method through grinding after machining. If the distance from $M_3$ to any point on edge III is $l$, then the vector $\mathbf{r}_m^{(3)}(l)$ of part III in the coordinate system $S_n$ can be expressed as

$$
\mathbf{r}_m^{(3)}(l) = \begin{bmatrix}
M_{2x} - l \cos \delta_r \\
M_{2y} \mp l \sin \delta_r \\
0 \\
1
\end{bmatrix}
$$

(8)

Here $\left( M_{2x}, M_{2y}, 0 \right)$ is the coordinate value of the point $M_2$ in the coordinate system $S_n$, and $M_{2x} = t_f \cos \alpha - t_f^2 \sin \alpha / 2p + L \sin \alpha$, $M_{2y} = \pm \left( t \sin \alpha + t_f^2 \cos \alpha / 2p - L \cos \alpha + l_f \right) + j/2$; $\delta_r$ is the angle between the line $\overline{M_3M_2}$ and coordinate axis $x_n$.

The upper and lower signs in the Eqs. (6)-(8) correspond to the left-side and right-side profile in the coordinate system $S_n$, respectively.

Similarly, the rack cutter surface $\Sigma_p$ in the coordinate system $S_p$ is determined by vector function

$$
\mathbf{r}_m^{(0)}(l, u_r) = \mathbf{M}_m(u_r) \mathbf{r}_m^{(0)}(l)
$$

(9)

$$
\mathbf{r}_m^{(1)}(t, u_r) = \mathbf{M}_m(u_r) \mathbf{r}_m^{(1)}(t)
$$

(10)

$$
\mathbf{r}_m^{(3)}(\theta_r, u_r) = \mathbf{M}_m(u_r) \mathbf{r}_m^{(3)}(\theta_r)
$$

(11)

Where $\mathbf{M}_m(u_r)$ is the transformation matrix from the coordinate system $S_n$ to $S_p$, and it can be expressed as

$$
\mathbf{M}_m(u_r) = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & \cos \beta & \sin \beta & u_r \sin \beta \\
0 & -\sin \beta & \cos \beta & u_r \cos \beta \\
0 & 0 & 0 & 1
\end{bmatrix}
$$

(12)

3. Geometry of the gears generated by rack cutters

3.1 Principle of tooth surface generation

According to the kinematics method of gear geometry theory, the gear tooth surface can be obtained based on the original tooth surface and the given relative motion: the points on the conjugate meshing tooth surface are the results of
during the generation process, the rack cutter is translated with line velocity $v_p$, perpendicular to the rotation axis of the pinion blank whereas the pinion blank is rotated with angular velocity $\omega_1$. The rack cutter pitch plane remains tangent to the pinion pitch cylinder. The pinion tooth surfaces are generated as the envelope to the family of positions of the rack cutter cutting edges in its rolling without sliding relative movement over the pinion pitch cylinder. When the pinion blank rotates counterclockwise with angle $\varphi_1$, the rack cutter will translate towards the left with a distance $r_1\varphi_1$.

3.2 Toh surface generation of pinion

3.2.1 Applied coordinate systems

As shown in Fig. 4, the fixed coordinate system $S_o(O_o-x_0, y_0, z_0)$ is rigidly connected to the pinion. The movable coordinate systems $S_p(O_p-x_p, y_p, z_p)$ and $S_1(O_1-x_1, y_1, z_1)$ are rigidly connected to the rack cutter and the pinion, respectively.

![Coordinate systems for generating pinion](image)

During the generation process, the rack cutter is translated with line velocity $v_p$, perpendicular to the rotation axis of the pinion blank whereas the pinion blank is rotated with angular velocity $\omega_1$. The rack cutter pitch plane remains tangent to the pinion pitch cylinder. The pinion tooth surfaces are generated as the envelope to the family of positions of the rack cutter cutting edges in its rolling without sliding relative movement over the pinion pitch cylinder. When the pinion blank rotates counterclockwise with angle $\varphi_1$, the rack cutter will translate towards the left with a distance $r_1\varphi_1$.

3.2.2 Geometry of the pinion

Suppose that the conjugate point on the cutting edge I is $P_a$, the normal vector $n_p$ of the conjugate point can be expressed in the coordinate system $S_p$ as

$$n^p_{\rho_a}(\theta_a, u_a) = \begin{bmatrix} \frac{\partial \rho_a}{\partial \theta_a} \times \frac{\partial \rho_a}{\partial u_a} \\ \frac{\partial \rho_a}{\partial \theta_a} \times \frac{\partial \rho_a}{\partial u_a} \\ \frac{\partial \rho_a}{\partial \theta_a} \times \frac{\partial \rho_a}{\partial u_a} \end{bmatrix} = \begin{bmatrix} \pm \cos \theta_a \\ -\sin \theta_a \cos \beta \\ \sin \theta_a \sin \beta \end{bmatrix}$$  \hspace{1cm} (14)

$$n^p_{\alpha_a}(\alpha_a, u_a) = \begin{bmatrix} \frac{\partial \alpha_a}{\partial \alpha_a} \times \frac{\partial \alpha_a}{\partial u_a} \\ \frac{\partial \alpha_a}{\partial \alpha_a} \times \frac{\partial \alpha_a}{\partial u_a} \\ \frac{\partial \alpha_a}{\partial \alpha_a} \times \frac{\partial \alpha_a}{\partial u_a} \end{bmatrix} = \begin{bmatrix} \pm \sin \alpha_a \\ -\cos \alpha_a \cos \beta \\ -\cos \alpha_a \sin \beta \end{bmatrix}$$  \hspace{1cm} (15)

The relative velocity $v_p$ of the conjugate point in the coordinate system $S_p$ can be expressed as

$$v^p_{\rho_a}(\theta_a, u_a, \varphi_1) = \begin{bmatrix} \alpha_1 (r_1\varphi_1 \pm [\rho_a \sin \theta_a - (\rho_a + \rho_u) \cos \alpha_{\rho_a} + l_1 \cos \beta - u_a \sin \beta] \\ \alpha_1 (-(\rho_a + \rho_u) \cos \theta_a + (\rho_a + \rho_u) \sin \alpha_{\rho_a}) \end{bmatrix}$$  \hspace{1cm} (16)
By substituting Eqs. (14) and (16), Eqs (15) and (17) into the meshing equation, respectively, analytic expressions of meshing equations corresponding to being-generated gear tooth surfaces are derived, which are shown in Eqs. (18) and (19), respectively.

\[
\Phi (\alpha_x, u_x, \varphi_1) = \pm r_x \varphi_1 + (\rho_x \cos \alpha_x - l_x) \mp u_x \sin \beta - \rho_x \cos \alpha_x \cos \beta = 0
\]  

(18)

\[
\Phi (\alpha_x, u_x, \varphi_1) = \pm r_x \varphi_1 + (\rho_x \cos \alpha_x - l_x) \mp u_x \sin \beta - \rho_x \cos \alpha_x \cos \beta = 0
\]  

(19)

When equations of meshing (18) and (19), for the fillet and active tooth surfaces, respectively, are satisfied, the envelope to the family of generating cutter surfaces in coordinate system \( S_f \) will exist and the pinion tooth surfaces can be obtained

\[
\begin{align*}
[r^{(ii)}_f (\theta_o, u_x, \varphi_1)] &= M_{fp} (\varphi_1) [r^{(ii)}_p (u_x, \theta_o)] \\
\Phi (\theta_o, u_x, \varphi_1) &= 0
\end{align*}
\]  

(20)

\[
\begin{align*}
[r^{(ii)}_f (\alpha_x, u_x, \varphi_1)] &= M_{fp} (\varphi_1) [r^{(ii)}_p (\alpha_x, u_x)] \\
\Phi (\alpha_x, u_x, \varphi_1) &= 0
\end{align*}
\]  

(21)

Where \( M_{fp} (\varphi_1) \) represents the coordinate transformation for the coordinate system \( S_p \) to \( S_f \), and it can be expressed as

\[
M_{fp} (\varphi_1) = \begin{bmatrix}
\cos \varphi_1 & -\sin \varphi_1 & 0 & r_1 (\cos \varphi_1 + \varphi_1 \sin \varphi_1) \\
\sin \varphi_1 & \cos \varphi_1 & 0 & r_1 (\sin \varphi_1 - \varphi_1 \cos \varphi_1) \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]  

(22)

3.3 Tooth surface generation of gear

3.3.1 Applied coordinate systems

As shown in Fig. 5, the pitch line of the rack cutter is tangential to the pitch circle of gear. The movable coordinate systems \( S_p (O_p-x_p, y_p, z_p) \) and \( S_2 (O_2-x_2, y_2, z_2) \) are rigidly connected to the rack cutter and the gear, respectively. The fixed coordinate system \( S_o (O-x_o, y_o, z_o) \) is rigidly connected to the rack cutter.

During the generation process, the gear blank is rotated with angular velocity \( \omega_2 \), and the rack cutter translates with line velocity \( v_p \). When the helical gear rotates counterclockwise with angle \( \varphi_2 \), the rack cutter will translate towards the left with a distance \( r_s \varphi_2 \).

![Fig. 5 Coordinate systems for generating gear](image-url)

3.3.2 Geometry of the gear

If the conjugate point on the cutting edge I is \( P_f^{(i)} \), the unit normal vector of \( P_f^{(i)} \) in the coordinate system \( S_p \), can be expressed as
Similarly, the normal vectors of the conjugate points $P_i^{(m)}$ and $P_j^{(m)}$ in $S_p$ can be expressed as

$$\mathbf{u}_p^{(m)}(\theta_j, u_j) = \begin{vmatrix} \frac{\partial \mathbf{r}_p}{\partial \theta_j} \times \frac{\partial \mathbf{r}_p}{\partial u_j} \\ \frac{\partial \mathbf{r}_p}{\partial \theta_j} \times \frac{\partial \mathbf{r}_p}{\partial u_j} \end{vmatrix} = \begin{bmatrix} \pm \cos \theta_j \\ \sin \theta_j \cos \beta \\ -\sin \theta_j \sin \beta \end{bmatrix}$$

The relative velocity of the conjugate points in the coordinate system $S_p$ can be expressed respectively as

$$\mathbf{r}_p^{(m)}(\theta_j, u_j, \phi_2) = \begin{vmatrix} -a_2 \left[ r \phi_2 - \left( \mp \rho_{L_i} \sin \theta_j \mp \frac{j}{2} \right) \cos \beta + u_j \sin \beta \right] \\ 0 \\ -a_2 \left[ r \phi_2 \mp \left( \frac{\sin \alpha + \frac{j}{2} \cos \alpha}{p} \right) \cos \beta - u_j \sin \beta \right] \end{vmatrix}$$

$$\mathbf{r}_p^{(m)}(u_j, \phi_2) = \begin{vmatrix} -a_2 \left[ r \phi_2 \mp \left( \frac{\sin \alpha + \frac{j}{2} \cos \alpha}{p} \right) \cos \beta + u_j \sin \beta \right] \\ 0 \\ 0 \end{vmatrix}$$

By substituting Eqs. (23) and (26), Eqs. (24) and (27), and Eqs. (25) and (28), into the meshing equation, analytic expressions of meshing equations corresponding to being-generated gear tooth surfaces are derived

$$\Phi(\theta_j, u_j, \phi_2) = \mp r \phi_2 + \left( h_j - \rho_{L_i} \right) \cos \beta \tan \theta_j \pm u_j \sin \beta \mp \frac{j}{2} \cos \beta = 0$$

$$\Phi(u_j, \phi_2) = \pm r \phi_2 + \left( \sin \alpha + \frac{j}{2} \cos \alpha \right) \pm u_j \sin \beta \left( \sin \alpha + \frac{j}{2} \cos \alpha \right) - \left( \frac{1}{2} \cos \alpha \right) \cos \beta = 0$$

When the equation of meshing are satisfied, the envelope of the family of generating cutter surfaces in the coordinate system $S_z$ can be expressed as

$$\begin{cases} r_{z_p}^{(m)}(\theta_j, u_j, \phi_2) = M_{2p} \left( \phi_2 \right) M_{2p}^{(m)} \left( \theta_j, u_j \right) \\ \Phi(\theta_j, u_j, \phi_2) = 0 \\ \Phi(u_j, \phi_2) = 0 \\ \Phi(\theta_j, u_j, \phi_2) = 0 \\ \Phi(u_j, \phi_2) = 0 \\ \Phi(\theta_j, u_j, \phi_2) = 0 \end{cases}$$
Here $M_{2p}(\varphi_2)$ represents the matrix transformation from the coordinate system $S_p$ to $S_2$, and it can be expressed as

$$M_{2p}(\varphi_2) = \begin{bmatrix} \cos \varphi_2 & \sin \varphi_2 & 0 & -r_z (\cos \varphi_2 + \varphi_2 \sin \varphi_2) \\ -\sin \varphi_2 & \cos \varphi_2 & 0 & r_z (\sin \varphi_2 - \varphi_2 \cos \varphi_2) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (35)$$

### 4. Stress analysis

To determine the evolution of contact and bending stressed along the cycle of meshing, investigate the formation of the bearing contact and detect the areas of severe contact stress, the stress analysis was carried out in this work. The finite element analysis of three examples of the proposed gear drives with different pressure angles and the circular-arc gear drive with the pressure angle of $\alpha_n = 25^\circ$ is applied. Application of finite element method requires the development of the finite element model formed by the finite element mesh, the definition of contact surfaces, and the establishment of boundary conditions to load the gear drive with the desired torque. In this section, the stresses of an engagement pair tooth along the cycle of meshing for four gear drives are investigated.

#### 4.1 Development of finite element models and boundary conditions

The solid modes are established by the computer software Matlab and UG. The Matlab program is developed to solve the equations of the tooth surfaces, and then the point coordinate values are imported into the UG software to establish the solid models. The increase of the point number is in favor for the precision of solid models. In this work, there are $200 \times 200$ points on a single side tooth surface to assure the accuracy of the solid models. One of the gear drives is shown in Fig. 6, the main design parameters of the gear drive and cases of design investigated are shown in Tables 1 and 2, respectively. It must be note that four gear drives are of the same pinion and the transmission ratio, and the only difference is the normal profile of the gear. The meshing of the pinion and gear of four gear drives are shown in Fig. 7.

---

**Fig. 6 Gear drive**

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**Table 1 Main design parameters of the gear drives**

<table>
<thead>
<tr>
<th>Centre distance $a$ (mm)</th>
<th>130</th>
</tr>
</thead>
<tbody>
<tr>
<td>Normal module $m_n$ (mm)</td>
<td>6</td>
</tr>
<tr>
<td>Transmission ratio $i_{11}$</td>
<td>0.2</td>
</tr>
<tr>
<td>Number of teeth of pinion $Z_1$</td>
<td>6</td>
</tr>
<tr>
<td>Number of teeth of gear $Z_2$</td>
<td>30</td>
</tr>
<tr>
<td>Helix angle $\beta$ (°)</td>
<td>33.822</td>
</tr>
<tr>
<td>Face width $B$ (mm)</td>
<td>40</td>
</tr>
</tbody>
</table>
Table 2 Design parameters of normal tooth profiles

<table>
<thead>
<tr>
<th></th>
<th>Example 1</th>
<th>Example 2</th>
<th>Example 3</th>
<th>Circular arc</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intersection angle $\theta$ (°)</td>
<td>0</td>
<td>2.5</td>
<td>5</td>
<td>-</td>
</tr>
<tr>
<td>Apex angle $\alpha$ (°)</td>
<td>25</td>
<td>25</td>
<td>25</td>
<td>-</td>
</tr>
<tr>
<td>Pressure angle $\alpha_p$ (°)</td>
<td>25</td>
<td>22.5, 27.5</td>
<td>20, 30</td>
<td>25</td>
</tr>
<tr>
<td>Radius of the convex tooth surface $\rho$ (mm)</td>
<td>9</td>
<td>9</td>
<td>9</td>
<td>9</td>
</tr>
</tbody>
</table>

Fig. 7 Meshing of pinion and gear: (a) Circular arc, (b) $\theta = 0°$, (c) $\theta = 2.5°$, (d) $\theta = 5°$

To save calculation resource, finite element models of three pairs of contacting teeth are used to study the influence of the load sharing on the pinion and gear tooth surfaces, as shown in Fig. 8. The developed approach for the finite element models is accomplished as follows (HUSTON 1994; Zhang, Hua et al. 2010; Fuentes, Iserte et al. 2011; Nenadic, Wodenscheck et al. 2011):

Step 1. The solid models with the format of Parasolid are imported into ANSYS.

Step 2. Compatible linear finite element solid 185 is used to mesh the solid mode of the pinion and gear. The total number of elements is 128800 with 139026 nodes. The material is steel with the properties of Young’s modulus $E = 2.07 \times 10^5$ MPa and Poisson’s ratio 0.30. Torque of 250Nm has been applied to the pinion.

Step 3. Three contact pairs (a contact pair consists of a target surface and a contact surface) are performed at the contact points, as shown in Fig. 9. The convex tooth surface of the pinion and the concave tooth surface of the gear are defined as the contact surface and the target surface respectively.

Step 4. Setting of boundary conditions is accomplished in the cylindrical coordinate systems as shown in Fig. 8. The fixed displacement constraint is applied at the nodes on the bottom rim of the gear and the radial and axial degrees of freedom of the nodes on the bottom of the pinion are also fixed. A tangential distributed load is applied at the nodes on the bottom rim of the pinion. The distributed force $F$ is defined as the following
\[
F = \frac{T}{n \times r_d}
\]

Here, \( T \) is the application torque to the pinion; \( n \) is the number of nodes on the bottom rim of the pinion; \( r_d \) is the radius of the bottom circular arc.

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4.2 Discussion of obtained results

Fig. 10 shows the contact area moving at the driving side of the pinion in one tooth engagement of example 1 (\( \theta = 0^\circ \)). In the process of meshing, the contact area on the tooth flank moves axially across the full face width of the pinion. The contact area is travels along a line which is parallel to the axes of rotation, and it does not travel within the transverse cross section of the pinion.

Fig. 11 shows the contact stresses of four examples at the driving side of the pinion when the pinion’s rotational angle is 60\(^\circ\). The contact stresses of the proposed gear drives (\( \sigma_{contact} = 753.998\text{MPa}, 759.197\text{MPa}, 820.963\text{MPa} \)) are smaller than that of the circular-arc gear drive (\( \sigma_{contact} = 1159.04\text{MPa} \)). And the contact stresses of the proposed gear drives are well distributed. The largest contact stress of the proposed gear drive occurs when the pressure angles are 20\(^\circ\) and 30\(^\circ\) (\( \theta = 5^\circ \)). The maximum contact stress will move towards to the fillet of the gear tooth when the intersection angle increases.
The evolution of contact stresses on the active surface of the pinion is shown in Fig. 12. The contact stress of the gear drive with normal parabolic curve profile is 23.8%-35.3% lower than that of the gear drive with the gear circular arc teeth. And the contact stress reduces by decreasing the intersection angle of the two contact point in parabolic curve. The gear drives with parabolic curve teeth have higher contact stress at the beginning of the meshing due to the higher contact position, after that the contact stress reduces quickly due to the spread of the contact area. The maximum contact stress of the circular-arc gear drive occurs when there is only one engagement contact pair. The maximum stress of the proposed gear drives occur at the beginning of meshing.
Fig. 13 shows the von Mises contact stresses on the active surface of the driving side of the pinion tooth surfaces for the four gear drives when the pinion’s rotational angle is 60°. The maximum stress of the circular-arc gear drive occurs at the contact position near the middle of the tooth flank. The maximum stress of the proposed gear drives occur at the middle of the tooth flank. The contact areas of the proposed gear drives ($\sigma_{\text{contact}} = 434.543\text{MPa}, 432.819\text{MPa}, 439.76\text{MPa}$) are smaller than that of the circular-arc gear drive ($\sigma_{\text{contact}} = 572.432\text{MPa}$). And the contact stresses of the proposed gear drives are well distributed and the contact areas are larger than that of the circular-arc gear drive. The contact stress of the proposed gear drive is the largest when the pressure angles are 20° and 30° ($\theta = 5^\circ$).

![Fig. 13](image)

**Fig. 13** Von Mises contact stress at the rotation angle of 60°. (a) Circular arc ($\sigma_{\text{contact}} = 572.432\text{MPa}$), (b) $\theta = 0^\circ$ ($\sigma_{\text{contact}} = 434.543\text{MPa}$), (c) $\theta = 2.5^\circ$ ($\sigma_{\text{contact}} = 432.819\text{MPa}$), (d) $\theta = 5^\circ$ ($\sigma_{\text{contact}} = 439.76\text{MPa}$).

Fig. 14 shows the evolution of von Mises contact stresses of on the active surface of the driving side of the pinion tooth surface. At the position where only one engagement tooth pair, the contact stress of the gear drive with the gear parabolic curve teeth is 23.2%-29.5% lower than that of the gear drive with the gear circular arc teeth. At beginning of the meshing, the contact stresses of the gear drives having the parabolic curves are a litter higher than that of the circular-arc gear drive. Even the contact stress of the example 3 is the highest one, no remarkable difference in the von Mises contact stress has been found when the parameter $\theta$ is difference.
The evolution of bending stresses at the fillet of the driving side of the pinion tooth surface is shown in Fig. 15. The bending stresses are significantly increasing at first and then decreasing. The bending stresses of the proposed gear drives are about 6.4% to 11.5% higher than that of circular-arc gear drives.

Fig. 15 Evolution of bending stresses

Fig. 16 shows the evolution of contact deformation on the slave side of the driven gear of one engagement tooth. The circular-arc gear drive is of 7% to 14% larger than the proposed gear drives when there are only one engagement tooth surface. At the end of the meshing, the contact deformation increase quickly because of the edge contact.

Fig. 16 Evolution of contact deformation
5. Conclusion

Based on the performed research work, the following conclusions can be drawn:

(1) A novel helical gear drive with the pinion circular arc teeth and the gear parabolic curve teeth is investigated based on the application of mismatched rack cutters. The geometry of the rack cutters surfaces that will generate the pinion and the gear respectively has been developed, and the equations of the gear tooth surfaces are derived.

(2) The parabolic curve profile can reduce 23.8%-35.3% contact stress or 23.2%-29.5% von Mises contact stress compared to the circular arc profile. And the contact areas of the proposed gear drives are larger and well distributed. The higher of the intersection angle of the two pressure angle will reduce the contact stress while there is no remarkable difference in the von Mises contact stress.

(3) The bending stresses of the proposed gear drives are of 4.9% to 9.6% larger than the circular-arc gear drive due to the higher contact position. The contact deformations of the proposed gear drives are of 7% to 14% lower than the circular arc gear drive.

(4) The results obtained by finite element analysis confirm the longitudinal path of contact and importance of the avoidance of edge contact. The tip reliefs to gear tooth surface will be studied forward to avoid the edge contact, the appearance of severe areas of contact stresses and to get a smooth load transfer between consecutive pairs of contacting teeth.

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Appendix A. Derivation of the working edge of the rack cutter with concave tooth profile

Suppose that the coordinate system $S_i(O; x_i, y_i, z_i)$ is connected to the rack cutter for the generation of the gear, as shown in Fig. A.1. The origin $O_i$ is on the vertex of the parabolic curve. And the parabolic curve is tangent to a circular arc with radius $\rho$ at the points $P_1$ and $P_2$. $Q$ is the centre of the circular arc. The coordinate value of the tangent points in the coordinate system $S_i$ are $P_1(-\rho\sin\theta, L-\rho\cos\theta, 0)$ and $P_2(\rho\sin\theta, L-\rho\cos\theta, 0)$. $L$ is the distance between the point $O_i$ and $Q$.

The equation of parabolic curve can be expressed in $S_i$ as

![Fig. A.1 Coordinate systems](image-url)
According to the differential geometry theory, the slope of the point $P_2$ is

$$k = \tan \theta = \frac{\rho \sin \theta}{p}$$ (A.2)

Here $\theta = (\alpha_2 - \alpha_1)/2$.

Solving Eq. (A.2)

$$p = \rho \cos \theta$$ (A.3)

Substituting Eq. (A.3) into Eq. (A.1), the equation of the parabolic curve can be expressed in $S_t$ as

$$r_i = \begin{bmatrix}
t \\
t^2/2p \\
n 0 \\
1 \\
\end{bmatrix}$$ (A.4)

At the tangent point $P_2$, the follow equation is established

$$\left( \rho \sin \theta \right)^2 = 2p(L - \rho \cos \theta)$$ (A.5)

Solving Eq. (A.5)

$$L = \frac{\rho \sin \theta^2}{2\cos \theta} + \rho \cos \theta$$ (A.6)

According to the principle of coordinate transformation, the parabolic curve in the coordinate system $S_n$ can be expressed as

$$r_i(t) = M_n r_i(t)$$ (A.7)

Here, $\alpha = \alpha_1 + \theta = (\alpha_2 + \alpha_1)/2$; $M_n$ is the transformation matrix from the coordinate system $S_i$ to $S_n$, and it can be expressed as

$$M_n = \begin{bmatrix}
\cos \alpha & -\sin \alpha & 0 & L \sin \alpha \\
\sin \alpha & \cos \alpha & 0 & -L \cos \alpha + l + j/2 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}$$ (A.8)

References


