Mathematical model and manufacturing of a human or a robotic knee joint

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Abstract
This paper proposed two imaginary planar rack-cutters to create a spherical gear pair with ring-involute teeth. The spherical gear pair can be used to a knee joint mechanism. The imaginary planar rack cutter consists of two kinds of conical tooth. One is convex conical teeth to create a spherical gear with concave ring-involute teeth; the other is concave conical teeth to create a spherical gear with convex ring-involute teeth. Based on two-parameter envelope theory and coordinate transformation matrix, the planar rack cutter with concave teeth or convex teeth can derive the mathematical model of the spherical gears with convex teeth or concave teeth, respectively. Through the obtained mathematical model, a computer-aided software package and a 5-axis CNC machine, the real prototype of the spherical gear is manufactured. The real prototype of the spherical gear is used to replace knee joint mechanism of a skeleton.

Key words: Mathematical model, Knee, Mechanism, Joint

1. Introduction

A spherical gear pair mechanism can be used to a human or a robotic joint mechanism. The kind of the mechanism has a double degree of freedoms (DOFs). Today, a human knee joint mechanism is used to replace worn away or broken knee joint, as shown in Fig. 1. However, a dislocation is usually happen. The mechanism cannot afford a big force. A sliding is very easy happen. In precision industry, the mechanical arm has replaced some techniques, such as assembling, mating and welding, as shown in Fig. 2 (Roshemi, 1994). This mechanism has many elements including many helical and bevel gears. All of these gears are single DOF. A similar elbow mechanism (Shyue-Cheng, 2007) with a three degrees of freedom was presented and shown in Fig. 3. By comparing Figs. 2 and 3, it can be seen that the traditional robotic arm’s wrist mechanism needs many more elements and positions to achieve their function. Moreover, compared to Figs. 1 and 3, it is clear that the joint mechanism of the spherical gear with ring-involute teeth has a better degree of freedom and good bearing load.

The spherical gear mechanism was first used in the flexible wrist of the industrial robot developed by Trallfa company in Norway. Zhiquan et al., (1990) proposed a designed method of a spherical gear mechanism. A geometry model of spherical gears with distributed convex or concave teeth on a half-spherical blank was presented by (Shyue-Cheng, 2002a, 2002b, 2003 and 2007). Using an imaginary rack cutter to generate a spherical gear was presented by (Shyue-Cheng, 2005), too. Huran (2009a and 2009b) used the concave cone teeth and its cutting tool to manufacture a spherical gear. Ting and Cunyun (2008, 2009a and 2009b) used grinding process to generate the spherical gear. However, this kind of spherical gear is easily to slide when z-axial of this mechanism was rotated. It was not suitable applied to a joint mechanism. The meshing principles and the manufacturing methods for spherical gears with concave cone surface teeth were presented by (Huran, 2009). A transmission characteristic of the spherical gear mechanism was studied by (Cunyun et al., 2006). Chensheng et al., (2011) presented a conical concave or convex of a spherical gear with two-degree of freedom. Nayak and Shunmugam (2012) carried out on a five-axis CNC machining centre using a specially developed conical end-mill cutter having a semi-cone angle of 20° to machine a spherical gear with ring-involute tooth. However, using imaginary planar rack-cutters to generate a spherical gear pair with
ring-involute teeth is rarely investigated.

In this paper, the mathematical model of an imaginary planar rack-cutter with convex teeth or concave teeth is designed. First, a mathematical model of the imaginary planar rack cutter having convex teeth or concave teeth is derived using geometric relations and coordinate transformation. After the relationship between the coordinate system of the imaginary planar-rack cutter and that of the spherical gear pair is set up, a family of imaginary planar rack-cutter surfaces is obtained by the homogeneous coordinate transformation matrix. Based on the two-parameter envelope theory (Litvin and Seol, 1996), the equations of meshing between the planar rack-cutter and the spherical gear are determined. Substituting the equation of meshing into the family of imaginary planar rack cutter surfaces, the mathematical models of the spherical gear pair are obtained and their geometries are plotted using an in-house software package. A script file for the CAD simulation of the proposed spherical gear pair is used to generate the CNC program for machining the proposed gear. The actual machining of the spherical gear pair is manufactured by a five-axis CNC machining centre using a spherical end-mill cutter. The actual small-size physical model is used to replace the knee joint on a skeleton’s knee joint. Results from these mathematical models should have applications in the design of a knee joint mechanism.
2. The produce process of the spherical gear mechanism

2.1 Imaginary planar rack-cutter design

The cross section of an imaginary rack cutter was shown in Fig. 4. The rack cutter \( \Sigma_t \) is used to create a spherical gear. The rack cutter \( \Sigma_c \) is used to create a spherical pinion. The concept of an imaginary planar rack-cutter comes from the imaginary rack-cutter by distributing conical teeth along \( x \)-axis and \( y \)-axis. The part profile of one tooth on planar rack-cutter was shown in Fig. 5. The cutting curve of the part tooth was divided five segments \( \overline{AB}, \overline{BC}, \overline{CD}, \overline{DE} \) and \( \overline{EF} \).

![Fig. 4 The imaginary rack cutter.](image)

![Fig. 5 The parameters of the one tooth profile.](image)

In Fig. 5, symbol \( b_c \) is equal to \( 0.25m \). Because the knee joint of the human body is smaller space, the modulus \( m=2 \) is chosen so that the teeth on the spherical gear are smaller. \( r \) is the radius of the root fillet for the tooth profile. \( a_c \) is the vertical distance from point D to \( O_1 \) or point \( O_1 \) to C, respectively. \( \phi_c \) is the gear’s pressure angle. The five segments \( \overline{AB}, \overline{BC}, \overline{CD}, \overline{DE} \) and \( \overline{EF} \) of the imaginary planar rack-cutter in the coordinate system \( O_1 \xi \zeta_1 \) are represented as:

\[
\mathbf{R}^m_{\xi_1} = \begin{bmatrix}
(-\frac{P}{2} + t)\cos \theta + pc, \\
(-\frac{P}{2} + t)\sin \theta + pc, \\
-a_c + r \sin \phi_c - r
\end{bmatrix}, \quad 0 \leq t \leq \overline{P} - b_c - a_c \tan \phi_c - r \cos \phi_c, \quad 0 \leq \theta \leq 2\pi.
\]
where symbol $p$ is the diameter pitch of the tooth profile and $p=mn$. $t$ is the curvilinear parameter at each segment. The mathematical model of the surface rack cutter with conical tooth surface was got when the cross section of the tooth profile in Fig. 5 rotates along the $z_1$ axis about angle $\theta$. The symbol $c_x$ is used to duplicate the conical tooth along the $x_1$-axis direction, $c_x=0, 1, 2\ldots$ Similarly, $c_y$ is used to duplicate the conical tooth along the $y_1$-axis direction $c_y=0, 1, 2\ldots$. Using Eqs. (1)-(5) and the values of the Table 1, a complete planar rack-cutter with conical teeth was displayed by Mathematica software package, as shown in Fig. 6.

Table 1 The parameters of ring-involute gear.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Symbol</th>
<th>Numerical</th>
</tr>
</thead>
<tbody>
<tr>
<td>Module</td>
<td>$m$</td>
<td>2mm</td>
</tr>
<tr>
<td>Number of teeth</td>
<td>$N_t$</td>
<td>36</td>
</tr>
<tr>
<td>Vertical distance</td>
<td>$a_{c}$</td>
<td>1m</td>
</tr>
<tr>
<td>Pitch Radius</td>
<td>$r_{p1}$</td>
<td>$mN_t/2$</td>
</tr>
<tr>
<td>Pressure angle</td>
<td>$\phi_{c}$</td>
<td>20°</td>
</tr>
<tr>
<td>Pitch</td>
<td>$p$</td>
<td>$m\pi$</td>
</tr>
</tbody>
</table>

Fig. 6 The complete model of imaginary planar rack cutter with conical teeth
2.2 The motion between the imaginary planar rack cutter and spherical blank

The gear manufacture processes have hobbing, shaping and others. The mathematical model of a gear manufactured by these processes can be simulated by an imaginary rack cutter to cut a gear’s blank. In this paper, an innovation simulation concept is to use imaginary planar rack cutter to cut a spherical blank. As shown in the right side figure of Fig. 7, the imaginary planar rack cutter move $V_x$ along the X axis and the spherical blank rotated along the Y axis about the angle $\phi_Y$. Similarly, the imaginary planar rack cutter move $V_Y$ along the Y axis and the spherical blank rotated along the X axis about the angle $\phi_X$. Based on the two moving processes, a complete spherical gear with concave ring-involute teeth will be obtained.

![Fig. 7 The moving relationship between the imaginary planar rack cutter and the spherical gear blank.](image)

2.3 Coordinate transformation

An imaginary planar rack cutter with convex tooth can create the spherical gears with concave tooth and vice versa, the imaginary planar rack cutter with concave teeth can produce the spherical gears with convex one. Here, we just describe the mathematical model of the spherical gear with convex ring-involute teeth. Using the same method, we can obtain the mathematical model of the spherical gear with concave ring-involute teeth. The relationship between the coordinate systems of the imaginary planar rack cutter with convex teeth and the spherical blank is shown in Fig. 8.

![Fig. 8 The coordinate systems between the rack cutter and the spherical blank.](image)

At the start, the imaginary planar rack cutter and the spherical blank need to be prepared to move to the position of machining process, like the concept of dividing head. The coordinate system $S_1(O_x, x_1, y_1, z_1)$ is rigidly connected to imaginary planar rack cutter. The coordinate system $S_2(O_x, x_2, y_2, z_2)$ is rigidly connected to the spherical blank and rotated with spherical blank. The coordinate systems $S_z(O_x, x_y, y_1, z_0)$, $S_z(O_x, x_1, y_z, z_0)$ and $S_z(O_x, x_1, y_z, z_0)$ are auxiliary coordinate systems. The coordinate system $S_1(O_{1f}, x_{1f}, y_{1f}, z_{1f})$ is a fixed coordinate system. The coordinate transformation matrix from coordinate system $S_1(O_x, x_1, y_1, z_1)$ to coordinate system $S_2(O_x, x_2, y_2, z_2)$ can be divided into following steps:

1. The imaginary planar rack cutter on the coordinate system $S_1(O_x, x_1, y_1, z_1)$ is first translated the linear displacements $c_x$ and $c_y$ along the $x_1$-axis and $y_1$-axis to coordinate system $S_2(O_{1h}, x_{1h}, y_{1h}, z_{1h})$, respectively.
Then, the coordinate system $S_h(O_h,x_h,y_h,z_h)$ is translated the linear displacements $s_x$ and $s_y$ along the $x_h$-axis and $y_h$-axis to coordinate system $S_i(O_i,x_i,y_i,z_i)$.  

2. The coordinate system $S_i(O_i,x_i,y_i,z_i)$ is translated the linear displacements $s_z$ along the $z_i$-axis to coordinate system $S_f(O_f,x_f,y_f,z_f)$. Then, the coordinate system $S_f(O_f,x_f,y_f,z_f)$ is rotated along the $x_f$-axis and $y_f$-axis about the angles $\phi_x$ and $\phi_y$, respectively, to the coordinate system $S_j(O_j,x_j,y_j,z_j)$.  

3. Final, the coordinate system $S_j(O_j,x_j,y_j,z_j)$ is rotated along the $x_j$-axis and $y_j$-axis about the angles $\phi_{i\phi}$ and $\phi_{i\varphi}$, respectively, to the coordinate system $S_2(O_2,x_2,y_2,z_2)$.  

Following steps 1 to 3, we can obtain the matrix $M_{i2}$ from coordinate systems $S_1$ to $S_2$ and be represented as: 

$$M_{i2} = M_{i1}M_{i1}M_{i1}$$  

where 

$$M_{i1} = \begin{bmatrix}
1 & 0 & 0 & c_xp + s_x \\
0 & 1 & 0 & -c_yp + s_y \\
0 & 0 & 1 & s_z \\
0 & 0 & 0 & 1
\end{bmatrix}$$  

$$M_{i\phi} = \begin{bmatrix}
\cos \phi_x & \sin \phi_x & 0 & 0 \\
-\sin \phi_x & \cos \phi_x & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}$$  

$$M_{i\varphi} = \begin{bmatrix}
\cos \phi_{i\phi} & \sin \phi_{i\phi} & 0 & 0 \\
-\sin \phi_{i\phi} & \cos \phi_{i\phi} & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}$$

where $\phi_x$ and $\phi_y$ are called the two parameters of motion. $M_{i1}$ is used to represent coordinate transformation matrix from coordinate system $S_i$ to coordinate system $S_f$. $M_{i\phi}$ is coordinate transformation matrix from coordinate system $S_f$ to coordinate system $S_j$. $M_{i\varphi}$ is coordinate transformation matrix from coordinate system $S_j$ to coordinate system $S_2$. Parameters $s_x$ and $s_y$ are equals to $r_{\phi} \phi_x$ and $r_{\varphi} \phi_y$, respectively. Parameters $\phi_{i\phi}$ and $\phi_{i\varphi}$ are equal to $c_x p/r_{\phi}$ and $c_y p/r_{\varphi}$, respectively. Using Eqs. (1) to (6), two-parameter family of the imaginary planar rack cutter surfaces can be obtained and represented by:

$$R^i_2(\phi_x,\phi_y,t,\theta) = M_{i2}(\phi_x,\phi_y)R^i_1(t,\theta),$$

$$i = AB, BC, CD, DE, EF.$$  

The position vector $R^i_2$ is the family of the imaginary planar rack-cutter surfaces. $\phi_x$ and $\phi_y$ are called parameters of motion. The position vector $R^i_1$ was presented in Eqs.(1) to (5). The second superscript $i$ is the segments $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DE}$ and $\overline{EF}$.  

3. Equations of meshing with two-parameter envelope  

Equations of meshing are the necessary condition for determining the conjugate condition between the imaginary planar rack cutter and spherical blank. Equations of meshing can be obtained by the method of differential geometry. This paper use two-parameter envelope theory to determine the equations of meshing between the imaginary planar rack cutter and spherical blanket. The equations of meshing can be determined by differential geometry method when the family of imaginary planar rack-cutters surfaces was obtained. The equations of meshing by two-parameter envelope theory can be represented by:

$$h(\phi_x, \phi_y, t, \theta) = \left\langle \frac{\partial R^i_2}{\partial t} \times \frac{\partial R^i_2}{\partial \theta} \right\rangle \frac{\partial R^i_2}{\partial \phi_x} = 0$$  

$$k(\phi_x, \phi_y, t, \theta) = \left\langle \frac{\partial R^i_2}{\partial t} \times \frac{\partial R^i_2}{\partial \theta} \right\rangle \frac{\partial R^i_2}{\partial \phi_y} = 0$$
Through Eqs. (11) to (12), the equations of meshing can be represented by:

\[
 h(\phi_x, \phi_y, t, \theta) = \phi_x - \frac{n_{i_z} y_i^l - c_n n_{i_z} p - n_{i_y} z_i^l}{r_p n_{i_z}}
\]

\[
 k(\phi_x, \phi_y, t, \theta) = \phi_y + \frac{E \cos \phi_x + F \sin \phi_x - r_p n_{i_x}}{r_p n_{i_x} \sin \phi_x - r_p n_{i_x} \cos \phi_x}
\]

where \( E = r_p n_{i_z} - n_{i_z} x_i^l + n_{i_z} z_i^l + c_n n_{i_z} p \), \( F = r_p n_{i_y} - c_n n_{i_y} p + n_{i_y} x_i^l + c_n n_{i_y} z_i^l \). The symbols \( n_{i_x}, n_{i_y} \) and \( n_{i_z} \) are the components of normal vector to the imaginary planar rack-cutter surface at \( x, y \) and \( z \) direction. One simultaneously solves the equations (10), (13), (14) and using SolidWorks software package and Table 1, the geometric model of spherical gear with convex ring-involute teeth is shown in Fig. 9.

![Fig. 9 The geometric model of the spherical gear with convex ring-involute teeth (1:1).](image)

Using the above same method, another spherical gear with concave ring-involute teeth can be obtained and shown in Fig. 10.

![Fig. 10 The geometric model of the spherical gear with concave ring-involute teeth.](image)

4. Real small scale spherical gear pair

To minimize human error in the design and manufacture of the geometric surface of a spherical gear pair with concave or convex ring-involute teeth, the generated cutting path was verified before actual machining. In this section, the cutter path is determined for milling the spherical gear with a spherical end cutter. In the milling process, the spherical blank is held on a rotary table of a milling machine. Fig.11 shows a spherical end cutter machining a curved surface. The cutter moves on a parallel curved surface with an offset to the spherical end cutter of radius \( R \). As plotted in Fig. 11, the spherical gear with convex ring-involute teeth is \( R_x(\beta, \theta) \). The tip of cutter follows the path is

\[
 r = R_x(\beta, \theta) + R(N - a_s)
\]

where \( N \) is the unit normal vector to the spherical gear at \( \beta = \beta_0 \) and \( a_s \) is a unit vector along the cutter axis \( a_s \). [Faux and Pratt, 1979].
Using CNC five-axis machine, post-process file of the spherical gear with convex or concave ring-involute teeth, and FANUC 18iMb5 controller, the complete spherical gear pair with ring-involute teeth can be manufactured. To illustrate, a small scale element is done by the CNC five axes machine in our Lab. The cutting conditions for manufacturing the spherical gears had listed in the Table 2. The scale prototype of the complete assembly for spherical gear with ring-involute teeth used to knee joint of a skeleton was shown in Fig. 12.

![Fig. 11 Spherical end cutter turning 3D surface.](image)

![Fig. 12 The spherical gear pair with ring-involute teeth used to the skeleton's knee joint (1:0.25).](image)

<table>
<thead>
<tr>
<th>Used tool</th>
<th>Tungsten Carbide Ball End Mill SØ2</th>
<th>Tungsten Carbide Ball End Mill SØ8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Feed speed</td>
<td>500 mm/min</td>
<td>800 mm/min</td>
</tr>
<tr>
<td>Cutting speed</td>
<td>40 m/min</td>
<td>60 m/min</td>
</tr>
<tr>
<td>Spindle speeds</td>
<td>5000 rpm</td>
<td>2400</td>
</tr>
<tr>
<td>CAM software</td>
<td>Unigraphics</td>
<td></td>
</tr>
</tbody>
</table>

5. Conclusions

In this study, the geometry mathematical model of imaginary planar rack cutter with convex ring-involute teeth is proposed. The advantage of the imaginary planar rack is easier to generate the mathematical model of a spherical gear pair with convex or concave ring-involute teeth. The homogeneous coordinate transformation matrix develops the family of the imaginary planar rack-cutter surfaces. Based on two-parameter envelope theory, the equation of meshing is determined by differential geometry method. The mathematical models of the spherical gear with convex teeth and
concave teeth were obtained by the equation of meshing and the family of the imaginary planar rack-cutter surfaces. A tool path generation method for determining the cutter location for CNC five-axis machining of the spherical gear pair with ring-involute teeth was implemented. A complete artificial knee joint mechanism of the spherical gear pair with ring-involute teeth used to a skeleton’s knee joint was developed. The proposed mathematical model analysis for a spherical gear pair should be helpful in the design and production of a knee joint mechanism.

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