Multistage stochastic programming model and solution algorithm for the capacity expansion of railway network

Takayuki SHIINA*, Tomoaki TAKAICHI*, Yige LI*, Susumu MORITO* and Jun IMAIZUMI**

* School of Creative Science and Engineering, Waseda University
3-4-1 Okubo, Shinjuku-ku, Tokyo 169-8555, Japan
E-mail: tshiina@waseda.jp

** Faculty of Business Administration, Toyo University
5-28-20 Hakusan, Bunkyo-ku, Tokyo 112-8606, Japan

Abstract
We consider a capacity expansion problem for a railway network under uncertainty. In our approach, integer and stochastic programming provide a basic framework. We develop a multistage stochastic programming model in which some of the variables are restricted to integer values. Given the distribution of the number of customers in a scenario, the problem of minimizing the expected value of the total investment cost is considered. The problem is reformulated as a problem with first stage integer variables and continuous second stage variables. An L-shaped algorithm is proposed to solve this problem.

Keywords: Railway planning, Capacity expansion, Optimization, Multistage stochastic programming, L-shaped method

1. Introduction

The railway business is creating an investment plan to increase safety and improve the service level. Looking outside of Japan, the establishment of new railroads was considered, and large-scale investments are taking place. Demand for cargo transport via railway in North America is on the rise. According to Lai and Barkan (2011), the demand is expected to increase by 84% in 2035, compared to 2007. Lai and Shih (2013) dealt with the capacity expansion problem of railways as a stochastic programming problem to maximize the fulfilled demands for railways, and to extend routes within a limited budget. The problem was formulated as multistage stochastic programming in which demand fluctuation over multiple periods was considered. To deal with the capacity expansion problems, Ahmed et al. (2003) and Ahmed and Sahinidis (2002) proposed an approximate solution by reformulation of the problem.

In Japan, a few new railway lines were constructed and the abolition of trams continued with the progress of motorization. The declining population being considered, introduction of a light rail transit (LRT) is being studied in order to consolidate homes and workplaces into the city center in the future, and design a new city. Although it is difficult to construct a new railway or subway, the introduction of the LRT complements the conventional public transport network and an LRT functions effectively as an auxiliary network. In order to evaluate the introduction of such a new transportation system, we propose a stochastic programming model which provides an economically effective introduction plan as an optimal solution. The model deals with a network consisting of nodes and arcs, whose nodes represent important points such as a station and a branch point, and arcs express railroads. Each arc has its capacity, which is the maximum number of trains that can run on the railroad corresponding to the arc. For each OD pair, a traffic demand, which is practically the number of (potential) customers, is given as a random variable following a distribution scenario. The objective of the model is minimizing the expected value of the total investment cost to satisfy all demands by capacity expansion of arcs.

The capacity expansion problems in the electric power field can be mentioned as related research. In capacity expansion problems of electric power, it was requested to introduce facilities so as to satisfy the total electric power demand. The multistage stochastic programming models were developed by Shiina and Birge (2003) and Shiina (2011)
especially for power generation problems. In the railway expansion problem dealt with in this paper, passenger demand is given for each OD pair. In order to satisfy the demand for each OD pair, we have to extend the capacity of arcs; however, to calculate how much we should extend it, we have to know how the train routes will change due to the expansion. In other words, we have to decide the train routes in the (tentatively) expanded network. In addition, our proposed model considers the demand fluctuation for all OD pairs and, furthermore, includes several decision variables, e.g., the construction or abolition of equipments. Those variables are practically important and expressed as integer variables. In general, a model with integer variables is difficult to solve; however, our model has the property called block separable recourse, which is introduced by Louveaux (1986). Due to this property, one can decompose the capacity expansion problem into the master problem with integer variables and subproblem for each scenario, and moreover, solve it by so-called the L-shaped method.

Next, the basic framework of the problem is described. Stochastic programming (Birge (1997), Birge and Louveaux (1997)) studies optimization under uncertainty. An important problem in stochastic planning is a two-stage stochastic program. First of all, a decision is made at the first stage, and then second stage decisions are made according to the realizations of the random variables. The L-shaped method by Van Slyke and Wets (1969) is well-known as a solution method. This method was applied to the concentrator location problem (Shiina (2000a,b)). Variatnts of the nested decomposition method have been developed for multistage problems by Birge (1985), Louveaux (1980) and Birge et al. (1996).

As for the solution method, the following modifications are made. In the ordinary L-shaped method, it is necessary to add feasibility cuts for the recourse problem to be feasible. In order to generate feasibility cuts, we need to find an extreme ray in the dual problem of the recourse problem. It is necessary to solve the optimization problem separately for generating an extreme ray, which takes time and effort for calculation. Since this calculation is complicated, a penalty term for unsatisfied demand is added to the objective function of the recourse problem. The introduction of the penalty term allows a solution not to satisfy the demand. However, due to this, the modified problem always has a feasible solution, which is called a complete recourse. This avoids generating feasibility cuts in the conventional L-shaped method. In our L-shaped method, a parameter controls the penalty value. In this paper, we show how to set the value of the parameter so as to eventually satisfy the demands, which has not been mentioned clearly in the previous research by Lai and Shih (2013). By adding the optimality cut, which approximates the penalty, to the master problem, the capacity expansion is chosen to reduce the penalty for unsatisfied demand. Repeating this procedure, one can decrease the amount of unsatisfied demand and finally obtain a solution that satisfies all the demands.

In section 2, we show that the capacity expansion problem is reformulated as a problem with integer variables in the first stage integer variables and continuous variables in the second stage. In section 3, we present numerical results obtained by applying our solution approach to test problems. In appendix, we review the multistage stochastic programming problems.

2. Multi-stage Capacity Expansion Problem

Under the following assumptions, we deal with an investment decision problem regarding the railroad for multiple periods. Let the customer traffic demand follow a random distribution and consider plans to expand the railway network. The section capacity of a railroad network depends on the length of the section, the presence or absence of a saving line, the number of signals, the average speed of trains, and so on. For example, saving lines, the number of traffic lights, and double tracking are capacity expansion options. For each link (section), a combination of determining the number of evacuation lines and traffic lights to be installed, and whether to draw double tracks, is an option. For this multi-period plan, we consider which sections are to be expanded and in which order to expand, and minimize the sum of costs per period with the demand fluctuation being considered.

The preconditions for this problem are shown below:

(1) The railroad lines are defined as a network consisting of node (station) set and arc (rail) set.
(2) Traffic demand is given for each OD pair, and it is defined as a random variable.
(3) The capacity of an arc means the maximum number of trains that can run on the (railroad corresponding to the)
arc.
(4) The amount of the increase in demand is assumed to be uncertain. The pattern of the demand increase per period is a random variable that follows a known distribution (e.g. 3%, 5%, 7% increase with the equal probability of 1/3 from the first to second term demand).
(5) Increase in capacity is prepared for each arc in the form of extension work (increase traffic lights, create escape lines, etc.).
(6) The objective function is to minimize the sum of the cost of capacity increase, the cost of train running, and the penalty cost for not satisfying demand.
(7) For constraints, there are budget constraints, constraints on the number of extensions, demand constraints, and so on.

The following symbols are used:

**Definition of sets**
- $N$: set of nodes (stations)
- $A$: set of link $(i, j)$
- $K$: set of OD pair $k$
- $Q$: set of choice $q$ for capacity expansion
- $T$: set of planning periods
- $O(i)$: set of arcs which emanate from node $i$
- $I(i)$: set of arcs which enter into node $i$

**Parameters**
- $B^t$: Cumulative budget at stage $t$
- $C_{ij}$: Transportation cost for arc $(i, j)$
- $H^q_{ij}$: Capacity expansion cost of engineering alternative $q$ for arc $(i, j)$
- $U_{ij}$: Initial capacity for arc $(i, j)$
- $a^q_{ij}$: Additional capacity of engineering alternative $q$ for arc $(i, j)$
- $\alpha$, $\beta$: Weight for objective function
- $\tau$: Penalty for unfilled demand
- $d^k$: Random traffic demand for OD pair $k$ at stage $t$
- $d^k_{ts}$: Realization of traffic demand of OD pair $k$ at stage $t$ under scenario $s$
- $p^t_s$: Probability of scenario $s$ at stage $t$
- $o_k$: Origin of OD pair $k$
- $e_k$: Destination of OD pair $k$

**Decision variables**
- $y^k_{ij}$: Number of trains of OD pair $k$ running on arc $(i, j)$ at stage $t$ under scenario $s$
- $u^q_{ij}$: 1, if engineering alternative $q$ is adopted for arc $(i, j)$ at stage $t$, otherwise 0
- $d^k_{ij}$: Fulfilled traffic demand of OD pair $k$ at stage $t$ under scenario $s$

Using the above symbols, the problem can be formulated as a multi-period stochastic programming problem:

$$\min \sum_{(i, j) \in A} \sum_{t \in T} \sum_{q \in Q} \alpha^q h^q_{ij} u^q_{ij} + \beta \sum_{t \in T} \sum_{s=1}^K p^t_s \sum_{(i, j) \in A} \sum_{k \in K} c_{ij} d^k_{ij}$$ (1) subject to

$$\sum_{(i, j) \in A} \sum_{t \in T} \sum_{q \in Q} h^q_{ij} u^q_{ij} \leq B^t, t \in T$$ (2)

$$\sum_{t \in T} u^q_{ij} \leq 1, (i, j) \in A$$ (3)

$$\sum_{k \in K} y^k_{ij} \leq U_{ij} + \sum_{t \in T} \sum_{s=1}^K u^q_{ij} d^k_{ij}, \quad (i, j) \in A, t \in T, s = 1, \ldots, K_t$$ (4)

$$\sum_{(i, j) \in O(i)} y^k_{ij} - \sum_{(j, i) \in I(i)} y^k_{ij} = \begin{cases} d^k_{ij} & i = o_k \\ -d^k_{ij} & i = e_k \\ 0 & \text{otherwise} \end{cases}, \quad t \in T, k \in K, s = 1, \ldots, K_t$$ (5)

$$y^k_{ij} \geq 0, (i, j) \in A, t \in T, k \in K, s = 1, \ldots, K_t$$

$$u^q_{ij} \in [0, 1], (i, j) \in A, t \in T, q \in Q$$
The objective function (1) minimizes the sum of the expansion construction costs, and train running expenses. The inequalities (2) represent the constraint that expansion costs must not exceed the predetermined budget. The constraints (3) ensure that the expansion work of the arc \((i, j)\) is, at most, once. The inequalities (4) are the constraint that the total number of trains running on arc \((i, j)\) is less than or equal to the sum of the original capacity and expanded capacity. The equations (5) express the actual traffic volume that is filled with the \(k\)th OD pair at stage \(t\) under scenario \(s\).

The decision variable \(w_{ij}^q\) is an aggregate level decision, and the decision variable \(q_{ij}^s\) having a subscript \(s\) is defined as a detailed level decision. The structure of the constraints is illustrated in Fig 1.

\[
\begin{align*}
\text{Aggregate level decision} & \quad \text{Detailed level decision} \\
\sum_{t \in T} \sum_{q \in Q} h_{ij}^q w_{ij}^q & = 0 & 0 \\
\sum_{t \in T} w_{ij}^q & = 0 & 0 \\
\sum_{t \in T} \sum_{q \in Q} u_{ij}^q & = 0 & 0 \\
0 & = 0 & 0 \\
\end{align*}
\]

Fig. 1 Structure of the problem

The L-shaped method is often applied to this type of problem. Indeed, in Lai and Shih (2013), it was applied to the railway expansion problem. It is true that they showed its effectiveness by numerical experiment, but they did not provide the theoretical backgrounds such as a proof of convergence. It became clear that the problem has a property of the block separable recourse from Proposition 4.1. The capacity expansion problem for a railway line can be transformed into a problem that has first stage integer variables and continuous second stage variables. This problem can be solved using the L-shaped method. To solve the problem, the following master problem is formulated.

**Master Problem for Multistage Stochastic Railway Expansion Problem**

\[
\begin{align*}
\min & \quad \sum_{(i,j) \in A} \sum_{t \in T} \sum_{q \in Q} \alpha_t h_{ij}^q w_{ij}^q + \sum_{s=1}^{K_1} \beta_s \theta_s^1 + \ldots + \sum_{s=1}^{K_T} \gamma_s T_s T_s \\
\text{subject to} & \quad \sum_{(i,j) \in A} \sum_{t \in T} \sum_{q \in Q} h_{ij}^q w_{ij}^q \leq B_t, t \in T \\
& \quad \sum_{t \in T} \sum_{q \in Q} u_{ij}^q \leq 1, (i, j) \in A \\
& \quad w_{ij}^q \in \{0, 1\}, (i, j) \in A, t \in T, q \in Q
\end{align*}
\]

The recourse functions \(Q_t(w_{ij}^q, \ldots, w_{ij}^q, q \in Q), t \in T, s = 1, \ldots, K_t\) are not given explicitly in advance. Describing a recourse function completely is equivalent to adding many valid inequalities. Instead, to approximate \(\hat{\theta}_t^s \geq Q_t(w_{ij}^q, \ldots, w_{ij}^q, q \in Q)\) the optimality cuts are added to the formulation. After solving the mixed integer programming master problem, the optimal solution is obtained. The optimal solution to the master problem is denoted by \(w_{ij}^{*q}, (i, j) \in A, t \in T, q \in Q\). Then, using the optimal solution of the master problem, the recourse problem for stage \(t\) scenario \(s\) can be solved.

**Recourse Problem for Stage \(t\) Scenario \(s\)**

\[
\begin{align*}
Q_t^s(w_{ij}^q, \ldots, w_{ij}^q, q \in Q) = \min & \quad \beta \left( \sum_{(i,j) \in A} \sum_{k \in K} c_{ij} y_{ij}^{ks} \right) + \sum_{s=1}^{K_1} \sum_{(i,j) \in A} \sum_{q \in Q} u_{ij}^q w_{ij}^q \theta_s^1, (i, j) \in A \\
& \quad \sum_{(i,j) \in A} \sum_{q \in Q} y_{ij}^{ks} \leq U_{ij} + \sum_{(i,j) \in A} \sum_{q \in Q} u_{ij}^q w_{ij}^q, (i, j) \in A \\
& \quad y_{ij}^{ks} = \begin{cases} d_{ij}^{ks} & i = o_k \\
- d_{ij}^{ks} & i = e_k \\
0 & \text{otherwise} \end{cases}, k \in K \\
& \quad y_{ij}^{ks} \geq 0, (i, j) \in A, k \in K
\end{align*}
\]

In the solution procedure of the stochastic programming, if the recourse problem is infeasible, the feasibility cut is added to the formulation of the master problem. The dual problem to the recourse problem for the stage \(t\) scenario \(s\) is formulated.
as follows.

\[(\text{Dual to Recourse Problem for Stage } t \text{ / Scenario } s)\]

\[
\begin{align*}
\max & \quad \sum_{(i,j) \in A} (U_{ij} + \sum_{r \in T_p} \sum_{q \in Q} u_{ij}^{q,r} s_{ij}) \lambda_{ij} + \sum_{k \in K} (d_{ij}^{ck} \mu_{ik}^k - d_{ij}^{ck} \tilde{\mu}_i^k) \\
\text{subject to} & \quad \lambda_{ij} + \sum_{(i,j) \in A} \sum_{k \in K} \mu_{ik}^k - \sum_{(i,j) \in K} \mu_{ik}^k \leq \beta c_{ij}, (i,j) \in A, k \in K \\
& \quad \lambda_{ij} \geq 0, (i,j) \in A
\end{align*}
\]

If the objective value of the dual problem is infinite, then the primal recourse problem is not feasible. There exists an extreme ray \((\lambda, \tilde{\mu})\) such that \(\Sigma_{(i,j) \in A} (U_{ij} + \sum_{r \in T_p} \sum_{q \in Q} u_{ij}^{q,r} s_{ij}) \lambda_{ij} + \sum_{k \in K} (d_{ij}^{ck} \mu_{ik}^k - d_{ij}^{ck} \tilde{\mu}_i^k) > 0\), \(\lambda_{ij} \in \Sigma_{(i,j) \in A} \Sigma_{k \in K} \mu_{ik}^k - \Sigma_{(i,j) \in K} \mu_{ik}^k \leq 0\), and \(\tilde{\lambda} \geq 0\). The following inequality, called feasibility cut, excludes solutions \(w^{ek}, \ldots, w^{ek}, q \in Q\), that make the recourse problem infeasible since \(\Sigma_{(i,j) \in A} (U_{ij} + \sum_{r \in T_p} \sum_{q \in Q} u_{ij}^{q,r} s_{ij}) \lambda_{ij} + \sum_{k \in K} (d_{ij}^{ck} \mu_{ik}^k - d_{ij}^{ck} \tilde{\mu}_i^k) > 0\).

\[
\sum_{(i,j) \in A} (U_{ij} + \sum_{r \in T_p} \sum_{q \in Q} u_{ij}^{q,r} s_{ij}) \lambda_{ij} + \sum_{k \in K} (d_{ij}^{ck} \mu_{ik}^k - d_{ij}^{ck} \tilde{\mu}_i^k) \leq 0
\]

Generating feasibility cuts requires to solve the problem of generating extreme rays. Therefore, the model is modified so that the recourse problem can always be feasible. The additional decision variable \(d^{ck}\) is defined as the actual traffic amount for the traffic demand \(d^{ck}\) and the penalty term for the unfulfilled demand is added to the objective function, where \(\tau\) is a penalty parameter. The actual traffic amount satisfies the inequality \(d^{ck} \leq d^{ck}\). The modified recourse problem is shown as follows.

\[(\text{Modified Recourse Problem for Stage } t \text{ / Scenario } s)\]

\[
Q^t_i(w^{1,q}, \ldots, w^{q}, q \in Q) = \min \left( \beta \sum_{(i,j) \in A} c_{ij} \gamma_{ij}^{ks} + \tau \sum_{k \in K} (d_{ij}^{ck} - d^{ck}) \right) \text{ subject to } \gamma_{ij}^{ks} \leq U_{ij} + \sum_{r \in T_p} \sum_{q \in Q} u_{ij}^{q,r} s_{ij}, (i,j) \in A \\
\sum_{(i,j) \in A} \sum_{k \in K} \gamma_{ij}^{ks} - \sum_{(i,j) \in K} \gamma_{ij}^{ks} = \begin{cases} \sum_{k \in K} \gamma_{ij}^{ks} & i = o_k \\
\sum_{k \in K} \gamma_{ij}^{ks} & i = e_k, k \in K \\
d^{ck} \leq d^{ck}, k \in K \\
\gamma_{ij}^{ks} \geq 0, (i,j) \in A, k \in K \end{cases} \text{ otherwise,} \\
d^{ck} \geq 0, k \in K
\]

It is obvious that this problem always has a feasible solution, which is called a complete recourse.

After solving the modified recourse problem, if the recourse problem has a finite optimal primal solution \(y^{o,ck}, a^{o,ck}\), \((i,j) \in A, k \in K\), the optimal dual solution \(\lambda_{ij}^{ck}, \mu_{ik}^{ck}, \nu^{ck}\) can be calculated. By using these optimal dual solutions, the recourse function \(Q^t_i(w^{1,q}, \ldots, w^{q}, q \in Q)\) is approximated by a set of valid inequalities.

\[
\theta^t_i \geq Q^t_i(w^{1,q}, \ldots, w^{q}, q \in Q) = \min \left( \beta \sum_{(i,j) \in A} c_{ij} \gamma_{ij}^{ks} + \tau \sum_{k \in K} (d_{ij}^{ck} - d^{ck}) \right) \text{ subject to } \gamma_{ij}^{ks} \leq U_{ij} + \sum_{r \in T_p} \sum_{q \in Q} u_{ij}^{q,r} s_{ij}, (i,j) \in A \\
\sum_{(i,j) \in A} \sum_{k \in K} \gamma_{ij}^{ks} - \sum_{(i,j) \in K} \gamma_{ij}^{ks} = \begin{cases} \sum_{k \in K} \gamma_{ij}^{ks} & i = o_k \\
\sum_{k \in K} \gamma_{ij}^{ks} & i = e_k, k \in K \\
d^{ck} \leq d^{ck}, k \in K \\
\gamma_{ij}^{ks} \geq 0, (i,j) \in A, k \in K \end{cases} \text{ otherwise,} \\
d^{ck} \geq 0, k \in K
\]

The following optimality cut cuts off the solution, such that \(\theta^t_i \leq Q^t_i(w^{1,q}, \ldots, w^{q}, q \in Q)\).

\[
\theta^t_i \geq \sum_{(i,j) \in A} (U_{ij} + \sum_{r \in T_p} \sum_{q \in Q} u_{ij}^{q,r} s_{ij}) \lambda_{ij} + \sum_{k \in K} d_{ij}^{ck} \nu^{ck} + \tau \sum_{k \in K} (d_{ij}^{ck} - d^{ck}) 
\]

However, setting the value of the penalty parameter \(\tau\) small, the demand is not satisfied since the cost of expanding the capacity in the first stage is smaller than the penalty. Although the feasibility of the problem is satisfied, the demand may not be satisfied in the conventional method by Lai and Shih (2013). To satisfy the demand constraints, the value of
the penalty parameter $\tau$ must be set to sufficiently large. If $d^{iks} - a^{iks} > 0$ for some $k \in K$, the value of $\tau$ can be set so that it satisfies the following inequality.

$$\tau(d^{iks} - a^{iks}) > h^i_j, (i, j) \in A, q \in Q$$

(8)

This inequality signifies that the penalty for the demand shortage is greater than the capacity expansion cost as shown in Fig. 2. Since this value of $d^{iks} - a^{iks}$ is a positive integer, the value of $\tau$ never grows infinitely.

![Fig. 2 Setting penalty parameter](image)

**Step 1. Solve Master Problem**
Solve the mixed integer programming master problem by the branch and bound method. Let $w^{1q}, \ldots, w^{sq}, q \in Q$, $\theta^t_s, s = 1, \ldots, K_t, t = 1, \ldots, T$ be the optimal solution to the master problem.

**Step 2. Solve Recourse Problem**
Solve the modified recourse problem for the stage $t$ scenario $s, s = 1, \ldots, K_t, t = 1, \ldots, T$ to obtain $a^{iks}, y^{iks}$. If $d^{iks} - a^{iks} > 0$, set the penalty parameter to satisfy (8) and repeat Step 2.

**Step 3. Add Optimality Cuts**
Calculate $Q^t_s(w^{1q}, \ldots, w^{sq}, q \in Q), s = 1, \ldots, K_t, t = 1, \ldots, T$. If $\theta^t_s < (1 - \varepsilon)Q^t_s(w^{1q}, \ldots, w^{sq}, q \in Q)$, the optimality cut (7) is added to the formulation of the master problem ($\varepsilon > 0$: tolerance). Go to Step 1.

**Step 4. Convergence Check**
If no optimality cuts are added, then stop.

Because all of the integer variables are aggregate level decision variables, the master problem becomes a mixed integer programming problem. The upper bound value for the optimal objective value can be adopted as the approximate optimal objective value. If the initial value of the parameter $\tau$ is too small, the demand may not be satisfied in step 2. In this case, the value may be set using (8). The epigraph of the recourse function $Q_s^t$ for the large $\tau$ indicated by a thick solid line is a subset of the epigraph when the value of $\tau$ is small as shown in Fig. 3. Therefore the optimality cut for a small value of $\tau$ indicated by a dotted line is valid even after the value of $\tau$ is set to a large value. This L-shaped method differs from the conventional method in the following point. First, in the conventional method, a two-stage problem is defined so as to satisfy constraints for each scenario. In this method, constraint violation is permitted in each scenario. The second stage problem is defined to be feasible as a mathematical programming problem by introducing the penalty. In this paper, a method of setting the penalty parameter is presented to satisfy the relaxed constraint in the second stage problem. Even when it is difficult to set a penalty value, it finally becomes possible to satisfy the relaxed constraint by increasing the value successively.
3. Numerical Experiments

The L-shaped method for the multistage stochastic programming problem was implemented using AMPL (Fourer et al., (1993)) on a 2.00 GHz Xeon E5507 with two processors and 12.0 GB memory. The whole framework of the algorithm was coded in AMPL. The mathematical programming problems were solved by linear programming/branch-and-bound solver CPLEX 12.4. Numerical experiments were performed with the following settings. Let the number of nodes \(|N| = 15\), the number of arcs \(|A| = 44\), the number of OD pairs \(|K| = 14\) and the number of alternative choices of expansion \(|Q| = 5\). The network shown in Fig. 4 is based on Lai and Shih (2013), where the number on the arc denotes the number of arcs in forward direction (that in backward direction is in parentheses).

![Railway network](image)

Fig. 4 Railway network

The origin and the destination of OD pair is shown in Table 1.

Demand data was set as follows. A standard demand value \(\bar{d}^k\) is given for each OD pair \(k \in K\). In the network under consideration, expansion work is not necessary to satisfy the standard demand data. If demand fluctuation after the first period makes it impossible to meet demand, expansion work is needed. We define the number of demand realizations \(k_t\) in each period as fixed values regardless of the period. The number of realizations in each period were varied as \(k_t = 2, 3, 4, 5\) for \(t = 1, \ldots, T\). The total number of scenarios are shown as shown in Table 2.

At the same time, the demand fluctuation coefficient \(r^s\) for \(s = 1, \ldots, k_t\) are given to the standard demand value and the realization of the demand in period \(t\) is defined as (9) and (10), where \(s - 1 \mod k_t\) represents the remainder obtained by dividing \(s - 1\) by \(k_t\).

\[
d_t^{k_1} = (1 + r^s)d_t^k, \quad s = 1, \ldots, k_1
\]

(9)

\[
d_t^{k_2} = (1 + r^{s - 1 \mod k_t} + 1)d_t^{p-1, k_{t-1}, t}, \quad s = 1, \ldots, K_t, t = 2, \ldots, |T|
\]

(10)
Demand fluctuation coefficients $r^s, s = 1, \ldots, k_t$ are shown in Table 3 based on the maximum demand increase rate $r^{\text{max}} > 0$. The value of the maximum demand increase rate $r^{\text{max}} > 0$ was altered as $r^{\text{max}} = 1\%, 5\%, 10\%, 15\%$.

We set a budget for capacity expansion so that demand can be satisfied under every scenario in every period. As the expansion capacity increased, the scale economy worked and the incremental extension cost was set to be smaller. The L-shaped method is applied to solve the problem. From the optimal solution of the master problem, the upper bound for the optimal objective value can be calculated. The results of the numerical experiments are shown in Table 4, 5, 6, 7 for $r^{\text{max}}=1\%, 5\%, 10\%, 15\%$, respectively. A deterministic equivalent problem means a large-scale problem in which demand satisfaction conditions are described as constraints for all scenarios. For the instances with a large array of scenarios, the L-shaped method shows its efficiency.

Table 1 The origin and the destination of OD pair

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<thead>
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<th>OD pair</th>
<th>Origin</th>
<th>Destination</th>
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<td>13</td>
<td>15</td>
<td>6</td>
</tr>
<tr>
<td>14</td>
<td>15</td>
<td>3</td>
</tr>
</tbody>
</table>

Table 2 The number of realizations and total number of scenarios

<table>
<thead>
<tr>
<th>Number of realizations $k_t$</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total number of scenarios $\Pi^1_{k_t}$</td>
<td>32</td>
<td>243</td>
<td>1024</td>
<td>3125</td>
</tr>
</tbody>
</table>

Table 3 Setting of demand fluctuation coefficient $r^s$

<table>
<thead>
<tr>
<th>Number of realizations $k_t$</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s = 1$</td>
<td>+$r^{\text{max}}$</td>
<td>+$r^{\text{max}}$</td>
<td>+$r^{\text{max}}$</td>
<td>+$r^{\text{max}}$</td>
</tr>
<tr>
<td>$s = 2$</td>
<td>-$r^{\text{max}}$</td>
<td>0</td>
<td>+$r^{\text{max}}/2$</td>
<td>+$r^{\text{max}}/2$</td>
</tr>
<tr>
<td>$s = 3$</td>
<td>-</td>
<td>-$r^{\text{max}}$</td>
<td>-$r^{\text{max}}/2$</td>
<td>0</td>
</tr>
<tr>
<td>$s = 4$</td>
<td>-</td>
<td>-</td>
<td>-$r^{\text{max}}$</td>
<td>-$r^{\text{max}}/2$</td>
</tr>
<tr>
<td>$s = 5$</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>+$r^{\text{max}}$</td>
</tr>
</tbody>
</table>

Table 4 Computational Results ($r^{\text{max}}=1\%$)

<table>
<thead>
<tr>
<th>Total number of Scenarios</th>
<th>Computing Time (s): Deterministic Equivalent</th>
<th>Computing Time (s): L-shaped</th>
</tr>
</thead>
<tbody>
<tr>
<td>32</td>
<td>6</td>
<td>12</td>
</tr>
<tr>
<td>243</td>
<td>36</td>
<td>93</td>
</tr>
<tr>
<td>1024</td>
<td>167</td>
<td>449</td>
</tr>
<tr>
<td>3125</td>
<td>797</td>
<td>2934</td>
</tr>
</tbody>
</table>

Table 5 Computational Results ($r^{\text{max}}=5\%$)

<table>
<thead>
<tr>
<th>Total number of Scenarios</th>
<th>Computing Time (s): Deterministic Equivalent</th>
<th>Computing Time (s): L-shaped</th>
</tr>
</thead>
<tbody>
<tr>
<td>32</td>
<td>8</td>
<td>22</td>
</tr>
<tr>
<td>243</td>
<td>37</td>
<td>119</td>
</tr>
<tr>
<td>1024</td>
<td>284</td>
<td>580</td>
</tr>
<tr>
<td>3125</td>
<td>1175</td>
<td>2889</td>
</tr>
</tbody>
</table>
Table 6 Computational Results ($r_{max}=10\%$)

<table>
<thead>
<tr>
<th>Total number of Scenarios</th>
<th>Computing Time (s): Deterministic Equivalent</th>
<th>Computing Time (s): L-shaped</th>
</tr>
</thead>
<tbody>
<tr>
<td>32</td>
<td>17</td>
<td>35</td>
</tr>
<tr>
<td>243</td>
<td>169</td>
<td>185</td>
</tr>
<tr>
<td>1024</td>
<td>1132</td>
<td>731</td>
</tr>
<tr>
<td>3125</td>
<td>6548</td>
<td>3128</td>
</tr>
</tbody>
</table>

Table 7 Computational Results ($r_{max}=15\%$)

<table>
<thead>
<tr>
<th>Total number of Scenarios</th>
<th>Computing Time (s): Deterministic Equivalent</th>
<th>Computing Time (s): L-shaped</th>
</tr>
</thead>
<tbody>
<tr>
<td>32</td>
<td>25</td>
<td>95</td>
</tr>
<tr>
<td>243</td>
<td>344</td>
<td>329</td>
</tr>
<tr>
<td>1024</td>
<td>2875</td>
<td>1067</td>
</tr>
<tr>
<td>3125</td>
<td>14272</td>
<td>4131</td>
</tr>
</tbody>
</table>

The numbers of expansion of each problem in which the value of the maximum demand fluctuation is altered are shown in Table 8.

Table 8 Maximum demand fluctuation and expansion times

<table>
<thead>
<tr>
<th>Maximum demand fluctuation $r_{max}$</th>
<th>1%</th>
<th>5%</th>
<th>10%</th>
<th>15%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Times of expansion</td>
<td>3</td>
<td>3</td>
<td>7</td>
<td>9</td>
</tr>
</tbody>
</table>

When the demand fluctuation is small, the number of times of capacity expansion is small. Since many variables $y$ take the value 1, the problem of small fluctuation is almost like LP and the number of subproblems is small. Therefore the L-shaped method takes more time.

As the maximum demand fluctuation increases, the number of extensions also increases. Taking into consideration the integer constraint of the 0-1 variable $y$, the computation time increases with the increase in the number of extensions in the conventional solution method. On the other hand, the L-shaped method does not increase the calculation time compared with the conventional solution, even if the number of times of expansion increases. The effect of decomposition is remarkable. As the number of scenarios and the number of extensions increase, the computational efficiency of the solution by the L-shaped method is higher than that of the conventional solution.

4. Concluding Remarks

We developed a multistage stochastic programming model for the railway line capacity expansion problem. By utilizing the L-shaped method, the capacity expansion problem can be solved effectively. The planning method developed in this paper can be applied to the introductory plan of light rail transit, which complements the municipal traffic in Japan. As for a solution method, the penalty for constraint violation is defined, and a solution algorithm using only optimality cuts is shown. At the same time, how to set the parameters of the penalty is derived. Since it is not necessary to solve the problem of generating feasibility cuts and the calculation has been reduced, this method is effective for multi-period stochastic programming.

Appendix: Multistage Stochastic Programming Problem

The multistage stochastic linear programming problem with stages $t = 0, 1, \ldots, H$ is formulated as follows.

(Multistage Stochastic Linear Programming Problem)

\[
\begin{align*}
\min & \ c^0 x^0 + E_{\xi_1} \left[ \min \ c^1 x^1 + \ldots + E_{\xi_{H-1}} \left[ \min \ c^H x^H \right] \right] \\
\text{subject to} & \ T^{0}_{\cdot,0} x^{0} = h^0 \\
& \ T^{t}_{\cdot,0} + W^{t}_{\cdot} x^{t} = h^{t}, \ \text{a.s.} \\
& \quad \vdots \\
& \ T^{H}_{\cdot,H-1} x^{H-1} + W^{H}_{\cdot} x^{H} = h^{H}, \ \text{a.s.} \\
& \ x^{0} \geq 0, x^{t} \geq 0, t = 1, \ldots, H, \ a.s.,
\end{align*}
\]

In this formulations, $c^0$ is a known vector in $R^m$, $h^0$ is a known vector in $R^n$, and each $W^t$ is a known $m_t \times n_t$ matrix, and bold face vectors and matrices are possibly stochastic, where $c^t, h^t, T^t$ are in $R^n$, $R^m$, $R^{m_t} \times R^{n_t}$, respectively.
decision vector in $\mathbb{R}^{n_t}$ for stage $t = 1, \ldots, H$ is denoted as $x^t$. The decisions are made so that the constraints hold almost surely (denoted a. s.).

It is assumed that the stochastic elements are defined over a finite discrete probability space $(\Xi, \sigma(\Xi), P)$, where $\Xi = \Xi^1 \times \cdots \times \Xi^H$ is the support of the random data in each stage with $\Xi^t = (\xi^t_s, (T^t, h^t, c^t_s), s = 1, \ldots, k_t)$ and $(T^t, h^t, c^t_s)$ is a realization of $((T^t, h^t, c^t_s))$. The scenarios are defined as the possible sequences of the realization of random variables $(\xi^1, \ldots, \xi^H)$, and are often described using a scenario tree. In intermediate stages $t \leq H$, we have a limited number of possible realizations, which we call stage $t$ scenarios. In a scenario tree, the stage $t$ scenario connected to the stage $t - 1$ scenario $s$ is referred to as a successor of the stage $t - 1$ scenario $s$.

The concept of block separable recourse was introduced by Louveaux (1986).

**Definition 4.1** A multistage stochastic linear program has block separable recourse if, for all stages $t = 1, \ldots, H$, the decision vectors $x^t$ can be written as $x^t = (w^t, y^t)$, where $w^t$ represents aggregate level decisions and $y^t$ represents detailed level decisions. Moreover, the constraints follow these partitions:

1. The stage $t$ objective contribution is $c^t x^t = r^t w^t + q^t y^t$.
2. The constraint matrix $W^t$ is a block diagonal: $W^t = \begin{pmatrix} A^t & 0 \\ 0 & B^t \end{pmatrix}$, where $A^t$ is associated to the vector $w^t$ and $B^t$ to the vector $y^t$.
3. The other components of the constraints are random and we assume that $T^t$ and $h^t$ can be written: $T^t = (R^t \begin{pmatrix} 0 \\ 0 \end{pmatrix})$ and $h^t = \begin{pmatrix} b^t \\ d^t \end{pmatrix}$ to conform with the $(w^t, y^t)$ separation.

The multistage program with block separable recourse is proved to be equivalent to the two-stage program.

**Proposition 4.1 (Louveaux 1986)** A multistage stochastic program with block separable recourse is equivalent to a two stage stochastic program, where the first-stage is an extensive form of an aggregate level problem, and the value function of the second stage is the sum (weighted by the appropriate probabilities) of the detailed level recourse functions for all stage $t$ scenarios, $t = 1, \ldots, H$.

From the proposition, the multistage stochastic program with block separable recourse can be transformed into a two stage stochastic program with recourse. The problem proves to be a problem that has first stage integer variables and continuous second stage variables.

**References**


