Magnetic-head positioning control of HDDs with improved \( \mathcal{H}_\infty \) controller by Robust Controller Bode (RCBode) plot

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Received: 29 July 2018; Revised: 7 October 2018; Accepted: 24 October 2018

Abstract
In this paper, we present a robust controller design method for a magnetic-head positioning system in a hard disk drive (HDD) based on the \( \mathcal{H}_\infty \) control theory and the Robust Controller Bode (RCBode) plot. The RCBode plot represents robust performance criteria as allowable and forbidden areas on Bode diagrams for a controller. Using the RCBode plot, we can improve an existing \( \mathcal{H}_\infty \) controller with visualized guidelines in order to compensate for disturbances against plant perturbations. We show utilities of the proposed method by applying it to a popular benchmark problem for the HDDs. Simulation results with the HDD benchmark problem showed that the proposed method enables us to improve positioning accuracy of the magnetic-head position by about 15% without losing stability margins from the original \( \mathcal{H}_\infty \) controller.

Keywords: Robust control, Loop shaping, Hard disk drive, Positioning control, Visualized design

1. Introduction
In the 21st century, technologies for data-storage fields are important factors for Internet services, cloud computing, Big Data analytics, and social networking. In the data storage systems, the most widely used device is a hard disk drive (HDD), and positioning accuracy of magnetic heads in HDDs must be improved to meet increasing demands for larger storage capacity (Atsumi et al., 2013; Abrahamson and Huang, 2015; Atsumi, 2016a). To do so, a magnetic-head positioning system has to compensate for disturbances in the positioning system against plant perturbations (Yamaguchi and Atsumi, 2008). It is known as the robust control problem (Doyle et al., 1992).

To design the controller for mechatronic systems, we can employ notch filters which are designed independently from a main controller. In this method, design results of the depth or width of notch filters depend on the designer’s experiences strongly. This means that performances of the control system also depend on the designer’s experiences strongly.

To design the robust controller systematically, the \( \mathcal{H}_\infty \) control theory is widely used because the small gain theorem makes the robust stability problem easy to handle (Hirata et al., 1992; Semba, 2001; Ohno and Hara, 2005; Hirata and Hasegawa, 2007). However, the \( \mathcal{H}_\infty \) control theory doesn’t handle robust performance problem. As a result, the designed \( \mathcal{H}_\infty \) controllers may not keep nominal control performances against plant perturbations.

On the other hands, a loop-shaping methodology called “Robust Controller Bode (RCBode) plot” is developed for the robust control design (Atsumi and Messner, 2010; Taylor and Messner, 2011; Atsumi and Messner, 2012; Taylor and Messner, 2014; Taylor and Messner, 2015). The RCBode plot represents robust performance criteria as allowable and forbidden areas on Bode diagrams for a controller. Using the RCBode plot, we can improve an existing controller with visualized guidelines in order to keep control performances against plant perturbations. Moreover, we can easily consider the effects of structured uncertainties in control objects without using complicated generalized plants.

However, we cannot design an initial controller by using the RCBode plot because it is a loop-shaping technique. This means that the loop-shaping design with the RCBode plot requires an initial robust controller, and it is not clear that how to remove a conservativeness in the initial controller for the RCBode plot.
In this study, to overcome the above-mentioned problem, we propose a design method of a robust controller based on the $H_{\infty}$ control theory and the RCBode plot for the magnetic-head positioning system in HDDs. By using the proposed method, we can design a robust controller which does not have an excessive conservativeness. We show the utility of the proposed method by applying it to a benchmark problem for the magnetic-head positioning control system in the HDD (Yamaguchi et al., 2012).

2. Discrete-time $H_{\infty}$ control

\[ G[z] \]

\[ C[z] \]

Fig. 1 Discrete-time $H_{\infty}$ control problem

The $H_{\infty}$ control problem for a discrete-time system employs a generalized plant $G[z]$ and a discrete-time controller $C[z]$ in Fig. 1, and is defined by the following inequality:

\[ \| G_{\text{sw}}[z] \|_{\infty} < \gamma \]

(1)

where, $G_{\text{sw}}$ is a closed loop transfer function from $w(i)$ to $z(i)$, and $\gamma$ is a kind of performance criterion (Yamaguchi et al., 2012). The $H_{\infty}$ control problem can handle the mixed sensitivity problem by satisfying the following inequality:

\[ \left\| \frac{W_s[z]S_{\text{nl}}[z]}{W_u[z]T_{\text{nl}}[z]} \right\|_{\infty} < 1 \]

(2)

where, $S_{\text{nl}}[z]$ is a sensitivity function with a nominal plant $P_{\text{nl}}[z]$ for the $H_{\infty}$ controller design, and $T_{\text{nl}}[z]$ is a complementary sensitivity (co-sensitivity) function with the $P_{\text{nl}}[z]$. $W_s[z]$ is a weighting function for $S_{\text{nl}}[z]$, and $W_u[z]$ is a weighting function for $T_{\text{nl}}[z]$ and plant perturbations.

3. Robust controller design based on $H_{\infty}$ control theory and RCBode plot

3.1. Robust performance problem

Any mathematical model of an actual controlled object of a physical system will have characteristics not represented by that model. These unmodeled characteristics fall into the two broad categories of (a) structured uncertainty and (b) unstructured uncertainty (Doyle et al., 1992). Fig. 2 shows the block diagram of a control system which includes the structured uncertainty and the unstructured uncertainty as multiplicative uncertainties. Here, $\Delta_s$ is a structured uncertainty, $\Delta_u$ is an unstructured uncertainty, $P_n$ is a nominal discrete-time plant, $P_r$ is a real discrete-time plant including the structured uncertainty, $\tilde{P}$ is a real discrete-time plant including the structured uncertainty and the unstructured uncertainty.

The structured uncertainty is a discrete set of plants. Therefore, the characteristics are given by one or more frequency responses. For example, the structured uncertainty includes variations of the resonance frequencies, variations of the damping ratios, or variation of plant gain. The structured uncertainty is characterized by both gain frequency responses and the phase frequency responses.
The unstructured uncertainty is regarded as a disk uncertainty in the complex plane, which is characterized only by gain information. The unstructured uncertainty includes not only parameter variations of the plant but also some stochastic uncertainties, for example, the reliability of measured signals given by SNR (signal-to-noise ratio), the effect of non-linearity in the controlled object or the effect of discretization error (Franklin et al., 1998).

The relationships between \( P_n, P_r, P \) are

\[
P^r(\omega, i_m) = P_n(\omega) (1 + \Delta(\omega, i_m)), \quad i_m = 1, \ldots, n_m, \quad (3)
\]

\[
\hat{P}(\omega, i_m) = P_r(\omega, i_m) (1 + \Delta_r(\omega)) = P_n(\omega) (1 + \Delta_n(\omega)), \quad i_m = 1, \ldots, n_m. \quad (4)
\]

where \( n_m \) is the number of frequency responses of the real plant, and \( i_m \) is the model number of the real plant, and \( \Delta_n \) is a multiplicative uncertainty that includes the structured uncertainty and the unstructured uncertainty. To simplify our analysis, the gain of \( \Delta_n \) is defined by

\[
|\Delta_n(\omega, i_m)| := |\Delta_s(\omega, i_m)| + \sup |\Delta_m(\omega)| + \sup |\Delta_r(\omega, i_m)|, \quad i_m = 1, \ldots, n_m. \quad (5)
\]

because \( \Delta_n \) has phase uncertainty and \( \sup |\Delta_m(\omega)| \) can be given by a real number.

The frequency responses of the nominal sensitivity function \( S_n \), and the nominal co-sensitivity function \( T_n \) are

\[
S_n(\omega) = \frac{1}{1 + P_n(\omega)C(e^{j\omega t})}, \quad T_n(\omega) = \frac{L_n(\omega)}{1 + P_n(\omega)C(e^{j\omega t})}. \quad (6)
\]

The frequency responses of the real sensitivity function \( S_r \), and the real co-sensitivity function \( T_r \) are

\[
S_r(\omega, i_m) = \frac{1}{1 + P_r(\omega, i_m)C(e^{j\omega t})}, \quad T_r(\omega, i_m) = \frac{P_r(\omega, i_m)C(e^{j\omega t})}{1 + P_r(\omega, i_m)C(e^{j\omega t})}, \quad i_m = 1, \ldots, n_m. \quad (7)
\]

In robust control design, we use two weighting functions \( W_u(\omega) \) and \( W_p(\omega) \). \( W_u \) is a function for the co-sensitivity function which specifies the uncertainty in the plant model (Doyle et al., 1992). \( W_p \) specifies the sensitivity function performance. \( W_u \) is designed to satisfy

\[
|W_u(\omega)| > \max_{i_m} |\Delta_u(\omega, i_m)| = \max_{i_m} \left| \frac{P(\omega, i_m)}{P_n(\omega)} - 1 \right|, \quad \forall \omega. \quad (8)
\]

To achieve robust stability against \( \Delta_u \), the control system needs to satisfy the following inequality (Doyle et al., 1992).

\[
|W_u(\omega)|^{-1} > |T_n(\omega)|, \quad \forall \omega. \quad (9)
\]

The weighting function \( W_p \) specifies the robust performance such that

\[
|W_p(\omega)|^{-1} > \max_{i_m} \left| \frac{1}{1 + P(\omega, i_m)C(e^{j\omega t})} \right|, \quad \forall \omega. \quad (10)
\]

If the open-loop transfer function \( P_n(\omega)C(e^{j\omega t}) \) is stable and the nominal sensitivity function and nominal co-sensitivity function satisfy

\[
|W_u(\omega)T_n(\omega)| + |W_p(\omega)S_n(\omega)| < 1, \quad \forall \omega,
\]

then the control system achieves (9) and (10).

3.2. Robust Controller Bode (RCBode) plot

![Fig. 3](image-url) Basic concept of loop-shaping design with the RCBode plot.
The RCBode plot is a Bode diagram which shows relationship between a controller and the robust performance problem (Atsumi and Messner, 2012). Fig. 3 shows the basic concept of the loop-shaping design method with the RCBode plot. First, we choose the frequency responses for the nominal plant, for the performance weighting function, and for the uncertainty weighting function. Second, we design the initial controller and plot the RCBode plot where the intersections between forbidden regions and the controller frequency response indicate the frequencies at which the initial controller does not meet specific robust performance criteria. Third, we design a loop-shaping filter to eliminate the intersections between the forbidden regions and the controller frequency response on the RCBode plot. At the third step, iterations may be necessary to eliminate all intersections between the forbidden regions and the controller frequency response. To draw the RCBode plot, we have to define the frequency responses of the nominal plant $P_{ns}(\omega, i_m)$ which include the effect of structured uncertainty, the digital controller $C[z]$, the performance weighting function $W_p(\omega)$, and the uncertainty weighting function $W_u(\omega)$.

In this method, the real plant consists of a plurality of frequency responses that include the effect of structured uncertainty. The nominal plant also consists of a plurality of frequency responses that sample the set of structured uncertainties. The real plant consists of a plurality of frequency responses that include the effect of structured uncertainty.

Solving (13) for $|C[e^{j\omega t}]|$ transforms the robust performance criterion of controller gain characteristics to the following quadratic inequality.

$$\left(1 - |W_u(\omega)|^2\right)|P_{ns}(\omega, i_m)|^2|C[e^{j\omega t}]|^2 + 2|P_{ns}(\omega, i_m)||C[e^{j\omega t}]|(|\cos(\angle P_{ns}(\omega, i_m)C[e^{j\omega t}]) - |W_u(\omega)||W_p(\omega)|) + 1 - |W_p(\omega)|^2 > 0, \quad i_m = 1, \ldots, n_m, \forall \omega. \quad (14)$$

Solving (13) for $\cos(\angle C[e^{j\omega t}])$ transforms the robust performance criterion of controller phase characteristics to

$$\cos(\angle C[e^{j\omega t}] + \angle P_{ns}(\omega, i_m)) > Q(\omega, i_m), \quad (15)$$

where

$$Q(\omega, i_m) = \frac{|W_p(\omega)|^2 - 1}{2|P_{ns}(\omega, i_m)C[e^{j\omega t}]|^2} + |W_u(\omega)||W_p(\omega)| + \left(\left|W_u(\omega)\right|^2 - 1\right)|P_{ns}(\omega, i_m)C[e^{j\omega t}]|, \quad i_m = 1, \ldots, n_m, \forall \omega. \quad (16)$$

In the RCBode plot, we plot the frequency response of the controller and forbidden regions that indicate frequencies at which the controller does not meet specific robust performance criteria given by (14) and (15). It is important that elimination of the intersections on the gain plot will simultaneously result in elimination of the intersections on the phase plot, and vice versa. Note that, in this paper, the forbidden regions are shown by gray area on the RCBode plot.

### 3.3. Improving $H_\infty$ controller using RCBode plot

In this paper, we propose an improving method for the $H_\infty$ controller using the RCBode plot. The proposed method follows steps below.

**Step 1.** Design an $H_\infty$ controller as an initial controller $C_0$.

**Step 2.** Define frequency responses of the nominal plant $P_{ns}$ for the RCBode plot. The $P_{ns}$ consists of a plurality of frequency responses that include the effect of structured uncertainty.

**Step 3.** Calculate the amplitude spectrum of disturbance signal $d$ from amplitude spectrum of $e$ and the gain frequency response of $S$, using $C = C_0$ in Fig. 2.

**Step 4.** Determine the frequency response of weighting functions $|W_p|$ and $|W_u|$. Then plot the RCBode plot using $P_{ns}$, $|W_u|$, $|W_p|$, and $C = C_0$. 

Step 5. Design a new controller $C_1$ by removing a conservative part from the initial controller $C_0$ based on the RCBode plot in the Step 4. Then plot the RCBode plot using $P_{ns}$, $|W_u|$, $|W_p|$, and $C = C_1$.

Step 6. Iteratively design a modified notch filter $C_2$ so that $C = C_1C_2$ satisfy the robust performance criterion in a high frequency range on the RCBode plot using the same $P_{ns}$, $|W_p|$, and $|W_u|$. In this step, we eliminate intersections between the forbidden regions and the frequency response of $C_1C_2$ in the high frequency range on the RCBode plot.

Step 7. Iteratively design a loop-shaping filter $C_3$ so that $C = C_1C_2C_3$ satisfy the robust performance criterion on the RCBode plot using the same $P_{ns}$, $|W_p|$, and $|W_u|$. In this step, we eliminate intersections between the forbidden regions and the frequency response of $C_1C_2C_3$ on the RCBode plot for all frequencies.

4. Magnetic-head positioning system in HDDs

4.1. Features of magnetic-head positioning control system

Fig. 4 shows a picture of an HDD without a top cover. The HDD is comprised of disks, a spindle motor, magnetic heads, and a voice coil motor (VCM). In the magnetic-head positioning control systems of the HDD, the control input is an input command value to a VCM driver, and the observed output is the magnetic-head position (Atsumi, 2016b). Note that we can use many second-order filters for the magnetic-head position control system (Atsumi et al., 2007).

To encourage studies about the HDDs, an open-source HDD benchmark problem has been developed on MAT-LAB/Simulink, and has been widely used in HDD research groups (Yamaguchi et al., 2012). The magnetic-head positioning control system has two primary modes of operation: track seek for moving the magnetic head from one track to another, and track follow for keeping the read-write head on a single track with a high degree of accuracy. During the seek, the objective is the control of the transient characteristics of the magnetic-head position. During track follow the objective is the control of the steady-state characteristics of the magnetic-head position subject to various disturbances. This paper focuses the development of the track following controller and employs the benchmark problem Ver. 1. In this benchmark problem, the disturbance signal $d$ includes the periodic disturbances caused by disk rotation known as repeatable runout (RRO), the effects of the torque noise, mechanical vibrations, and aerodynamic drag forces from the airflow induced by the spinning disks. In this benchmark software, to calculate the positioning error signals, all users have to do is setting the controller’s parameters.
The magnetic-head positioning control system in this benchmark problem is the sampled-data control system shown in Fig. 5. Here, $S$ is sampler, $H$ is zero-order-hold, $P_c$ is the continuous-time plant. In this benchmark problem, the sampling times of $S$ and $H$ are both 37.9 $\mu$s. The continuous-time plant $P_c$ has 18 transfer functions to represent the variable characteristics of the mechanical system. Each transfer function consists of seven mechanical resonant modes and a time-delay of 10.0 $\mu$s. As a result, the controlled object $P_r$ (the real plant) in Fig. 2 can be given as a discretized version of $P_c$ with the zero-order-hold. Fig. 6 shows the frequency responses of $P_c$ and Fig. 7 shows the frequency responses of $P_r$.

5. Control system design

In the Step 1, we employ an initial $H_\infty$ controller which is shown in the reference (Yamaguchi et al., 2012). Fig. 8 shows the generalized plant used in this design. Here, $P_{all}$ is a nominal plant for the $H_\infty$ controller design, $W_m$ is a weighting function for control performance, $W_r$ is a weighting function for plant perturbation, and $\varepsilon$ is a small constant number (0.04) to solve this $H_\infty$ problem. According to the reference (Yamaguchi et al., 2012), $P_{all}$, $W_m$, and $W_r$ are set as shown in Fig. 9 (solid line: $P_{all}$, dashed line: $W_r$, dot-dashed line: $W_m$). Note that the nominal plant $P_{all}$ for the $H_\infty$ controller design includes a rigid-body mode and a large mechanical resonant mode at 4 kHz. By using the generalized plant, we can get the transfer function of $C_0[z]$ as follows:

$$C_0[z] = \frac{0.022462(z - 0.9748)(z - 0.9968)(z^2 + 0.43253z + 0.2866)^2(z^2 - 1.1z + 0.618)}{(z - 1)(z + 0.3202)(z + 0.49)(z + 0.09472)(z + 0.003697)(z^2 - 0.2785z + 0.07575)(z^2 - 0.9781z + 0.689)}.$$ (17)

Fig. 10 shows the frequency response of the initial $H_\infty$ controller.

In the Step 2, we define the frequency responses of the nominal plant $P_{ns}$ for the RCBode plot. In this case study, $P_{ns}$ is the real plant $P_r$ shown in Fig. 7. Note that this nominal plant includes large aliasing characteristics from 7 kHz to the Nyquist frequency.

In the Step 3, we calculate the amplitude spectrum of the disturbance signal $d$. In this case study, we already know characteristics of $d$ because the benchmark problem shows the characteristics of $d$. However, in an actual control system, we cannot measure the disturbance signal directly. Thus, in this paper, we intentionally calculate the amplitude spectrum.
of the disturbance signals \(d\) by the following equation:

\[
d_A(\omega, i_m) = \frac{e_A(\omega, i_m)}{|S_j(\omega, i_m)|}\]

where, \(d_A\) is the amplitude spectrum of the disturbance signal \(d\), and \(e_A\) the amplitude spectrum of the error signal \(e\). Fig. 11 shows the gain frequency responses of \(|S_j|\) (solid lines) and \(|T_i|\) (dashed lines) with \(C = C_0\). Fig. 12 also shows the simulation result of \(e_A\) with \(P_r, C = C_0\), and \(r = 0\). Fig. 13 shows the calculation result of \(d_A\).

In the Step 4, we determine the frequency responses of weighting functions \(|W_p|\) and \(|W_u|\). We obtained the frequency response of \(|W_p(\omega)|\) by taking a 51-point moving average of the \(\sum_{m=1}^{\infty} d_A(\omega, i_m)/n_m\) except for the first three RROs frequency (120 Hz, 240 Hz, and 360 Hz), and then multiplying the result by 0.0005 so that the initial controller satisfies inequality (13) below 100 Hz. For 120 Hz, 240 Hz, and 360 Hz, \(|W_p(\omega)| = 0.0005 \sum_{m=1}^{\infty} d_A(\omega, i_m)/n_m\). The solid line in Fig. 14 shows the frequency response of \(|W_p|\). Figs. 6 and 7 show that \(P_{ns}\) includes large aliasing characteristics from 7 kHz to the Nyquist frequency (13200 Hz) (Atsumi, 2017b). Therefore, we chose \(|W_u|\) to be

\[
|W_u(\omega)| = \begin{cases} 
0.01, & 0 \leq \omega \leq 2\pi \cdot 7000 \\
0, & 2\pi \cdot 7000 < \omega \leq 2\pi \cdot 13200
\end{cases}
\]

(19)

The dashed line of Fig. 14 shows the frequency response of \(|W_u|\). This weighting function indicates that the control system will accept unmodeled dynamics in the estimate of frequency response of the real plant is less than 1% below 7 kHz, but may be as large as 200% from 7 kHz to the Nyquist frequency. Fig. 15 shows the RCBode plot with the frequency responses of \(P_{ns}, C = C_0, |W_p|, \) and \(|W_u|\).

In the Step 5, we design a new controller \(C_1\) by removing a conservative part from the initial controller \(C_0\). Fig. 15 indicates that the original \(H_{so}\) controller \(C_0\) has the excessive conservativeness around 4 kHz because the frequency response of the controller is away from the forbidden regions around 4 kHz. Moreover, the gain of the original \(H_{so}\) controller should be decreased above 5 kHz because there are some intersections between the frequency response of the controller and the forbidden regions. Therefore, we divided the initial controller \(C_0\) into two parts: a PI-lead part \(C_1\), and a notch-filter part \(C_0\). The transfer function of \(C_1[z]\) and \(C_0[z]\) are given as follows:

\[
C_1[z] = \frac{0.022462(z - 0.9748)(z - 0.9968)(z^2 + 0.43253z + 0.2866)}{(z - 1)(z + 0.3202)(z + 0.49)(z + 0.09472)(z + 0.003697)}.
\]

(20)
As a result, we can set $C_2[z]$ that is discretized $C_{ntc}(s)$ using a matched pole-zero method. Fig. 18 shows the RCBode plot where the feedback controller is $C = C_1C_2$. This figure shows that the control system with $C = C_1C_2$ satisfies the robustness criterion on the RCBode plot from 3 to 13.2 kHz.

\[ C_{01}[z] = \frac{(z^2 + 0.43253z + 0.2866)(z^2 - 1.1z + 0.618)}{(z^2 - 0.2785z + 0.07575)(z^2 - 0.9781z + 0.689)}. \]  

(21)

Fig. 16 shows frequency responses of $C_1$ (solid line) and $C_{01}$ (dashed line). Fig. 17 shows the RCBode plot with the frequency responses of $P_{ntc}$, $C = C_1$, $|W_p|$, and $|W_u|$. In the Step 6, we iteratively design an modified notch filter $C_2$ so that $C = C_1C_2$ satisfy the robust performance criterion on the RCBode plot in a high frequency range. Fig. 17 shows the controller gain from 7 to 13.2 kHz must be decreased because there is no allowable region on the phase plot at this frequency range. Similarly, the controller gain around 5 kHz must be decreased because there is a little allowable region on the phase plot. This figure also shows that the gain around 3 kHz should be a little lower. Therefore, to avoid the intersection above 3000 Hz, we employ following a cascade of notch filters.

\[ C_{ntc}(s) = \prod_{i=1}^{6} \frac{s^2 + 2\zeta_{ntc}\omega_n + \omega_n^2}{s^2 + 2\zeta_{ntd}\omega_d + \omega_d^2}, \]  

(22)

where

- $\omega_n1 = 2\pi \cdot 3000, \zeta_{ntc1} = 0.02, \zeta_{ntd1} = 0.015$, $\omega_n2 = 2\pi \cdot 5000, \zeta_{ntc2} = 0.03, \zeta_{ntd2} = 0.65$,
- $\omega_n3 = 2\pi \cdot 7000, \zeta_{ntc3} = 0.021, \zeta_{ntd3} = 0.08$, $\omega_n4 = 2\pi \cdot 8230, \zeta_{ntc4} = 0.021, \zeta_{ntd4} = 0.08$,
- $\omega_n5 = 2\pi \cdot 9470, \zeta_{ntc5} = 0.021, \zeta_{ntd5} = 0.08$, $\omega_n6 = 2\pi \cdot 12000, \zeta_{ntc6} = 0.06, \zeta_{ntd6} = 0.25$.

As a result, we can set $C_2[z]$ that is discretized $C_{ntc}(s)$ using a matched pole-zero method. Fig. 18 shows the RCBode plot where the feedback controller is $C = C_1C_2$. This figure shows that the control system with $C = C_1C_2$ satisfies the robustness criterion on the RCBode plot from 3 to 13.2 kHz.
In the Step 7, we iteratively design a loop-shaping filter $C_3$ so that $C = C_1C_2C_3$ satisfy the robust performance criterion on the RCBode plot for all frequencies. Fig. 18 shows that there are intersections between the forbidden regions and the frequency response of $C_1C_2$ at 120, 240, 360, 760, 900, 1000, 1200, and 1800 Hz.

To eliminate the intersections at 120, 240, and 360 Hz, we must increase the controller gain at these frequencies because there is no allowable region on the phase plot. Therefore, we employ following a cascade of the complex phase-lead filters.

$$C_{pl}(s) = \prod_{i=1}^{3} \frac{s^2 + 2\zeta_{p1}\omega_{p1}s + \omega_{p1}^2}{s^2 + 2\zeta_{p2}\omega_{p2}s + \omega_{p2}^2},$$

where

$$\omega_{p1} = 2\pi \cdot 120, \zeta_{p1} = 0.00025, \omega_{p2} = 2\pi \cdot 240, \zeta_{p2} = 0.008, \omega_{p3} = 2\pi \cdot 360, \zeta_{p3} = 0.0005, \omega_{p3} = 0.005.$$

To eliminate the intersections at 760, 900, 1000, 1200, and 1800 Hz, we have several possible approaches. In this case study, we focus on the phase characteristics at these frequencies. Fig. 18 shows that we can eliminate the intersections from 700 to 2000 Hz by increasing the phase in narrow bands. In such a case, a complex lead filter is well suited to this purpose because it realizes a narrow band phase-lead effect by using complex poles and zeros (Messner et al., 2007). Therefore, we employ following a cascade of the complex phase-lead filters.

$$C_{cl}(s) = 1.0928 \prod_{n=1}^{5} \frac{s^2 + 2\zeta_{cl}\omega_{cl}s + \omega_{cl}^2}{s^2 + 2\zeta_{cl}\omega_{cl}s + \omega_{cl}^2},$$

where

$$\omega_{cl} = \omega_{n} \left( -\zeta_n \tan(\phi_n) + \sqrt{-\zeta_n^2 \tan^2(\phi_n) + 1} \right), \omega_{cl} = \omega_{n} \left( \zeta_n \tan(\phi_n) + \sqrt{-\zeta_n^2 \tan^2(\phi_n) + 1} \right),$$

$$\omega_{cl} = 21^\circ, \zeta_{cl} = 0.012, \omega_{5} = 2\pi \cdot 1805.$$
the mechanical resonance at 4 kHz in \( P_{nh} \) (the nominal plant for \( \mathcal{H}_\infty \) controller design) doesn’t help very much in the robust performance improvements.

Fig. 21 shows the frequency responses of \(|S_r|\) and \(|T_r|\) for \( C = C_1C_2C_3 \). In this figure, solid lines indicate the results of \(|S_r|\) and dashed lines indicate the results of \(|T_r|\). Fig. 22 shows the Nyquist diagram of the open-loop characteristics with the original controller \( C = C_1 \) and the real plant \( P_r \). Fig. 23 shows the Nyquist diagram of the open-loop characteristics with the modified controller \( C = C_1C_2C_3 \) and \( P_r \). Table 1 lists the worst cases of \(||S_r||_\infty\) and \(||T_r||_\infty\) for two controllers (the case with the original controller: the maximum gain of in Fig. 11, the case with the modified controller: the maximum gain in Fig. 21). From comparison between Figs. 11 and 21, we can see that using the modified controller improves the worst cases of \(||S_r||_\infty\) and \(||T_r||_\infty\) by about 9 dB.

### Table 1

<table>
<thead>
<tr>
<th>Characteristics of control systems</th>
<th>Original controller</th>
<th>Modified controller</th>
</tr>
</thead>
<tbody>
<tr>
<td>max (</td>
<td></td>
<td>S_r</td>
</tr>
<tr>
<td>max (</td>
<td></td>
<td>T_r</td>
</tr>
<tr>
<td>Positioning accuracy [% of track width]</td>
<td>11.6</td>
<td>9.8</td>
</tr>
</tbody>
</table>

We performed time-domain simulations by using the control systems with \( C = C_1C_2C_2 \). Fig. 24 shows the amplitude spectrum of \( e \) with \( C = C_1C_2C_3 \). Table 1 lists the positioning accuracies (three standard deviations of \( e \)) for the initial controller and the modified controller. From these results, we can see that the proposed method enables us to improve the positioning accuracy by about 15% from that with the original \( \mathcal{H}_\infty \) controller without losing stability margins.

7. Conclusion

In this paper, we propose a design method of a robust controller for the magnetic-head positioning system in HDDs based on the \( \mathcal{H}_\infty \) control theory and the RCBode plot. In the proposed method, we employ a loop-shaping methodology called “RCBode plot” that enables us to handle the robust performance problems with graphical guidelines. We demonstrate utility of the proposed method by applying it to the magnetic-head positioning control system in the HDD benchmark problem. Simulation results showed that using the proposed method can improve the positioning accuracy by about 15% without losing stability margins.
Acknowledgement

This work is supported by JSPS KAKENHI Grant Number JP18K04210.

References

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