Effect of Cross Transfer Function on Chatter Stability in Plunge Cutting*

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Abstract
An accurate analytical model for regenerative chatter vibration in plunge cutting is proposed. Effects of the cross transfer function and the cutting force ratio on chatter stability are considered in the proposed model. Equivalent transfer function is defined, which is useful to understand those effects on chatter stability. Basic plunge cutting experiments were conducted to verify the proposed model. Although the critical widths of cut in the CW and CCW rotation processes were significantly different from each other in the experiment, even when the other conditions were the same, chatter stability limits were predicted accurately by the proposed model as compared with the conventional model.

Key words: Cutting, Chatter Vibration, Stability Analysis, Transfer Function, Regenerative Effect, Plunge Cutting

1. Introduction
Chatter vibration often results in poor surface finish, rapid tool wear, intense noise and destruction of mechanical structures, consequently limiting productivity. Therefore, the prediction of chatter-stable conditions is one of the most important themes in industry. In general, self-excited chatter vibration in plunge cutting is caused by regeneration. The most commonly used analytical model of plunge cutting with regenerative chatter vibration is formulated through representation of thrust direction with a direct transfer function and a thrust cutting force constant (1). However, it is known that the stability limits calculated by the conventional model sometimes disagree with the experimental results. As a typical example of this, there is often a significant difference between the stability limits in turning processes at clockwise (CW) and counter-clockwise (CCW) spindle rotations, even when the other conditions are the same. It is impossible to explain such a difference with the simple conventional model. This difference is essentially caused by dynamic fluctuation of principal force in synchronization with dynamic thrust force. In other words, it indicates that they affect the chatter stability significantly.

A two-dimensional (2D) analytical model is required to consider the effects accurately - where the relationship between dynamic displacement in the thrust direction and the dynamic principal force is formulated with the cross transfer function of the most flexible mechanical structure and the cutting force ratio of the thrust force to the principal force. Although a simple two-degree-of-freedom (2DOF) model for the stability analysis of plunge cutting has been proposed (2), past studies have not focused on these effects. Therefore, the importance of the cross transfer function and the cutting force ratio is...
generally unknown, and the conventional one-dimensional (1D) analytical model is utilized in most cases regardless of these significant effects.

In the research presented here, two kinds of formulations, which are based on one-degree-of-freedom (1DOF) and 2DOF systems, are proposed. In the proposed analytical model based on the 1DOF system, a 1D “equivalent transfer function” is defined by using the direct and cross transfer functions and the cutting force ratio, which is helpful to understand the effects of the cross transfer function and the cutting force ratio on chatter stability. Basic plunge cutting experiments are conducted to verify that the proposed analytical model provides accurate predictions as compared with the conventional analytical model.

Nomenclature

\[ [A] \]: cutting force coefficient matrix
\( a \): width of cut, m
\( a_{\text{lim}} \): critical width of cut, m
\( F_f \): dynamic cutting force in feed direction, N
\( F_c \): dynamic cutting force in cutting direction, N
\( F_x \): cutting force in \( x \)-axis, N
\( F_y \): cutting force in \( y \)-axis, N
\[ [G] \]: transfer function matrix, m/N
\( G_{xx}, G_{yy} \): direct transfer functions in \( x \) and \( y \) axes, m/N
\( G_{xy}, G_{yx} \): cross transfer functions, m/N
\( h \): dynamic uncut chip thickness, m
\( k \): integer number of full vibration cycles in a spindle revolution period
\( K_f \): cutting constant in feed direction, Pa
\( K_r \): constant ratio of thrust force to principal force
\( n \): spindle speed, min\(^{-1}\)
\( T \): spindle revolution period, s
\( t \): time, s
\( x \): displacement, m
\( \varepsilon \): phase difference between inner and outer modulations, rad
\( \Lambda \): eigenvalue of determinant, N/m
\( \Lambda_R \): real part of eigenvalue \( \Lambda \), N/m
\( \Lambda_I \): imaginary part of eigenvalue \( \Lambda \), N/m
\( \Phi \): 1D equivalent transfer function, m/N
\( \Phi_R \): real part of 1D equivalent transfer function \( \Phi \), m/N
\( \Phi_I \): imaginary part of 1D equivalent transfer function \( \Phi \), m/N
\( \omega_c \): chatter vibration frequency, rad/s

2. Stability Analysis of Regenerative Chatter Vibration in Plunge Cutting

2.1 Formulation Based on 2DOF System

Figure 1 shows the plunge cutting process with regenerative chatter vibration in the CW spindle rotation. When the mechanical structure vibrates, relative displacement between the tool and the workpiece fluctuates. The uncut chip thickness fluctuates by not only the present displacement but also the past displacement left on the surface in the previous rotation. The dynamic uncut chip thickness, i.e., difference between the present and previous relative displacements, causes the dynamic cutting force. The dynamic force generates the present vibration. This closed-loop or feedback system is modeled here to predict the stability limits accurately.

The dynamic uncut chip thickness \( h(t) \) can be derived from the displacement \( x(t) \) as follows:
\[ h(t) = x(t) - x(t - T), \quad (1) \]

where \( T \) is a spindle revolution period. The dynamic cutting force components in the feed and cutting directions, \( F_f(t) \) and \( F_c(t) \), can be expressed as:

\[
F_f(t) = aK_f h(t), \quad F_c(t) = \frac{F_f(t)}{K_r}, \quad (2)
\]

where the cutting forces are proportional to the cutting constant in the feed direction \( K_f \), the width of cut \( a \) and the dynamic uncut chip thickness. \( K_r \) is the constant ratio of the thrust force to the principal force. The dynamic cutting forces in the \( x \)-axis and \( y \)-axis directions, \( F_x(t) \) and \( F_y(t) \), are given in the CW rotation process by

\[
F_x(t) = -F_f(t), \quad F_y(t) = F_f(t). \quad (3)
\]

Substituting Eqs. (1) and (2) into Eq. (3), the dynamic cutting forces can be expressed at the chatter vibration frequency \( \omega_c \) as follows:

\[
\begin{bmatrix}
F_x(j\omega_c) \\
F_y(j\omega_c)
\end{bmatrix} = -aK_f \begin{bmatrix} 1 & -1 \\ -1/K_r & 1 \\ \end{bmatrix} h(j\omega_c) = -aK_f \left( 1 - e^{-j\omega_c T} \right) \begin{bmatrix} x(j\omega_c) \\
y(j\omega_c)
\end{bmatrix} \]

\[
[A] = \begin{bmatrix} 1 & 0 \\ -1/K_r & 0 \\ \end{bmatrix} \quad (4)
\]

On the other hand, the relationship between the exciting forces and the relative displacements can also be obtained by using the 2DOF transfer function of the most flexible mechanical structure \( [G(j\omega_c)] \).

\[
\begin{bmatrix} x(j\omega_c) \\
y(j\omega_c)
\end{bmatrix} = [G(j\omega_c)] \begin{bmatrix} F_f(j\omega_c) \\
F_y(j\omega_c)
\end{bmatrix} = \begin{bmatrix} G_{x}(j\omega_c) & G_{y}(j\omega_c) \\ G_{y}(j\omega_c) & G_{y}(j\omega_c)
\end{bmatrix} \begin{bmatrix} F_f(j\omega_c) \\
F_y(j\omega_c)
\end{bmatrix} \quad (5)
\]

Under a critical stability limit, the dynamic cutting force does not intensify or attenuate through the closed-loop, and thus, the following equation can be derived by substituting Eq. (4) into Eq. (5).

Fig. 1 Schematic of plunge cutting process
\[ \begin{align*}
&\left[ F_x(j\omega_c) \right] = -a_{\text{lim}} K_f \left[ I - e^{-j\omega_c \tau} \right] \begin{bmatrix} A \end{bmatrix} G(j\omega_c) \left[ F_x(j\omega_c) \right], \\
&\begin{bmatrix} F_x(j\omega_c) \end{bmatrix} = -a_{\text{lim}} K_f \left[ I - e^{-j\omega_c \tau} \right] \begin{bmatrix} A \end{bmatrix} G(j\omega_c) \left[ F_x(j\omega_c) \right].
\end{align*} \tag{6} \]

where \( a_{\text{lim}} \) is the critical width of cut. Eq. (6) has a nontrivial solution if its determinant is zero.

\[ \det \begin{bmatrix} I + a_{\text{lim}} K_f \left[ I - e^{-j\omega_c \tau} \right] \begin{bmatrix} A \end{bmatrix} G(j\omega_c) \end{bmatrix} = 0, \tag{7} \]

which can be expressed as

\[ \det \begin{bmatrix} I + \Lambda \begin{bmatrix} G_0(j\omega_c) \end{bmatrix} \end{bmatrix} = 0 \]

\[ \Lambda = a_{\text{lim}} K_f \left[ I - e^{-j\omega_c \tau} \right], \quad \begin{bmatrix} G_0(j\omega_c) \end{bmatrix} = \begin{bmatrix} A \end{bmatrix} G(j\omega_c). \tag{8} \]

By calculating the eigenvalue, the critical conditions are given by:

\[ a_{\text{lim}} = \frac{\Lambda_R}{2K_f} \left[ 1 + \left( \frac{\Lambda_I}{\Lambda_R} \right)^2 \right], \]

\[ \varepsilon = \pi - 2 \tan^{-1} \frac{\Lambda_I}{\Lambda_R}, \tag{9} \]

\[ n = \frac{60\omega_c}{2k\pi + \varepsilon} \]

where \( \Lambda_R \) and \( \Lambda_I \) are, respectively, real and imaginary parts of the eigenvalue \( \Lambda \). \( \varepsilon \) is the phase difference between the inner and outer modulations, \( n \) is the spindle speed and \( k \) is integer number of full vibration cycles in a spindle revolution period. This solution is similar to that found in milling operation (3).

\subsection*{2.2 Formulation Based on 1DOF System}

The fluctuation of the relative displacement between the tool and the workpiece in the cutting direction (y-axis direction) does not fundamentally affect the dynamic cutting force, while that in the thrust direction (x-axis direction) produces the dynamic cutting force. Although the dynamic force has two components (principal and thrust forces), these components change in synchronization. This enables the 2DOF formulation to be reduced to 1DOF when considering the two-directional forces as follows.

The motion equation (5) can be reduced to the following equations:

\[ x(j\omega_c) = G_x(j\omega_c) F_x(j\omega_c) + G_y(j\omega_c) F_y(j\omega_c) \]

\[ = \left[ G_x(j\omega_c) - G_y(j\omega_c) \right] \begin{bmatrix} K_x \end{bmatrix} F_x(j\omega_c) = \Phi(j\omega_c) F_x(j\omega_c), \tag{10} \]

where \( \Phi(j\omega_c) \) is the 1D “equivalent transfer function”. As the motion equation is described only in the thrust direction, the cutting process with chatter vibration can be modeled simply as a 1DOF system in a way similar to the conventional model. The stability limits can be calculated by solving the following simple equations. Substituting Eq. (10) into Eq. (4) yields

\[ F_x(j\omega_c) = -a_{\text{lim}} K_f \left[ I - e^{-j\omega_c \tau} \right] \Phi(j\omega_c) F_x(j\omega_c). \tag{11} \]

\[ \therefore 1 + a_{\text{lim}} K_f \left[ I - e^{-j\omega_c \tau} \right] \Phi(j\omega_c) = 0 \tag{12} \]

By solving the complex equation (12), the critical conditions are given by:

\[ a_{\text{lim}} = \frac{-1}{2K_f \Phi_R}, \]

\[ \varepsilon = 2\pi - 2 \tan^{-1} \frac{\Phi_I}{\Phi_R}, \tag{13} \]

\[ n = \frac{60\omega_c}{2k\pi + \varepsilon} \]
where \( \Phi_r \) and \( \Phi_i \) are real and imaginary parts of the equivalent transfer function \( \Phi(j\omega) \).

Both analytical models based on the 1DOF and 2DOF systems give the same solutions. The analysis based on the 1DOF system, however, gives the solutions easily and clarifies the effects of the cross transfer function and the cutting force ratio on chatter stability. Stability limits can be estimated from the vector diagram of the equivalent transfer function. The 2DOF model, on the other hand, is redundant and not useful in understanding the process.

2.3 Consideration of Spindle Rotation Direction

Figure 2 shows the cutting processes at CW and CCW spindle rotations. As shown in the figure, the principal force direction depends on the rotation direction, while the thrust force direction is independent from rotation direction. Because of this, the \([A]\) matrix in Eq. (4), which is formulated based on the 2DOF system, has to be modified for the CCW rotation process to

\[
[A] = \begin{bmatrix}
1 & 0 \\
1/K_c & 0 \\
\end{bmatrix}.
\]  
(14)

Accordingly, the equivalent transfer function in Eq. (10), which is defined based on the 1DOF system, is also changed to

\[
\Phi(j\omega) = G_{xx}(j\omega) + \frac{G_{xy}(j\omega)}{K_c}.
\]  
(15)

By using the above \([A]\) matrix or equivalent transfer function, the stability limits in the CCW rotation can be calculated.

Figure 3 shows a block diagram of the plunge cutting process with the regenerative chatter vibration, which is drawn based on the 1DOF system. As shown in the figure, the sign of the displacement excited by the cutting force in the \(y\) direction depends on the spindle rotation direction. This means that the effect of the cross transfer function \(G_{xy}(s)\)
on the equivalent transfer function $\Phi(s)$ depends directly on the spindle rotation direction.

Rotating cylindrical workpieces are machined in turning operations. In such cases, the natural frequencies in the two radial directions are nearly equal because the workpiece is usually asymmetric and rotates. The cross transfer functions are often large due to anisotropy of the mechanical structures. In addition, the effect of the cross transfer function is magnified by the cutting force ratio, as shown in Eqs. (10) and (15). Therefore, the effects of the cross transfer function and the cutting force ratio on chatter stability become considerably large when the rotating workpiece is the most flexible structure and the cutting force ratio is small.

3. Plunge Cutting Experiments

To verify the proposed model, basic plunge cutting experiments were conducted, as shown in Fig. 4. A cylindrical workpiece made of stainless steel was fixed to the lathe. Its projection length from the edge of the chuck was set to 300 mm. A cemented carbide tool, whose width of cut is 1.25 mm, was used. The feed rate and the spindle speed were set to 0.025 mm/rev and 563 min$^{-1}$, respectively. It was also confirmed that the workpiece is significantly flexible in the radial directions as compared with the other mechanical structures, such as the tool holder and its fixture. The specific cutting force $K_f$ and the cutting force ratio $K_r$ were identified as 864 GPa and 0.36 in a preliminary experiment, where the same workpiece and the same tool were used. It was confirmed that the differences of $K_f$ and $K_r$ are negligible between CW and CCW spindle rotation processes.

Figure 5 shows positions of cutting and measurement. In this experimental setup, the compliance of the workpiece depends on the position along the rotation axis, while its natural frequency is unchanged. Therefore, the critical width of cut in the plunge cutting
increases as the distance $L$ from the edge of the chuck to the cutting position decreases.

The transfer functions were measured at several positions $L$ by the impulse response method. Figure 6 shows positions of hammering and measurement. The workpiece was excited by an impact hammer at the cutting positions, and the impulse responses of the workpiece structure were measured by accelerometers. Vector diagrams of the direct and cross transfer functions, $G_{xx}$ and $G_{xy}$, measured at $L=160$ mm are shown in Fig. 7. The natural frequency was about 230 Hz. The equivalent transfer functions in the CW and CCW rotation processes, which are calculated by Eqs. (10) and (15), are also plotted in the same figure. As shown in the figure, the compliance of the cross transfer function is not small in comparison to the direct one. In fact, it is magnified 2.8 times due to the small cutting force ratio ($K_r=0.36$). As a result, the respective equivalent transfer functions in the CW and CCW processes and the direct transfer function $G_{xx}$ are considerably different from one another. It is also imaginable that the stability limits predicted by the proposed analysis are different from those found through conventional analysis.

In the cutting experiments, the tool was set at a position of $L=125$ or 160 mm in the CW process, and at $L=75$ or 115 mm in the CCW process. As it is difficult to measure the vibrations at the cutting positions directly, the eddy current displacement sensor was set near the cutting position as shown in the figure, and the displacement of the workpiece was measured during machining. When its

<table>
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<th>Table 1: Experimental conditions and measured cutting force coefficients</th>
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<td><strong>Rotation direction</strong></td>
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<td>Distance from chuck to cutting position $L$ mm</td>
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<td>Distance from chuck to measuring position of transfer functions $L$ mm</td>
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<td>Measured cutting force coefficients</td>
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amplitude exceeded 10 \( \mu m \), the system was judged unstable. The experimental conditions and the identified cutting force coefficients used in the analyses are summarized in Table 1.

4. Discussion

Figure 8 shows stability lobes, which were calculated with the transfer function measured at \( L=160 \) mm. As shown in the figure, asymptotic stability limits are significantly different from each other. The critical width of cut predicted by the conventional analysis is the largest, while that in the CCW process predicted by the proposed analysis is the smallest at most of the spindle speed. The critical width of cut is in inverse proportion to the absolute value of the negative real part of the transfer function as indicated by Eq. (13). The minimum values of the real parts of the transfer functions decrease in the following order, \( G_{xx}, \Phi_{CW}, \Phi_{CCW} \). As demonstrated in this example, the equivalent transfer function is useful to estimate the effect of the cross transfer function and the cutting force ratio on chatter stability.

Figure 9 shows the experimental results and the analytical asymptotic stability limits of the width of cut. The analytical results are shown by the dots and the dotted lines. The areas under the lines indicate “stable” conditions. The experimental results are shown by the circles and the crosses, which mean “stable” and “unstable”, respectively. It was confirmed that the amplitudes of the chatter vibrations measured in the unstable conditions increased to about 16 \( \mu m \) during machining, and measured chatter frequencies were close to the natural frequency. On the other hand, the vibration amplitudes measured in the stable conditions were less than 1.3 \( \mu m \) and they did not increase during machining. As shown in the figure, the analytical stability limits calculated by the proposed method agree with the
experimental results, and this shows that the chatter stability limits can be predicted accurately by the proposed analytical model when considering spindle rotation directions. Unlike the proposed model, the conventional analysis does not agree with the experiments.

5. Conclusion

In order to develop an accurate analytical model of the plunge cutting process with chatter vibration, two kinds of formulations, based on the 1DOF and 2DOF systems, are proposed in the research presented. In the formulation based on the 1DOF system, the “equivalent transfer function” is defined by using the direct and cross transfer functions and the cutting force ratio. The proposed 1D model is simpler than the 2D model, and the 1D model is useful in considering the effect of the cross transfer function and the cutting force ratio on chatter stability. Basic plunge cutting experiments were also carried out to verify the proposed model. The critical widths of cut in the CW and CCW rotation processes were significantly different from each other in the experiment, even when the other conditions were the same. Analytical results obtained by the proposed method agreed with the experimental results accurately, while those obtained through the conventional method disagreed. It is also noted that the cross transfer function is generally large in rotational movement using flexible workpieces, and its effect on chatter stability is further magnified as the cutting force ratio is small.

References

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