Geometric Constraint of an Evolved Deployable Ball Mechanism*

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Abstract

This paper presents a novel method based on the virtual sliders for constructing a deployable ball mechanism and reveals the geometric constraint of the 20-bar ball mechanism and the radially foldable linkage that were extracted from the Hoberman switch-pitch structure. This leads to the development of the chain structure equation to unravel the radial motion characteristics of the mechanism and to the construction of the 20-bar ball mechanism with simulation and rapid prototyping process. The paper further presents position analysis and Jacobian matrix of the mechanism and simulation and numerical analysis are used to illustrate the kinematic characteristics. The analysis corresponds to motion in both simulation and prototyped ball mechanism.

Key words: Ball Mechanism, Eight-Bar Linkage, Geometric Constraint, Kinematics, Hoberman Switch-Pitch Structure

1. Introduction

Foldable and deployable mechanisms have drawn great interest from researchers in the past decades due to its wide applications in the fields of machines⁽¹⁾, customer products, toys, architectural applications and space technologies. This can be traced back to 1947 when architect Fuller⁽²⁾ invented a foldable cubo-octahedron with twenty-four wooden dowels. In 1993, Baker⁽³⁾ investigated a foldable network of modified Bennett linkages. At the same time, Wohlhart⁽⁴⁾ presented three overconstrained spatial mechanisms as turning tower, screwing tower and folding octoid. Later from 1995 to 1999, Wohlhart⁽⁵⁾⁻⁽⁷⁾ presented a family of deformable gate mechanisms based on Heureka polyhedra⁽⁸⁾ including the star-cube transformer which is similar to the later development of a Hoberman geared expanding structure⁽⁹⁾ derived from the Hoberman sphere⁽¹⁰⁾. During that period, several Röscher polyhedrons⁽¹¹⁾⁻⁽¹⁴⁾ were studied. In 1997, You and Pellegrino⁽¹⁵⁾ developed a two-dimensional foldable structure and investigated the structure layouts of the flat and curved structure folded along their perimeter. In 1998, Dai and Rees Jones⁽¹⁶⁾ proposed a family of foldable/erectable mechanisms from the study of origami cartons and the reconfigurable packaging⁽¹⁷⁾. In 1999, Langbecker⁽¹⁸⁾ investigated geometric design of deployable scissor structures and formulated the foldability equation. In 2001, Agrawal et al⁽¹⁹⁾ developed a single mobility structure of a lattice of expanding polyhedral units. The built-in symmetries make the entire structure expandable and contractible. Between 2001 and 2005, Wohlhart⁽²⁰⁾⁻⁽²⁴⁾ further discovered a series of polyhedral linkages including the

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Zig-Zag linkages and presented their deployability properties. In 2004, Dai et al⁽²⁵⁾ investigated the Hoberman sphere using the screw system relationship theory⁽²⁶⁾⁽²⁷⁾. In 2005, from computational geometric point of view, Demaine and O'rourke⁽²⁸⁾ gave an extensively survey on folding and unfolding linkages, paper and polyhedra. In 2005 and 2007, Patel and Ananthasuresh⁽²⁹⁾⁽³⁰⁾ presented planar radially foldable linkages and expandable planar linkages. In 2006, Kusakari⁽³¹⁾ proposed new linkages coined as "reconfigurable radial trees". In 2007, You⁽³²⁾ investigated the motion structure of the Hoberman flight ring and identified the motion characteristics that could be used to develop structures for satellite solar panels and space antennas to shelters. In 2008, Dai and Cannella⁽³³⁾ identified a new foldable/erectable mechanism evolved from origami that imitates decorative art. At the same time, Kiper et al⁽³⁴⁾⁽³⁵⁾ presented polygons and polyhedral radial linkages based on Cardan motion which were defined as mobile assemblies, attaining different configurations to meet various geometric constraints.

Combination of the radially foldable linkage and the deployable polyhedral linkage led to the consumer product development in mid 2000s as Hoberman sphere[®], switch-kick[®] and switch-pitch^{®(36)}. The Hoberman switch-pitch structure has several loops each of which can be regarded as an eight-bar radial linkage with the entire vertices moving radially towards the centre of the ball while the ball folds and expands radially. Though mobility of the Hoberman switch-pitch structure was investigated by Wei et al⁽³⁷⁾, much of the geometric constraint that leads to construction of the structure and its evolved mechanism and linkages has not been investigated and the their kinematics is still required to be investigated.

This paper reveals the geometric constraint of the structure particularly its evolved 20-bar ball mechanism without bevel gears in vertexes and uses the constraint to study the construction of the ball mechanism with both simulation and prototyping. The geometric constraint and motion characteristics of the extracted radially foldable linkage are presented. The geometric constraint leads to identifying the intrinsic characteristics of the ball mechanism. The paper for the first time applies the chain structure equation to the study and obtains the Jacobian matrix of the ball mechanism. This reveals the motion characteristics of the overconstrained ball mechanism and its extracted radially foldable linkage.

2. Construction of the Evolved Ball Mechanism Based on the Virtual Sliders

Extracted from the Hoberman switch-pitch structure⁽⁹⁾⁽³⁶⁾ by replacing the bevel gears

Fig. 1 Structure and parameters of a radially deployable ball mechanism

with revolute joints, a ball mechanism consisting of eight vertexes and twelve links is obtained in Fig. 1⁽³⁷⁾. The mechanism is similar to the paper-fold form presented by

Fig. 2 A planar PRRP chain and a cube with a frame embedded in its diagonals

Wohlhart in 1995⁽⁵⁾ through making up of four modular Heureka linkages⁽⁸⁾ and through integrating CRRC chains⁽⁷⁾. The ball mechanism can also be constructed by arranging planar PRRP chains at diagonals of a cube in a symmetric way.

Figure 2a gives a diagram of a planar PRRP chain whose link POP is a frame and two sliders noted as P and link RR form movable parts. Four pairs of this frame link form a cube in Fig. 2b with frame links as diagonals of the cube. Hence angle γ between two frame links of the PRRP chain is the angle between two diagonals of the cube. Arrange eight slides along the frame links in Fig. 2b, sliders P form vertexes V_0 to V_7 in Fig. 3a. Arrange twelve identical RR links connecting these sliders, four pairs of PRRP chains are interconnected in a symmetrical way. This constructs a ball structure in Fig. 3b after removing frame link of the cube. This forms an overconstrained mechanism of mobility one with radially reciprocating motion in such a way that vertexes V_0, V_4, V_5 and V_6 move towards central point O_C and vertexes V_1, V_2, V_3 and V_7 move away from point O_C and vice versa. The mobility of this mechanism has been proved by using the screw-loop equation method⁽³⁷⁾ and the above presents a new method of constructing overconstrained deployable mechanisms. Using this method, a family of overconstrained mechanisms that perform transferring of rotation motion to radially reciprocating motion can be obtained⁽³⁸⁾.

3. Geometric Constraint of the Deployable Ball Mechanism

From the above construction of the ball mechanism, it can be seen that it is a highly overconstrained deployable mechanism consisting of twelve links and eight vertexes each of which contains three revolute joints connected to three links. Further, from the construction and motion properties of the mechanism and applying the spherical trigonometry, the geometric constraint can be obtained as the three angles between the

Fig. 3 Construction of the mechanism via a combination of PRRP chains

selected vertex and its adjacent vertex axes need to have the following relationship

$$\begin{cases} \mathbf{s}_k \cdot (\mathbf{s}_l + \mathbf{s}_m + \mathbf{s}_n) = \cos \gamma_{kl} + \cos \gamma_{km} + \cos \gamma_{kn} = 1 \\ \sqrt{d_k^2 + d_l^2 - 2d_k d_l \cos \gamma_{ik}} = l - 2h \quad (i = l, m, n) \\ d_l = d_m = d_n \end{cases} \quad (1)$$

where $\mathbf{s}_k$ is a unit vector presenting the vertex axis of the selected vertex V_k , and $\mathbf{s}_l$, $\mathbf{s}_m$ and $\mathbf{s}_n$ are unit vectors of its three adjacent vertex axes of vertexes V_l , V_m and V_n . Angles γ_{kl} , γ_{km} , and γ_{kn} are angles between vertex V_k and its adjacent vertexes V_l , V_m and V_n . d_k , d_l , d_m and d_n are the distance between vertexes V_k , V_l , V_m , and V_n and centre point O_C , h is the offset distance along the common perpendicular of the R joint and P joint and l is side-length of the virtual cube as indicated in Fig. 2.

This can be explained when taking vertex V_7 as a sample vertex with vertex axis $\mathbf{s}_7$ and its adjacent vertex axes are $\mathbf{s}_4$, $\mathbf{s}_5$ and $\mathbf{s}_6$. The angles between the selected vertex axis $\mathbf{s}_7$ and its adjacent vertex axes are γ_{47} , γ_{57} and γ_{67} . These vertexes must satisfy the following geometric constraint

$$\begin{cases} \mathbf{s}_7 \cdot (\mathbf{s}_4 + \mathbf{s}_5 + \mathbf{s}_6) = \cos \gamma_{47} + \cos \gamma_{57} + \cos \gamma_{67} = 1 \\ \sqrt{d_7^2 + d_i^2 - 2d_7 d_i \cos \gamma_{i7}} = l - 2h \quad (i = 4, 5, 6) \\ d_4 = d_5 = d_6 \end{cases} \quad (2)$$

From this geometric constraint, it is found that the combination of vertex vectors $\mathbf{s}_4$, $\mathbf{s}_5$ and $\mathbf{s}_6$ produces a vector in the same direction as that of $\mathbf{s}_7$. Since every vertex complies with the geometric constraint in Eq. (1), it can be concluded that all eight vertexes move radially along the directions of their normals and Eq. (1) presents geometric constraint for the ball mechanism. The motions of the ball mechanism has been simulated via CAE software in Fig. 4 and physically tested with prototype in Fig. 5.

Fig. 4 Typical configurations of the ball mechanism in simulation

Fig. 5 Typical configurations of the ball mechanism in prototype

The ball mechanism is symmetric with equal-sized vertexes and equal-length links. Thus from Eq. (1), it can be deduced that

$$\gamma_{lk} = \gamma_{mk} = \gamma_{nk} = \gamma. \quad (3)$$

The value of γ can be obtained in the following when the mechanism is in a virtual cubic configuration:

$$\gamma = \arccos \left(\frac{\left(\frac{\sqrt{3}l}{2} \right)^2 + \left(\frac{\sqrt{3}l}{2} \right)^2 - l^2}{2 \times \frac{\sqrt{3}l}{2} \times \frac{\sqrt{3}l}{2}} \right) = \arccos \frac{1}{3}, \quad (4)$$

where l is the side-length of the virtual cube as indicated in Fig. 2. Substitute the above to Eq. (1), it follows that

$$\cos \gamma_{lk} + \cos \gamma_{mk} + \cos \gamma_{nk} = 1/3 + 1/3 + 1/3 = 1. \quad (5)$$

Equations (3) and (5) indicate that all the angles between any two vertex axes are the same and their cosine values are all $1/3$.

The above geometric constraint leads to construction of the ball mechanism in Fig. 4 and in a physical form in Fig. 5. Two typical working configurations are presented in the Figs. 4 and 5. Both simulation and prototype models illustrate radial motion characteristics of the ball mechanism.

Further, in each vertex of the ball mechanism, angle between any two of the three revolute joint axes is equal to $\pi/3$. Therefore, the three revolute-joint axes form an equilateral triangle in each vertex and there exists

$$\arccos(\mathbf{s}_p \cdot \mathbf{s}_q) = \pi/3, \quad (6)$$

where $\mathbf{s}_p$ and $\mathbf{s}_q$ are the unit vectors of the directions of any two revolute joint axes in a vertex.

In addition, it should be pointed out that the Hoberman switch-pitch structure is the special case of the ball mechanism by introducing a set of bevel gears into the vertexes that coordinate the three revolute joints at each vertex.

4. Radial Motion of the Extracted Single-looped Foldable Linkage

Dismantling the ball mechanism by removing vertexes V_3, V_4, V_6, V_7 and the corresponding links, a single closed-looped chain linkage can be obtained as in Fig. 6. This is an eight-bar linkage and there are four links with two parallel joint axes at each end and four vertexes with two revolute joint axes being spread at both ends of the vertex by 60° . To reveal radial motion of this linkage, two of the vertex links V_0 and V_5 are picked up one as base and the other one as platform. The base and platform vertexes are connected by two

Fig. 6 Schematic and geometry of the single-looped linkage

identical limbs each of which is composed of two links, one triangular plate and four joints.

Since this is a linkage of mobility two. In order to fully define the configuration of the linkage, two inputs are required. Herein the two joints connected to the base are assigned to be actuated joints. Further, in order to carry out the kinematic analysis, a global coordinate frame (O_0, x_0, y_0, z_0) is attached to the base vertex V_0 at point O_0 , locating at the centre of the equilateral triangular base V_0 , x_0 axis is parallel to one side of the base, y_0 axis is perpendicular to another side and z_0 axis is perpendicular to vertex V_0 . A local coordinate frame (P_5, x', y', z') is attached to the platform vertex V_5 at point P_5 , locating at the centre of the vertex V_5 with x' axis parallel to x_0 axis and z_0 axis perpendicular to vertex V_5 as in Fig. 6.

Based on the chain structure equation proposed by McCarthy⁽³⁹⁾, the linkage can be split into two chains with two end points O_0 and P_5 . The right chain is referred to as limb 1 and the left chain as limb 2. Further, link lengths are defined as a_{jk} , link twist angle α_{jk} is defined as the angle of two joint axes at both end of the link and link offset d_{jk} is defined as the distance between two perpendicular vectors of three joint axes. Further, motion parameters are given as θ_{jk} . Herein, the first subscript for a_{jk} , α_{jk} , d_{jk} and θ_{jk} indicates the limb number and the second subscript denotes the joint number. The lengths of the links 2, 4, 6 and 8 are the same as $a_{11} = a_{13} = a_{21} = a_{23} = a$, the sizes of the vertexes V_0 , V_1 , V_2 and V_5 are identical and the lengths of the sides of the triangles of the vertexes are $2b$. The angle between joint axes at the same vertex is 60° , this leads to $d_{12} = d_{13}$ and $d_{22} = d_{23}$. From Fig. 6, it can be found that $|d_{jk}| = b$ and $h = \sqrt{3}b/3$. Based on D-H convention, these link and joint parameters are given in Table 1.

Table 1 Link parameters of the single-looped linkage

Joint k	Limb 1				Limb 2			
	α_{1k}	a_{1k}	d_{1k}	θ_{1k}	α_{2k}	a_{2k}	d_{2k}	θ_{2k}
0	$\pi/2$	h	0	$\pi/6$	$-\pi/2$	h	0	$\pi/6$
1	0	a_{11}	0	θ_{11}	0	a_{21}	0	$-\theta_{21}$
2	$-\pi/3$	0	$-d_{12}$	$-\theta_{12}$	$\pi/3$	0	$-d_{22}$	θ_{22}
3	0	a_{13}	d_{13}	θ_{13}	0	a_{23}	d_{23}	$-\theta_{23}$
4	$\pi/2$	h	0	θ_{14}	$-\pi/2$	h	0	$-\theta_{24}$
5	0	0	0	$\pi/6$	0	0	0	$\pi/6$

From the above structure parameters, chain structure equations for limb 1 and limb 2 of the eight-bar linkage can be constructed as,

$$\mathbf{D} = \mathbf{H}_{11}\mathbf{H}_{12}\mathbf{H}_{13}\mathbf{H}_{14}\mathbf{H}_{15}\mathbf{P}_1 = \begin{bmatrix} D_{11}^1 & D_{12}^1 & D_{13}^1 & D_{14}^1 \\ D_{21}^1 & D_{22}^1 & D_{23}^1 & D_{24}^1 \\ D_{31}^1 & D_{32}^1 & D_{33}^1 & D_{34}^1 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (7)$$

and

$$\mathbf{D} = \mathbf{H}_{21}\mathbf{H}_{22}\mathbf{H}_{23}\mathbf{H}_{24}\mathbf{H}_{25}\mathbf{P}_2 = \begin{bmatrix} D_{11}^2 & D_{12}^2 & D_{13}^2 & D_{14}^2 \\ D_{21}^2 & D_{22}^2 & D_{23}^2 & D_{24}^2 \\ D_{31}^2 & D_{32}^2 & D_{33}^2 & D_{34}^2 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (8)$$

Where the elements of the matrices in Eqs. (7) and (8) can be obtained through D-H rules and matrix operations.

Equalizing Eqs. (7) and (8) and comparing elements of the matrices, it can be found that when $\theta_{11} = \theta_{21}$, the angular displacement of the remaining joints in two limbs have the following geometric relation,

$$\theta_{12} = \theta_{22}, \theta_{13} = \theta_{23}, \theta_{14} = \theta_{24} . \quad (9)$$

In this case, the trajectory of P_5 is a straight line which is referred to as radial motion indicated by the following relations ⁽⁴⁰⁾,

$$\tan \alpha = \frac{P_{5z}}{P_{5y}} = \sqrt{2} = \text{constant} . \quad (10)$$

and

$$\cos \beta = \mathbf{z}' \cdot \mathbf{y}_0 = \frac{2\sqrt{2}}{3} = \text{constant} . \quad (11)$$

Where α is the angle between trajectory of point P_5 and y_0 -axis, and β is the angle between normal $\mathbf{z}'$ of vertex V_5 and y_0 -axis of the global frame. Since both α and β are constant that during the motion of the eight-bar linkage, one can find that once symmetric inputs are given for the above eight-bar linkage, its platform performs a radial motion.

5. Position Analysis and Link Parameters

In order to study kinematics of the ball mechanism, a global coordinate frame (O_C, x, y, z) is attached at point O_C at which the ball mechanism is imagined to be fixed to the ground as in Fig. 1. x -axis and y -axis are parallel to x_0 -axis and y_0 -axis respectively and z -axis aligned with z_0 -axis. Coordinate frame x_0 - y_0 - z_0 is established at vertex V_0 in the same way as that of the single-looped eight-bar linkage. Local coordinate frames are also established at joints A, B, C, D, E and F by following D-H convention⁽³⁹⁾. In order to calculate the displacement of vertexes V_1 , V_5 and V_7 , local coordinate frames x'_3 - y'_3 - z'_3 , x'_5 - y'_5 - z'_5 and x_7 - y_7 - z_7 are established at vertexes V_1 , V_5 and V_7 . An end-effector coordinate frame u - v - w is also attached at vertex V_7 with its u -axis aligned with x_7 -axis and w -axis aligned with z_7 -axis. Since the mobility of the ball variant is one, herein, link 1 (see Fig. 1) which connects vertexes V_0 and V_1 is chosen as the active link and is actuated at joint axis z_1 . Extending the study of the eight-bar linkage in Sections 3 by treating the ball mechanism been composed of five independent closed loops, the geometric analysis of the ball mechanism is presented as follows.

To begin with, Cartesian coordinates of point P_0 with respect to the global coordinate frame can be obtained directly from Fig. 1 as,

$$\mathbf{p}_0 = \mathbf{A}_0 \boldsymbol{\lambda} . \quad (12)$$

Where $\mathbf{A}_0 = [0 \ 0 \ 1]^T$ and $\boldsymbol{\lambda} = -b\sqrt{6}/3 - a(\sqrt{2}c\theta + 4s\theta)/4$, a is the length of the link in the mechanism, b is half of the width of the equilateral triangle which is formed by three joint axes in a vertex, and θ is the angular displacement of link 1.

Next, the D-H parameters for vertex V_1 with respect to the global coordinate frame are procured in Table 2.

Table 2 Link parameters to P_1 of vertex V_1

Joint j	α_j	a_j	d_j	θ_j
0	0	0	λ	0
1	$\pi/2$	h	0	$\pi/6$
2	0	a	0	θ
3'	$\pi/2$	h	0	$\gamma - \theta$

It should be noted that herein λ is obtained in Eq. (12) denoting the distance between points O_C and P_0 and in Table 2, $h = \sqrt{3}b/3$ is the same as aforementioned and γ is the same as obtained in Eq. (4).

Substitute the above parameters into the D-H transformation matrices which transform coordinates of P_1 - x'_3 - y'_3 - z'_3 to the global coordinate frame, Cartesian coordinates of point P_1 are

$$\mathbf{p}_1 = \mathbf{A}_1 \boldsymbol{\xi}, \quad (13)$$

where, $\mathbf{A}_1 = [\cos 30^\circ \quad \sin 30^\circ \quad -\sqrt{2}/4]^T$ and $\boldsymbol{\xi} = 4\sqrt{3}b/9 + a \cos \theta$.

Following the symmetric property, Cartesian coordinates of the centers of vertexes V_2 and V_3 , i.e. P_2 and P_3 can be obtained as

$$\mathbf{p}_2 = \mathbf{A}_2 \boldsymbol{\xi} \quad (14)$$

and

$$\mathbf{p}_3 = \mathbf{A}_3 \boldsymbol{\xi}, \quad (15)$$

where $\mathbf{A}_2 = [\cos 150^\circ \quad \sin 150^\circ \quad -\sqrt{2}/4]^T$, $\mathbf{A}_3 = [\cos 270^\circ \quad \sin 270^\circ \quad -\sqrt{2}/4]^T$, and $\boldsymbol{\xi}$ is the same as that of Eq. (13).

Further extending Table 2, the link parameters for vertex V_5 with respect to the global coordinate frame are given in Table 3.

Table 3 Link parameters to P_5 of vertex V_5

Joint j	α_j	a_j	d_j	θ_j
0	0	0	λ	0
1	$\pi/2$	h	0	$\pi/6$
2	0	a	0	θ
3	$-\pi/3$	0	$-b$	$-\pi/2 + \gamma - \theta$
4	0	a	b	$\pi/2 + \gamma - \theta$
5'	$\pi/2$	h	0	θ

Integrating these parameters into the D-H transformation matrix that transform coordinates of P_5 - x'_5 - y'_5 - z'_5 to the global coordinate frame, Cartesian coordinates of point P_5 can be obtained as

$$\mathbf{p}_5 = \mathbf{A}_5 \boldsymbol{\varepsilon}, \quad (16)$$

where, $\mathbf{A}_5 = [\cos 90^\circ \quad \sin 90^\circ \quad \sqrt{2}/4]^T$ and, $\boldsymbol{\varepsilon} = (4\sqrt{3}b + 3a \cos \theta + 6\sqrt{2}a \sin \theta)/9$.

Similarly, as vertexes V_4 , V_5 and V_6 are evenly distributed about z -axis of the global coordinate frame, Cartesian coordinates of centre points P_4 and P_6 are

$$\mathbf{p}_4 = \mathbf{A}_4 \boldsymbol{\varepsilon} \quad (17)$$

and

$$\mathbf{p}_6 = \mathbf{A}_6 \boldsymbol{\varepsilon}, \quad (18)$$

where $\mathbf{A}_4 = [\cos(-30^\circ) \quad \sin(-30^\circ) \quad \sqrt{2}/4]^T$, $\mathbf{A}_6 = [\cos(-150^\circ) \quad \sin(-150^\circ) \quad \sqrt{2}/4]^T$, and $\boldsymbol{\varepsilon}$ is the same as that of Eq. (16).

Finally, parameters from vertex V_7 with respect to the global coordinate frame are provided in Table 4.

Table 4 Link parameters to P_7 of vertex V_7

Joint j	α_j	a_j	d_j	θ_j
0	0	0	λ	0
1	$\pi/2$	h	0	$\pi/6$
2	0	a	0	θ
3	$-\pi/3$	0	$-b$	$-\pi/2+\gamma-\theta$
4	0	a	b	$\pi/2+\gamma-\theta$
5	$\pi/3$	0	b	$-\pi/2+\theta$
6	0	a	$-b$	$\pi/2+\theta$
7	$\pi/2$	h	0	$\gamma-\theta$
8	0	0	0	$\pi/2$

Substitute the above parameters to the transformation matrices which transform coordinate P_7 - u - v - w to the global coordinate frame, Cartesian coordinates of point P_7 in the seventh deltoid vertex can be obtained as

$$\mathbf{p}_7 = \mathbf{A}_7 \boldsymbol{\zeta}, \quad (19)$$

where, $\mathbf{A}_7 = [0 \ 0 \ 1]^T$ and $\boldsymbol{\zeta} = (4\sqrt{6}b - 9\sqrt{2}a \cos \theta)/12$.

It is noted that the positions of P_1 , P_2 , P_3 , P_4 , P_5 and P_6 are symmetric around the axis passing through points P_0 and P_7 .

6. Kinematics and Jacobian Matrix of the Ball Mechanism

It is found from the above analysis that if the input angle θ is given, the displacement of all the centers of the eight vertexes of the ball mechanism can be determined from Eqs. (12) throughout (19). From Eqs. (12) and (19), it can be found that the trajectories of points P_0 and P_7 are two straight line segments along z -axis. From Eq. (13), it is found that the trajectory of point P_1 is a segment in a plane passing through z -axis and the angle between it and x - z plane is 30° with a gradient of $-\sqrt{2}/4$. Similarly, Eqs. (14) and (15) show that trajectories of P_2 and P_3 are two line segments lie in two planes passing through z -axis and the angles between them and x - z plane are 150° and 270° , respectively; these two segments have the same slope of $-\sqrt{2}/4$. Further, trajectories of points P_4 , P_5 , and P_6 can also be found from Eqs. (17), (16), and (18) to be three segments whose gradients are all $\sqrt{2}/4$ and lie in three different planes passing through z -axis and the angles between them and the x - z plane are 90° , -30° , and -150° , respectively.

Further, substitute the position vectors $\mathbf{p}_4$, $\mathbf{p}_5$, $\mathbf{p}_6$ and $\mathbf{p}_7$ of vertexes V_4 to V_7 to the geometric constraint Eq. (1), it can be found that

$$\sqrt{(|\mathbf{p}_7| - h)^2 + (|\mathbf{p}_i| - h)^2 - 2(|\mathbf{p}_7| - h)(|\mathbf{p}_i| - h)\cos \gamma_{i7}} = a = l - 2h \quad (i = 4, 5, 6), \quad (20)$$

where $|\mathbf{p}_i|$ denotes the distance between the vertex V_i and the centre O_C . This equation implies that motions of all vertexes in the ball mechanism comply with the geometric constraint presented in Section 2.

The forward kinematics analysis of the ball mechanism consists of determination of the velocities of the centroids of the vertexes for a given velocity of the active link $\dot{\theta}$. Take differential of Eqs. (12) to (19), the velocities of points P_j ($j = 0, 1, \dots, 7$) can be obtained and Jacobian can be given as

$$\mathbf{V} = \mathbf{J} \dot{\theta}, \quad (21)$$

where, $\mathbf{V} = [\mathbf{v}_{p0}, \mathbf{v}_{p1}, \mathbf{v}_{p2}, \mathbf{v}_{p3}, \mathbf{v}_{p4}, \mathbf{v}_{p5}, \mathbf{v}_{p6}, \mathbf{v}_{p7}]$ and $\mathbf{J} = [\mathbf{J}_0, \mathbf{J}_1, \mathbf{J}_2, \mathbf{J}_3, \mathbf{J}_4, \mathbf{J}_5, \mathbf{J}_6, \mathbf{J}_7]$, sub-matrixes $\mathbf{J}_0, \mathbf{J}_1, \mathbf{J}_2, \mathbf{J}_3, \mathbf{J}_4, \mathbf{J}_5, \mathbf{J}_6, \mathbf{J}_7$ of the Jacobian matrix which can be obtained by

taking time differential of Eqs. (12) to (19).

It can be found that the resultant velocities of point P_1 , P_2 , P_3 and P_7 are the same, and the resultant velocities of point P_0 , P_4 , P_5 and P_6 are the same.

Further, once any position of the eight centroid points of vertexes, i.e. p_j ($j = 0, 1, \dots, 7$) of the ball mechanism is given, the input angle θ can be obtained. From any of equations (12) throughout (19), θ can be expressed as the functions of the positions of points P_j ($j = 0, 1, \dots, 7$). The most convenient one can be obtained from Eq. (19), when the position of point P_7 is known the input angle θ can be represented as

$$\theta = \arccos\left(\frac{4\sqrt{6}b - 12p_{7z}}{9\sqrt{2}a}\right), \quad (22)$$

where, p_{7z} is the z -axis component of vector p_7 .

Therefore, given position of vertex V_7 , input angle θ can be obtained via Eq. (22). for any given velocities of the center points of the vertexes, i.e., v_{pi} ($j = 0, 1, \dots, 7$), of the ball mechanism in a given configuration, from any row of Eq. (21), the velocity of the actuated joint can be procured. The most obvious one is from that of point P_7 which is depicted in the last row of Eq. (21). From which, the velocity of the actuated joint is

$$\dot{\theta} = \frac{12(v_{p7})_z}{9\sqrt{2} \sin \theta}, \quad (23)$$

where, $(v_{p7})_z$ is the z -axis component of velocity v_{p7} . It should be noted that when $\dot{\theta}$ is determined, the angular velocities of all the revolute joints can also be obtained and they have the same value in magnitude.

Let parameters of the ball mechanism be $a = 70\text{mm}$ and $b = 15\sqrt{2}\text{mm}$ and take the vertexes in the form of equilateral triangles, the value of θ_{11} for the eight-bar linkage and θ for the ball mechanism lie in the range of $[-(\pi/2 - \gamma), \pi/2]$ by considering the constraints of the mechanisms. The velocity of the actuated joint $\dot{\theta}$ for the ball mechanism is assumed to be $\dot{\theta} = 10\text{rad/sec}$. Based on these, the numerical analysis and simulation for the eight-bar linkage and the ball mechanism are developed.

With the parameters assumed above and substituting them into Eq. (7), the trajectory of the point P_5 in the platform of the linkage is obtained and it shows that the trajectory of the central point P_5 of the platform is a straight line segment on the y_0 - z_0 plane whose gradient is $\sqrt{2}$.

Similarly, substitute the structural parameters into Eq. (12) to (19), the trajectory of the centers of the vertexes of the ball mechanism, i.e. points P_j ($j = 0, 1, \dots, 7$) can be obtained. It is interesting to note that the trajectories of the points are eight straight line segments between two spheres. These eight segments intersect at a common point, i.e. the centre point of the ball mechanism O_C . The radii of the two spheres are determined by the structure parameters of the ball mechanism. The radius of the outer sphere is $R_{out} = (3a + 4\sqrt{3}b/3)/2\sqrt{2}$, and the radius of the inner sphere is $R_{in} = \sqrt{6}b/3$. In addition, it should also be noted that the trajectory segments of point P_0 and P_7 are on the same line and so are segments of P_1 and P_6 , P_2 and P_4 , and P_3 and P_5 .

With the structure and kinematic parameters substituted into Eqs. (21), for a specified angular velocity of the actuated joint $\dot{\theta}$, the velocities of the center points P_j ($j = 0, 1, \dots, 7$) can be deduced. It is found that points P_1 , P_2 , P_3 and P_7 have the same resultant in velocity and so do points P_0 , P_4 , P_5 and P_6 . It also reveals that singularities rise when the angle of the actuated joint is $\theta = 0$ or $\theta = \gamma$. Moreover, it can also be found that points P_0 and P_7 , P_1 and P_6 , P_2 and P_4 , and, P_3 and P_5 actually have symmetrical velocity curves.

7. Conclusions

Stemmed from the transformable device — Hoberman switch-pitch structure, this paper

presented a 20-bar evolved deployable ball mechanism and its single-looped eight-bar radially foldable linkage. The paper presented a method of using virtual sliders for construction of the ball mechanism and investigated geometric constraint associated to the angular relations. The paper further presented the chain structure equation and revealed the radial motion characteristics. The forward kinematics analysis and subsequently the Jacobian matrix of the ball mechanism were presented and applicable to the Hoberman switch-pitch structure. The simulation was carried out to demonstrate the motion characteristics and the prototype was made to verify the theoretical study. The paper presented a way that a highly overconstrained polyhedral mechanism can be made and the radial motion can be achieved from the revolute motion in a mechanism. The ball mechanism herein might find its application in the fields of machines, consumer products, architectural applications and space technologies.

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