Instability of circular array from slender helical vortices

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For the first time a stability of a multitude (couple, triplet, quad, etc) of the helical vortices with and without additional rotation from central rectilinear vortex and axial translation has been investigated analytically. In practice an existence of a couple or triplet has been observed in different vortex flows (e.g. in tornado cores; after vortex breakdown and in vortex chambers; as a model of the vortex couplings in turbulent flows etc.). A quad (and more) is absolutely unstable structure. In contrast to this the systems with central vortex (wakes behind propellers and turbines) are stable. All previous studies based on the stability analysis of point vortex systems conflicted with these data but all of these observations are in accord with the presented stability analysis of the helical vortex structures.

1. Introduction
The problem of stability of an equilibrium circular configuration comprising \( N \) vortex structures was theoretically studied only for polygonal arrays of \( N \) point vortices or \( N \) straight vortex filaments. According to the results of investigations performed by Kelvin, Thomson, and Havelock \([1]\), such \( N \)-vortex systems are stable when the number of point vortices is \( N < 7 \). In contrast of this the \((N+1)\)-vortex system with an additional hub vortex is unstable when strength of the hub vortex is negative \([2]\). This result, obtained in the simplest particular case (the limiting case of a real helical vortex with infinite pitch \( l \)) is at variance with experimental and natural observations. Indeed an existence of a couple or triplet has been many times observed in different vortex flows (e.g. in tornado cores; after vortex breakdown and in vortex chambers; as a model of the vortex couplings in turbulent flows and the like). A quad (and more) is absolutely unstable structure. In contrast to this these systems with negative hub vortex (wakes behind propellers and turbines) are usually stable. The reason of the contradiction lies in a helical form of the real vortices in the systems so the ultimate goal of this work is to analyze theoretically equilibrium conditions for a multitude of the identical \((N+1)\) helical slender vortices which rotate without change of their shape (the \( N \)-system is a particular case of the \((N+1)\)-system when hub vortex is zero).

2. The evaluation of velocity field induced by helical vortex
According to solution for a helical vortex filament \([3]\) outside of it’s cores the components of velocity induced by the \( N \)-left-handed helical vortices upon additional translation with constant speed \( u_0 \) and rotating from the hub vortex with strength \( \gamma \) can be written

\[
\begin{align*}
\mathbf{u}_r &= \frac{\gamma R}{2\pi r} \sum_{n=1}^{\infty} \ln \left\{ \frac{H^{(1)}_{1,1} (r|l,R|l,\gamma)}{H^{(1)}_{1,1} (R|l,r|l,\gamma)} \right\}, \\
\mathbf{u}_0 &= \frac{\gamma}{2\pi r} + \sum_{n=1}^{\infty} \left( \frac{\Gamma_n}{2\pi r} \right) \ln \left\{ \frac{H^{(1)}_{1,0} (r|l,R|l,\gamma)}{H^{(1)}_{1,0} (R|l,r|l,\gamma)} \right\}, \\
\mathbf{u}_z &= \mathbf{u}_0 - \sum_{n=1}^{\infty} \left( \frac{\Gamma_n}{2\pi r} \right) \ln \left\{ \frac{H^{(1)}_{1,1} (r|l,R|l,\gamma)}{H^{(1)}_{1,0} (R|l,r|l,\gamma)} \right\}
\end{align*}
\]

(1)

Here the upper and lower lines in the brackets correspond to the cases \( r < R \) and \( r \geq R \) respectively; \( R \) is radius of helix and \( \gamma = \pm 2\pi n/N \). The velocity field (1) was written in terms of a Kaptéyn series of the type

\[
H^{(1)}_{m,l}(x,y,\chi) = \sum_{m} m^{l} I^{(0)}_{m} (|x|) K^{(0)}_{m} (|y|) e^{-\chi |x|},
\]

(2)

where \( x \leq y; x \) or \( y \) are equal \( r/l \) or \( R/l \); \( I^{(0)}_{m} (|x|), K^{(0)}_{m} (|y|) \) - the Bessel functions; \( I^{(0)}_{m} (|x|), K^{(0)}_{m} (|y|) \) are the corresponding derivatives. The series (2) in (1) converges slowly, especially near the singular point \( (r \to R \) and \( \chi \to \chi_0) \). Therefore it is necessary to develop a simple and exact procedure for these evaluation. Certainly known asymptotic solutions for estimation of self-induced motion of helical vortex \([4-6]\) could be employed for the purpose. However each of them cannot provide an accuracy requirement in the entire variation range of the dimensionless helical vortex pitch \( \tau = l/R \). Next approach for the series evaluation is a separation of singularity in (2). For two cases of the series \( (l, J = 0, 1 \) or \( 0, l) \) in \([6]\), the singularities in the spatial variables were separated from (2) and described the main torsion effect of the helix using an integral remainder \( W \) (see Eq. (4.1) in \([6]\)). However, the remainder \( W \) cannot be expressed in the finite form and, eventually, the remainder was numerically calculated for the vortex system consist from one to four helical vortices only for 21 values of the dimensionless pitch \( \tau \) [6–7]. In this investigation another method \([8-9]\) has been used. According to the method the main part \( S^{(1)} \) was separated in the series (2)

\[
S^{(1)} = \lambda^{+J} \left( \frac{e^{\lambda \tau} - e^{-\lambda \tau}}{e^{\lambda \tau} + e^{-\lambda \tau}} \right) + \frac{a^{(1)}_{+J} \lambda^{+J} \left( e^{\lambda \tau} - e^{-\lambda \tau} \right)}{\lambda^{+1J} \left( e^{\lambda \tau} + e^{-\lambda \tau} \right)} + \frac{a^{(1)}_{+0J} \lambda^{+J} \left( e^{\lambda \tau} - e^{-\lambda \tau} \right)}{\lambda^{+1J} \left( e^{\lambda \tau} + e^{-\lambda \tau} \right)}
\]

(4)

in curved space with distorted distance between two points

\[
D = -e^{-\lambda \tau} \exp \left( 1 + (1 + \chi^2)^{-1} \right)
\]

with simple form for the singularity factors at \( x = y = 1/\tau \)

\[
\lambda^{(0)} = \frac{1}{\sqrt{1 + \tau^2}}, \quad \lambda^{(1)} = -\lambda^{(0)}, \quad \lambda^{(1)} = \frac{1}{\sqrt{1 + \tau^2}} + \frac{\tau}{2}\sqrt{1 + \tau^2};
\]

\[
a_{+J}^{(0)} = a_{-J}^{(0)} = 0, \quad a_{-J}^{(0)} = a_{0J}^{(0)} = 1; \quad a_{+J}^{(0)} = a_{0J}^{(0)} = 0; \quad a_{+J}^{(0)} = a_{0J}^{(0)} = 0;
\]

Thus, the singularities and their coefficients explicitly contain all information about the helical torsion or vortex pitch \( \tau \) and since the difference \( R^{(1)}_{\chi} \) between \( H^{(1)}_{\chi} \) and \( S^{(1)}_{\chi} \) is very small (less
than one per cent). Therefore, the problem under consideration is more effectively solved by representing series (2) through its main part $S_{\omega}^{\nu}$ only and small remainder $R_{\omega}^{\nu}$ will be neglected.

### 3. Motion of $(N+1)$ helical vortex system

The unperturbed $(N+1)$ helical vortex system uniformly moves along and rotates around the cylinder axis with total binormal velocity $u_{0}$. This velocity consists of the self-induced velocity of a fixed vortex [4] and the velocity defined by (1) where infinite terms corresponded to the fixed vortex are excluded from the sums. Replacing the series (2) by their main part (4) in the resulting expression and after some algebra by analogy with problem $N$-helical vortices [9-10], the required binormal velocity $u_{0}$ of a motion of the $(N+1)$-helical vortex system is found to be

$$4\pi R u_{0} = \frac{N(1-\tau^{2})}{\tau\sqrt{1+\tau^{2}}} + 2(\alpha r + b)$$

$$\frac{1}{1+\tau^{2}} \left[ \log(\tau) - \log(N\epsilon(\sqrt{1+\tau^{2}})) + \frac{3}{4} \right]$$

$$\frac{1}{1+\tau^{2}} \left[ \left( \tau^{4} - \tau^{2} - \frac{3}{8} \right) - \epsilon^{2} - 3\epsilon - \frac{3}{4} \right]$$

where $a = \gamma / \Gamma$ and $b = u_{0} 2\pi R / \Gamma$ is dimensionless hub circulation and speed translation; $\epsilon(3) = 1.202…$ is the Riemann zeta function; and $\epsilon'$ is vortex core radius dimensionless to $R(1+\tau^{2})$, stems from Ricca [4]. The equation (5) with zero $a$ and $b$ corresponds to the binormal velocity $u_{0}$ of a motion of the $N$-helical vortex system [9].

### 4. Stability of $(N+1)$ helical vortex system

In order to study instabilities of the $(N+1)$ helical vortex system, one turns to the helical variables $(r, \theta)$ with corresponding velocity projections $(u_{c}, u_{\theta}) = u_{c} + u_{\theta} / \tau$ [8-9]. The problem of linear stability of the equilibrium configuration to infinitesimal disturbance of the helical vortices with fixed hub vortex in the centre of the above system reduces to a two-dimensional case [1]. Let the $k$-th helical vortex is displaced from the equilibrium position to a point

$$R + i 2\pi k / N + j u_{\theta} / R$$

where $u_{\theta} = -u_{c} / R / \tau$ [4, 9]. Then, in the linear approximation, the perturbed equations of motion of the $k$-th vortex are as follows:

$$\frac{dA}{dt} = \beta(t) A(m)$$

$$\frac{dA}{dt} = \alpha(t) B(m)$$

(8)

Thus $\alpha$ and $\beta$ are proportional to $\exp(\sqrt{AB})$ and system stability depends on the sign of the $AB$ product.

$$\frac{16\pi R^{2}}{\Gamma} \left[ m(N-m) \right] + \frac{2N-3}{\tau^{2}}$$

$$\left[ \frac{\tau(1-\tau^{2})}{\sqrt{1+\tau^{2}}} \right] \left[ \frac{N-m}{N} - C - \psi\left(\frac{m}{N}\right) \right]$$

(9)

where $C = 0.577215…$ is the Euler constant; $\psi(\cdot)$ is psi-function. Note that (9) is for $\tau \to 0$ and $a = 0$ and $b = 0$ coincides with the product $m(N-m)(m(N-m) - 2N - 1)$ and for $\tau \to \infty; a = -N$ and $b = 0$ coincides with the product $m(N-m)(m(N-m) + 2)$ which are result for the cases of the $N$ and $(N+1)$ point vortex system [1, 2]. The first term $A$ in product (9) is positive for all $\tau$. Thus the system is unstable with disturbances growing exponentially in $t$ if $B > 0$ for any $0 < m < N - 1$, and if $B < 0$, for all $m$, the system is linearly stable. Analyzing of $B$ for various values of the pitch $\tau$ have been determined the unstable modes which show a more realistic pattern than does the solution for point vortices in [1-2].

### 5. Conclusion

Thus, the problem of stability of $(N+1)$ equilibrium configuration of helical vortices has for the first time been analytically studied.

### References