1. Introduction

We are concerned with the evaluating the implied volatility in the standard Black-Scholes formula.

As is well-known, one of the main issues raised by the standard Black-Scholes formula for a European put

$$
P(\sigma) = C(\sigma) + Ke^{-rT} - S
$$

with the function $d_1(\sigma)$ and $d_2(\sigma)$ given by

$$
\begin{align*}
(i) \quad d_1 &= \frac{\log \left( \frac{S}{K} \right) + (r + 0.5\sigma^2)T}{\sigma \sqrt{T}}, \\
(ii) \quad d_2 &= d_1 - \sigma \sqrt{T} = \frac{\log \left( \frac{S}{K} \right) + (r - 0.5\sigma^2)T}{\sigma \sqrt{T}},
\end{align*}
$$

where

$S = \text{current market price of the underlying stock}$ (\text{$S > 0$ assumed});

$T = \text{remaining life of the option}$ (\text{$T > 0$ assumed});

$N(\cdot) = \text{cumulative standard normal density function}$;

$K = \text{strike price}$ (\text{$K > 0$ assumed});

$r = \text{continuous constant known rate of interest}$; and

$\sigma^2 = \text{instantaneous variance of stock returns}$.

is the question of modelling volatility $\sigma$.

Before we can use the Black-Scholes formula to price options, we have to estimate $\sigma$.

The implied volatility $\sigma^*$ is the value of the instantaneous volatility of the stock's return which, when employed in the Black-Scholes formula for a European put, results in a model price equal to the market price $w$, that is, $w = P(\sigma^*)$.

Unfortunately, the implied volatility cannot be calculated explicitly, and previous researchers [2,3,4,5,6] have proposed numerical methods such as the Newton-Raphson method and its variants. These methods produce only one implied volatility for each market price, that is, discrete methods.

On the other hand, we will propose a global method which produces the implied volatility continuously by tracing the option pricing curve of a particular stock for the observation date.

Our method will be able to provide a starting value for the iteration of the Newton-Raphson method, too.

The paper [6] have presented an existence condition of positive implied volatility.

But the papers [2,3,4,5] are concerned with neither the existence analysis nor the error analysis of numerical results. Hence we cannot assure the number of accurate digits of their numerical results.

In the previous paper [1] we have established a verification theorem of implied volatility via European call.

In order to study the properties of European put, in the present paper we will show that we can indeed verify the existence of an exact implied volatility via European put and then we can assure the number of accurate digits from the error bound of numerical results.

2. Global method and Verification Theorem

As is well-known, the implied volatility $\sigma^*$ is solved by the Newton-Raphson iterative procedure

$$
\begin{align*}
\sigma_{n+1} &= \sigma_n - \frac{P(\sigma_n) - w}{\nu(\sigma_n)},
\end{align*}
$$

where

$$
\begin{align*}
\nu(\sigma) &= \frac{\partial P(\sigma)}{\partial \sigma}, \\
\nu &= \nu^{-1}dP.
\end{align*}
$$

The accuracy with which the implied volatility can be inferred from the observed price $w$ depends on the vega $\nu$, or rather its reciprocal $1/\nu$, by the formula

$$
dP \approx \nu dw.
$$

When vega $\nu$ is small enough, i.e. $\nu^{-1}$ is large enough, small errors $dP$ in put-option prices are magnified into large errors $d\sigma$ in the implied volatility, and one loses accuracy. This facts tell us that the Newton-Raphson iterative procedure (3) does not converge (See [2]).

In order to alleviate the above defects we propose a global method called the geometric method [7] to solve the transcendental equation

$$
\begin{align*}
w &= P(\sigma),
\end{align*}
$$

or

$$
\begin{align*}
f(w, \sigma) &= P(\sigma) - w = 0,
\end{align*}
$$

by tracing the curve (5). Since

$$
\begin{align*}
\nu(\sigma) &= \frac{\partial f(w)}{\partial \sigma} = S\sqrt{T}n(d_1) > 0,
\end{align*}
$$

where

$$
\begin{align*}
\text{\textbf{REFERENCES:}}
\end{align*}
$$

This is a note on computation of the implied volatility in the Black-Scholes formula to evaluate an accuracy of the computation.
\[ u(d) = \frac{1}{\sqrt{2\pi}} e^{-\frac{d^2}{2}}, \]

the equation (6) determines an increasing curve

\[ l: \begin{cases} 
\sigma = \sigma(\xi), \\
w = w(\xi), 
\end{cases} \]

for which from (6) we have

\[ (8) \quad \nu(\sigma) \frac{d\sigma}{d\xi} - \frac{dw}{d\xi} = 0, \]

that is,

\[ \frac{d\sigma}{d\xi} = \frac{dw}{\nu(\sigma)} = \lambda. \]

Then we have

\[ (9) \begin{cases} 
\frac{d\sigma}{d\xi} = \lambda, \\
\frac{dw}{d\xi} = \lambda \nu(\sigma), 
\end{cases} \]

where \( \lambda \) is an arbitrary parameter. Let us choose a parameter \( \xi \) so that \( \xi \) may be an arc length of the curve \( l \). Then we readily see that \( \lambda \) is given by

\[ (10) \quad \lambda = \pm \frac{1}{\sqrt{1+\nu^2(\sigma)}}. \]

Hence we have from (9),(10) a system of ordinary differential equations

\[ (11) \begin{cases} 
\frac{d\sigma}{d\xi} = \pm \frac{1}{\sqrt{1+\nu^2(\sigma)}}, \\
\frac{dw}{d\xi} = \pm \frac{\nu(\sigma)}{\sqrt{1+\nu^2(\sigma)}}. 
\end{cases} \]

Now we take a point \((\sigma_0, w_0)\) on the curve \( l \) and suppose that

\[ (\sigma(0), w(0)) = (\sigma_0, w_0), \]

where \( w_0 \) is given as the observed price \( w \) in the market and \( \sigma_0 \) is given by solving (5) using the Newton-Raphson procedure (3).

Then we can trace the curve \( l \) integrating numerically the differential equation (11) with the initial condition \((\sigma_0, w_0)\) by a step-by-step method, say, the Runge-Kutta method or embedded Runge-Kutta-Fehlberg method.

Fig.1 shows the curve \( l \) passing through the point \((\sigma(0), w(0)) = (0.195587, 700)\) in Numerical Example.

The accuracy with the implied volatility obtained by the above numerical procedure can be given by the following

**Verification Theorem.** Given an observed market price \( w \) in equation (6), we have an equation of the volatility \( \sigma \) as follows:

\[ (12) \quad F(\sigma) \equiv F(\sigma) - w = 0. \]

Assume that equation (12) possesses an approximate solution \( \hat{\sigma} \) in a region \( D = [0, \infty) \), and there are a positive constant \( \delta \) and a non-negative constant \( \kappa < 1 \) such that

\[ (13) \begin{cases} 
(i) \quad D_\delta = \{ \sigma ; |\sigma - \hat{\sigma}| \leq \delta \} \subset D, \\
(ii) \quad |\nu(\sigma) - \nu(\hat{\sigma})| \leq \frac{\kappa}{M} \text{ whenever } \sigma \in D_\delta, \\
(iii) \quad \frac{M\rho}{1-\kappa} \leq \delta, 
\end{cases} \]

where \( \rho \) and \( M(>0) \) are numbers such that

\[ (14) \quad |F(\hat{\sigma})| \leq \rho, \quad \nu^{-1}(\hat{\sigma}) \leq M. \]

Then the equation (12) possesses one and only one solution \( \sigma = \sigma^* \) in \( D_\delta \) such that

\[ (15) \quad \nu(\sigma^*) \neq 0, \]

and

\[ (16) \quad |\sigma - \sigma^*| \leq \frac{M\rho}{1-\kappa}. \]

**参考文献**