微分型シュレディンガー方程式の爆発解とプラズマ中の Alfvén 波の崩壊

Formation of singularities in solutions of the derivative nonlinear Schrödinger equation and collapse of Alfvén waves

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We consider finite energy solutions of the derivative nonlinear Schrödinger equation on the half line with 0-Dirichlet boundary condition. This type of equation arises in plasma physics, and is considered to describe the long wavelength dynamics of dispersive Alfvén waves propagating along an ambient magnetic field which is parallel to x-axis. We show that negative-energy solutions blow up in finite time, and that every blowup solution concentrates its $L^2$ density at some point on the half line.

1. The Derivative NLS

We will consider the following initial boundary value problem for the derivative nonlinear Schrödinger equation (DNLS):

$$
\begin{align*}
&\text{(DNLS) } 2i\partial_t\psi + \partial_x^2\psi + 2i\varepsilon_0\partial_x (|\psi|^2\psi) = 0, \\
&\text{(IV) } \psi(x,0) = \psi_0(x), \quad x \in I_+, \\
&\text{(IV) } \psi(0,t) = 0, \quad t \in [0,\infty),
\end{align*}
$$

where $\varepsilon_0$ is a real constant. In what follows, we call this initial boundary value problem as (IBP):

$$
\text{(IBP) } = \text{(DNLS)} + \text{(IV)} + \text{(IV)}.
$$

The derivative nonlinear Schrödinger equation on the whole line arises in plasma physics. It is considered to describe the long wavelength dynamics of dispersive Alfvén waves propagating along an ambient magnetic field which is parallel to x-axis (see, e. g., Mjölhus [4] and Mio-Ogino-Minami-Takeda [3]). The sign of the constant $\varepsilon_0$ indicates the direction of the wave propagation. The initial value problem for (DNLS) on the whole line $\mathbb{R}$ has been attracted both mathematicians and physicists in the last two decades (see, e. g., Kaup-Newell [2], Hayashi [1], Ozawa [11], M. Tsutsumi-Fukuda [12, 13] and references therein). However, the global existence or blowup for large solutions is still an open problem. In this note, we consider the blowup problem for (DNLS) in a somewhat restricted situation: the initial boundary value problem of (IBP). The boundary condition (IV) means that we have an infinite wall at $\{x = 0\}$.

2. Existence and Nonexistence

We start with the local existence of solutions for (IBP).

**Theorem 1** Suppose that $\psi_0 \in H_0^1(1\pm)$. Then there exists a unique solution of (IBP) in the space of

$$
C([0,T_m);L^2(1\pm)) \cap L^4((0,T_m);L^\infty(1\pm))
$$

for some $T_m \in (0,\infty)$ (maximal existence time) such that $\psi(t)$ satisfies the following three conservation laws of $L^2$, the momentum, and the energy $\mathcal{H}$, in this order:

$$
\|\psi(t)\| = \|\psi_0\|, \\
3 \int_{1\pm} \partial_x \psi(x,t) \overline{\psi(x,t)} dx + \varepsilon_0 \int_{1\pm} |\psi(x,t)|^4 dx = 3 \int_{1\pm} \partial_x \psi_0(x) \overline{\psi_0(x)} dx + \varepsilon_0 \int_{1\pm} |\psi_0(x)|^4 dx, \\
\mathcal{H}(\psi(t)) = \|\partial_x \psi(t)\|^2 - 3\varepsilon_0 \int_{1\pm} |\psi(x,t)|^2 \partial_x \psi(x,t) \overline{\psi(x,t)} dx + 2\varepsilon_0^2 \|\psi(t)\|^6 = \mathcal{H}(\psi_0),
$$

for $t \in [0,T_m)$, where $\|\cdot\|$ and $\|\cdot\|^6$ denotes the $L^2$ norm and $L^6$ norm with respect to the space variable $x$ respectively. Furthermore we have the following alternatives:

$$
T_m = \infty \quad (\text{global-in-time existence})
$$

or

$$
T_m < \infty \quad \text{and} \quad \limsup_{t \to T_m} \|\partial_x \psi(t)\| = \infty \quad (\text{blowup}).
$$
Next, we consider the existence of blowup solution of (IBP). As in an analogous way to the case of the nonlinear Schrödinger equation (see [9, 10]), we can prove the following theorem:

**Theorem 2** If \( \varepsilon_0 > 0 \), we consider (IBP) on \( I_+ \); and if \( \varepsilon_0 < 0 \), on \( I_- \). Then, every energy solution blows up in a finite time. Furthermore the family of Radon measures \( \{ |\psi(x,t)|^2 dx \} \) is tight on \( I_+ \cup \{ 0 \} \), i.e., for any \( \varepsilon > 0 \) there is a radius \( R > 0 \) for which we have
\[
\sup_{t \in [0,T_m]} \int_{I_+ \cap |x| > R} |\psi(x,t)|^2 dx < \varepsilon,
\]
and, we have: for sufficiently large \( R > 0 \),
\[
\int_0^{T_m} (T_m - t) \left( \int_{I_+ \cap |x| > R} |\partial_x \psi(x,t)|^2 dx \right) dt < \infty,
\]
\[
\int_0^{T_m} (T_m - t) \left( \int_{I_+ \cap |x| > R} |\psi(x,t)|^2 dx \right) dt < \infty.
\]

**Remark 1** We have an analogous result for the nonlinear Schrödinger equation of the 1-dimensional critical case. Actually, from the view point of the existence of blowup solution, our equation (DNLS) is considered to be corresponding to the following nonlinear Schrödinger equation:
\[
2i\partial_t \psi + \partial_x^2 \psi + \frac{3}{4} |\psi|^4 \psi = 0.
\]

2. Wave Collapse

By the tightness of \( \{ |\psi(x,t)|^2 dx \} \), one can see that there exists a positive measure \( \nu \in C_b(I_+) \) such that, in the sense of measures on \( I_+ \cup \{ 0 \} \),
\[
|\psi(x, t_n) |^2 dx \rightharpoonup \nu(dx)
\]
for some sequence \( t_n \uparrow T_m (n \to \infty) \). We can say that this measure \( \nu \) has purely discontinuous components, because we have:

**Theorem 3** Every blowup solution concentrates its \( L^2 \)-mass at some point. Precisely we have:
\[
A \equiv \sup_{R > 0} \left( \liminf_{t < T_m} \left( \sup_{y \in \mathbb{R}} \int_{|x-y| < \frac{R}{|\psi(y)|^2}} |\psi(x,t)|^2 dx \right) \right) \geq \mathcal{N}_c,
\]
where
\[
\mathcal{N}_c \equiv \inf_{\psi \in H^1(\mathbb{R}), \psi \neq 0} \left\{ \|\psi\|^2 \left| \left| \partial_x \psi \right| \right|^2 - \frac{1}{4} \|\psi\|^4 \leq 0 \right\}.
\]
Here we extend \( \psi \) on the whole line \( \mathbb{R} \) by 0.

**Corollary 1** If \( \varepsilon_0 > 0 \), we consider (IBP) on \( I_+ \); and if \( \varepsilon_0 < 0 \), on \( I_- \). Let \( \psi \) be a blowup solution of (IBP) with \( \mathcal{H}(\psi) < 0 \). Then, for any \( \varepsilon > 0 \), there exists a radius \( R > 0 \) and a bounded function \( y : [0,T_m] \to I_+ \) such that we have:
\[
\int_{|x-y(t)| < \frac{R}{|\psi(t)|^2}} |\psi(x,t)|^2 dx \geq (1 - \varepsilon) A,
\]
where \( A \) is defined in Theorem 3.

**Remark 2** (1) Theorem 3 holds for the blowup solutions of the initial value problem for (DNLS) on the whole line \( \mathbb{R} \).
(2) For the nonlinear Schrödinger equation, these results are already known (see [5, 8]). As well as the nonlinear Schrödinger equation case, we can show that \( y(t) \) is continuous, provided that
\[
A \geq \frac{1}{2} \|\nu\|^2
\]
or
\[
\mathcal{N}_c \leq \|\psi\|^2 \leq 2 \mathcal{N}_c.
\]
For this, see [6, 7].
(3) The variational value \( \mathcal{N}_c \) in (1.13) is attained by the nontrivial solution of
\[
\frac{d^2 \psi}{dx^2} - Q + \frac{3}{4} |\psi|^4 \psi = 0,
\]
\( Q \in H^1(\mathbb{R}) \).
For the variational problem (1.13), see Weinstein [14], and see also [8].

We do not know whether there is a case of having \( y(t) \leftarrow 0 \) or not. It seems that the blowup rate is relevant to the behavior of \( y(t) \). If it hits the boundary \( 0 \) at the blowup time, the blowup rate is keener than the other case.

References