Numerical Matching Scheme for MHD evolution equation
and Eigenvalue problem

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A new matching scheme for linear magnetohydrodynamic (MHD) stability analysis is proposed in a form that the numerical implementation is tractable. The scheme divides the plasma region into outer regions and inner layers, as in the conventional matching method. However the outer regions do not contain any rational surface as their terminal points; an inner layer contains a rational surface as an interior point. The Newcomb equation is therefore regular in the outer regions. The MHD equation in the layers is solved as an evolution equation in time, or as an eigenvalue problem. The matching conditions are derived from the conditions that the radial component of the solution in the layer be smoothly connected to those in the outer regions at the terminal points.

1. Introduction
It is well known that MHD equations, ideal or resistive, endow a high temperature, magnetically confined plasma such as tokamak with multi-scale structure. Especially for the linearized MHD equations, the structure is completely specified by the coefficients in them, which are functions of MHD equilibrium quantities. Among them the so-called safety factor $q$ and rational surfaces on each of which $q$ takes a rational number play an essential role in settling the multi-scale structure of a MHD mode close to the marginal stability, in which we are concerned. The multi-scale structure consist of outer regions and inner layers. An outer region is far from rational surfaces, where the inertial effects as well as other resistive effects can be neglected and then the mode is described by the Newcomb equation, inertia free linear ideal MHD equation [1]. On the other hand, an inner layer is a thin layer around a rational surface, where all effects should be retained. The width of a layer is determined by the plasma inertia (growth rate of an unstable ideal MHD mode) or the plasma resistivity; it asymptotically approaches zero as the growth rate or the resistivity goes to zero.

Such structure of the MHD mode naturally leads to the method of asymptotic matching for the MHD stability analysis [2]. However any computer code based on the asymptotic matching method has not been successfully developed that is practical and routinely used in the MHD stability analysis of tokamak. The Newcomb equation is singular at each rational surface where two independent solutions exist; one (dominant solution) is not square integrable and is strongly singular; the other (subdominant solution) is square integrable. It is difficult or impractical to treat numerically the dominant solution, which will be the main reason for the lack of such computer codes.

In the present paper we propose a new matching method that is practical from the numerical point of view. We illustrate the underlying idea by using the simplest MHD equations, the linear ideal MHD equations in cylindrical configuration $r \in (0, a)$; $r$ denotes the radial coordinate, $a$ the plasma radius.

2. New numerical matching scheme for Linear MHD Evolution Equation
We employ the linear ideal MHD equation
\[ \rho \frac{\partial^2 \xi}{\partial t^2} + F[\xi](r, t) = 0, \]
for the cylindrical coordinate system $(r, \theta, z)$ with $0 \leq r \leq a$; the infinitesimal displacement of plasma is assumed to be $\xi(r, t) \exp(\text{i}m \theta - \text{i}kz)$, $\rho$ the density of plasma, $F$ the force operator, which is well known [3]. We assume the fixed boundary condition $y(a, t) = 0$, where $y(r, t)$ is the radial component of $\xi(r, t)$. We solve Eq. (1) around the inner layer $(r_0, r_f)$, where the inertial term cannot be neglected, by using the full implicit scheme
\[ \rho \xi^{n+1} - (\Delta t)^2 F[\xi^{n+1}] = S_n[\xi], \]
where
\[ S_n[\xi] = \rho (2\xi^n - \xi^{n-1}), \]
and
\[ \xi^n(r) = \xi(r, n\Delta t) \]
and $\Delta t$ is the time step size. Equation (2) reduces to an inhomogeneous linear differential equation of second order in $y^{n+1}$, the radial component of $\xi^{n+1}$, that should be solved with appropriate boundary conditions. The two components of $\xi^{n+1}$ can be expressed by linear relations in $y^{n+1}$ and $dy^{n+1}/dt$; this is the characteristic feature of the ideal MHD equation, and is fully exploited in the present formulation.

In the outer regions $(0, r_0)$, $(r_f, a)$ far from the rational surface, where the inertia terms are regarded as small, we solve $F[\xi] = 0$, which reduces to the Newcomb equation. We can construct the solutions of Eq. (2) and the Newcomb equation in such a way that the radial components of the solutions take at $r = r_p (p = L, R)$ the same values $\xi_p^{n+1}$, which are yet unknown. We moreover impose on the solutions as the matching conditions that their
derivatives at the points are equal to each other. Those give us a linear equation on $e_p^{-1}$. They are easily solved; hence the matching problem can be solved numerically.

The finite element method uses the linear elements for Eq. (2) and the Newcomb equation. The following examples are computed for a uniform mesh with the fine mesh size of $4 \times 10^{-6}$ in order to capture the structure of a weakly unstable MHD mode, whereas the time step is large; $\Delta t = 0.1 \omega_{pa}(\omega_{pa}$ denotes the Alfven frequency at the plasma surface).

Figure 1 illustrates the present scheme applied to the $m = 1$ internal kink mode for $2\pi R_0 = 60(k = 2\pi / R_0)$. The safety factor $q$ is shown by dotted-broken line; the $q = 1$ surface locates at $r = 0.4$; the matching points are $r_L = 0.35$, $r_R = 0.45$ ($\Delta r =: r_R - r_L = 0.1$). The symbols *"o"*s denote the internal kink mode obtained by solving globally the linear ideal MHD equation Eq.(1) in the full range (0, 1). The inner layer solution is shown by the solid line, and the outer solutions are shown by the dotted lines. The solutions constructed by the matching procedure are indistinguishable from the global solution. The enlargements around the matching points show that the inner layer solution connects smoothly to each outer solution.

Figure 2 shows the time evolution of $\xi_L(t)$ shown by the broken line, and $\xi_R(t)$ by the dotted line, whereas the time evolution of the norm of the global solution is shown by the solid line. $\xi_L(t)$ and $\xi_R(t)$ grow with the growth rate $\gamma = 2.7 \times 10^{-7} \omega_{pa}$, which is close to the growth rate estimated from the norm of the global solution, $\gamma = 2.6 \times 10^{-7} \omega_{pa}$ (Alfven frequency at the plasma surface).

3. Conclusions

We have proposed to the MHD stability problems a new matching scheme that is tractable from the numerical point of view. The new scheme gives the matching conditions in the form of the linear equation, which can be easily solved, on the values of the radial displacement at the matching points. The new scheme is also used for the eigenvalue problems, which will be also reported at the congress.

When we adopt the present scheme, we can flexibly change the physical model, from the linear ideal MHD equation to more complex MHD equations, around an arbitrarily chosen rational surface. One of such MHD equations is the Frieman-Rosenberg equation that describes ideal MHD motion in a rotating plasma [4, 5], and that recently gets attention in the theory of resistive wall modes [6]. The present scheme therefore can be regarded as an effective approach to multi-scale, multi-physics problems in MHD.

References