On ODEs describing the intensity dynamics

Naoyuki ISHIMURA and Fumihiko KANDA
Graduate School of Economics, Hitotsubashi University
Kunitachi, Tokyo 186-8601, Japan.
E-mail: ishimura@econ.hit-u.ac.jp, kandajones@hotmail.co.jp

1 Introduction

We propose a family of ordinary differential equations (ODEs), which are derived from a model on the interaction among defaultable securities. Examples of this family include the next two systems. One is

\[ \begin{align*}
    u'(t) &= e^{-(a_1 + a_2 + \gamma)t}(1 + \beta v(t))(1 + \gamma w(t)) \\
    v'(t) &= e^{-(b_1 + a_2 + \gamma)t}(1 + \alpha u(t))(1 + \gamma w(t)) \\
    w'(t) &= e^{-(c_1 + a_2 + \gamma)t}(1 + \alpha u(t))(1 + \beta v(t)),
\end{align*} \]

(1)

where \( a_1, b_1, c_1, \alpha, \beta, \gamma \) are given constants, which are principally assumed to be positive, but there is an interpretation even if they are negative.

The other is

\[ \begin{align*}
    u''(t) + (a_1 + a_2 - c_2)u'(t) &= a_2e^{-(a_1 + c_1 + c_2)t}(1 + c_2 u(t))(1 + c_3 v(t)) \\
    v''(t) + (b_1 + b_2 - c_3)v'(t) &= b_1e^{-(b_1 + c_1 + c_2)t}(1 + c_2 u(t))(1 + c_3 w(t)),
\end{align*} \]

(2)

where \( a_1, a_2, b_1, b_2, c_1, c_2, c_3 \) are given constants.

The aim of this talk is to derive these systems of ordinary differential equations and show their various properties, which may clarify the character of the model (3). For our previous study we refer to [2].

2 Model

The underlying model of these equations is due to Jarrow and Yu [1]; they consider the case where each firm holds the other firm’s debt and the default of one firm makes jump the other firm’s default probability.

We extend this Jarrow-Yu model to three defaultable companies \( A, B, C \) so that the default intensities can be described by

\[ \begin{align*}
    \lambda^A &= a_1 + a_2 \mathbb{1}_{(t \geq \tau^A)} + a_3 \mathbb{1}_{(t \geq \tau^C)} \\
    \lambda^B &= b_1 + b_2 \mathbb{1}_{(t \geq \tau^A)} + b_3 \mathbb{1}_{(t \geq \tau^C)} \\
    \lambda^C &= c_1 + c_2 \mathbb{1}_{(t \geq \tau^A)} + c_3 \mathbb{1}_{(t \geq \tau^C)},
\end{align*} \]

(3)

where \( a_i, b_i, c_i (i = 1, 2, 3) \) are (positive) constants. Here \( \lambda_{A,B,C}^t \) denote the intensity of a doubly stochastic Poisson process \( N_{A,B,C}^t \), and \( \tau_{A,B,C}^t \) stand for the first jump time of \( N_{A,B,C}^t \), respectively. The indicator function \( \mathbb{1} \) is defined as

\[ \mathbb{1}_A = \begin{cases} 
1 & (x \in A) \\
0 & (x \notin A). 
\end{cases} \]

We introduce

\[ \begin{align*}
    F(t) := P(\tau^A \leq t), & \quad G(t) := P(\tau^B \leq t), \\
    H(t) := P(\tau^C \leq t),
\end{align*} \]

and compute

\[ \begin{align*}
    1 - F(t) &= P(\tau^A > t) = E \exp \left( - \int_0^t \lambda^A_s ds \right) \\
    &= e^{-(a_1 + a_2 + a_3)t} \left( 1 + a_2 \int_0^t e^{a_2s} (1 - G(s)) ds \right) \cdot \left( 1 + a_3 \int_0^t e^{a_3s} (1 - H(s)) ds \right).
\end{align*} \]
$1 - G(t) = e^{-(b_1 - b_2 + b_3)t} \left(1 + b_2 \int_0^t e^{b_3 s}(1 - F(s))ds\right) \cdot \left(1 + b_3 \int_0^t e^{b_2 s}(1 - H(s))ds\right)$.

$1 - H(t) = e^{-(c_1 + c_2 + c_3)t} \left(1 + c_2 \int_0^t e^{c_3 s}(1 - F(s))ds\right) \cdot \left(1 + c_3 \int_0^t e^{c_2 s}(1 - G(s))ds\right)$.

The equation (1) corresponds to the case $\alpha := b_2 = c_2$, $\beta := a_2 = c_3$, $\gamma := a_3 = b_3$; we put

$u(t) := \int_0^t e^{\alpha s}(1 - F(s))ds$

$v(t) := \int_0^t e^{\beta s}(1 - G(s))ds$

$w(t) := \int_0^t e^{\gamma s}(1 - H(s))ds$.

A calculation yields (1).

On the other hand, the equation (2) corresponds to the case $a_2 = b_2 = 0$; we put

$u(t) := \int_0^t e^{c_2 s}(1 - F(s))ds$

$v(t) := \int_0^t e^{c_3 s}(1 - G(s))ds$.

In this cases, after a little tedious computation, we obtain the equation (2).

3 Properties of the solutions

Here we just show one simple example where $\alpha = \beta = \gamma = a_1 = b_1 = c_1$ in (1).

Putting $U(t) := 1 + \alpha u(t)$, $V(t) := 1 + \alpha v(t)$, and $W(t) := 1 + \alpha w(t)$, we deduce that

$U'(t) = \alpha e^{-2\alpha t} V(t) W(t)$

$V'(t) = \alpha e^{-2\alpha t} U(t) W(t)$

$W'(t) = \alpha e^{-2\alpha t} U(t) V(t)$.

Invoking $U(0) = V(0) = W(0) = 1$, we infer that $U(t) \equiv V(t) \equiv W(t)$ and therefore

$U'(t) = \alpha e^{-2\alpha t} U(t)^2$

$- \frac{1}{U(t)} + 1 = \frac{1}{2}(1 - e^{-2\alpha t})$.

from which we see that $\lim_{t \to \infty} u(t) = \lim_{t \to \infty} v(t) = \lim_{t \to \infty} w(t) = 1/\alpha$.

Other properties will be given in our talk.

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