Consequences of the Volumetric-Distortional Decomposition of Deformation in Elasticity

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1. Introduction

The deformation gradient admits a multiplicative decomposition into a purely volumetric and a purely distortional component \(1^{15})\). In hyperelasticity, based on this decomposition, the elastic strain energy potential, the stress, and the elasticity tensor can be expressed in general as a function of both the volumetric deformation and the distortional deformation.

However, the volumetric-distortional decomposition of deformation has often been employed in a fully decoupled elastic strain energy potential, expressed as the sum of a term depending solely on the volumetric deformation and a term depending solely on the distortional deformation \(6^{11}4^{5}\).

This work \(3^{7}\) has three objectives. First, to derive the elasticity tensor in the general (non-decoupled) case, in its material, spatial, and linear forms; this is achieved by extensive use of fourth-order tensor algebra, and in particular of the deviatoric operator are idempotent and orthogonal. The tensors of the Piola transformation \(1\), indeed:

\[
I' = \chi'[I] = \frac{1}{4}(C^{-1} \otimes C^{-1} + C^{-1} \otimes C^{-1}),
\]

\[
\chi'[I] = \frac{1}{4}(C^{-1} \otimes C^{-1} + C^{-1} \otimes C^{-1}),
\]

\[
M' = \chi'[M] = \frac{1}{4}(C^{-1} \otimes C^{-1} + C^{-1} \otimes C^{-1} - C^{-1} \otimes C^{-1}).
\]

Both these tensors are such that ** ..., but *...

4. General Form of the Elasticity Tensor

For an elastic potential \(W(E) = \Psi(J, E)\), the second Piola-Kirchhoff stress (first derivative of \(W\) with respect to \(E\)) and the material elasticity tensor (second derivative) are given by

\[
S = -\frac{\partial W}{\partial E} J^{-1} + J^{-1} M : S = S_{\omega} + S,
\]

\[
C = (-p + K) J K' + 2\mu E' + J^{-1} \left[ (C^{-1} \otimes (M : Y)) + (M : Y) \otimes C^{-1} \right] +
\]

\[
J^{-1} M : \tilde{C} : M' + J^{-1} \left[ \partial \Psi / \partial J \right] (hydrostatic pressure),
\]

\[
\Psi = \partial \Psi / \partial E \] (large-strain bulk modulus),

\[
\Psi = \partial \Psi / \partial E \] (interaction tensor), and \(\tilde{C} = \partial \Psi / \partial E^2\).

The spatial identity, spherical, and deviatoric operators are defined by (tensor \(i\) is the spatial second-order identity) \(10^{11})\):

\[
i = \frac{1}{4}(i \otimes i + 3I), \quad \text{with } L_{ii} = \frac{1}{4}(\delta_{ii} \delta_{ij} + \delta_{ij} \delta_{ji}),
\]

\[
K = \frac{1}{4} i \otimes i, \quad \text{with } K_{ii} = \frac{1}{4} \delta_{ii} \delta_{ij},
\]

\[
M = I - K, \quad \text{with } M_{ii} = \frac{1}{2}(\delta_{ii} \delta_{ij} + \delta_{ij} \delta_{ji}) - \frac{1}{4} \delta_{ii} \delta_{jj}.
\]

Their material counterparts are obtained by pulling back the expressions \(i : \alpha, K : \alpha, \) and \(M : \alpha, \) to \(I : A, K : A, \) and \(M : A, \) where \(\alpha\) is a spatial tensor, and \(A\) is its pull-back:

\[
I' = \frac{1}{4}(i \otimes i + 3I), \quad \text{with } L_{ii} = \frac{1}{4}(\delta_{ii} \delta_{ij} + \delta_{ij} \delta_{ji}),
\]

\[
K = \frac{1}{4} i \otimes i, \quad \text{with } K_{ii} = \frac{1}{4} \delta_{ii} \delta_{ij},
\]

\[
M = I - K, \quad \text{with } M_{ii} = \frac{1}{2}(\delta_{ii} \delta_{ij} + \delta_{ij} \delta_{ji}) - \frac{1}{4} \delta_{ii} \delta_{jj}.
\]

Both the spatial and the material versions of the spherical and deviatoric operator are idempotent and orthogonal. The tensors obtained by pulling back \(i, K, \) and \(M\) do not correspond to \(\tilde{I}, \tilde{K}, \) and \(\tilde{M}\), indeed:

\[
I' = \chi'[I] = \frac{1}{4}(C^{-1} \otimes C^{-1} + C^{-1} \otimes C^{-1}),
\]

\[
\chi'[I] = \frac{1}{4}(C^{-1} \otimes C^{-1} + C^{-1} \otimes C^{-1}),
\]

\[
M' = \chi'[M] = \frac{1}{4}(C^{-1} \otimes C^{-1} + C^{-1} \otimes C^{-1} - C^{-1} \otimes C^{-1}).
\]

These tensors are such that ** ..., but *...

The deformation identity, spherical, and deviatoric components of second-order tensors is closely related to the deformation.

3. Decomposition and Fourth-Order Tensor Algebra

The algebra of the fourth-order tensors involved in the decomposition into spherical and deviatoric components of second-order tensors is closely related to the deformation.
Equations (11) and (12) are the general expressions of the material and spatial elasticity tensors derived from a potential \( W(E) \), seen as a function \( W(E) = \Psi(J, \mathbf{E}) \), and include those derived from the fully decoupled potential \( W(E) = U(J) + W(E) \), as the particular case in which the material and spatial interaction tensors, \( Y \) and \( y \), correspond to the mixed second derivatives of \( \Psi \) with respect to \( J \) and \( \mathbf{E} \), respectively, at zero strain.

The linear elasticity tensor is obtained from the spatial tensor, at zero deformation \( (J = I, \mathbf{E} = \mathbf{O}) \), and assuming a stress-free undeformed configuration \( (p = 0, \mathbf{\sigma} = \mathbf{0}, \mathbf{\sigma} = \mathbf{0}) \):

\[
\mathbf{L} = \kappa \mathbf{K} + \mathbf{I} \otimes (\mathbf{M} : \mathbf{a}) + (\mathbf{M} : \mathbf{a}) \otimes \mathbf{I} + \mathbf{\hat{L}} : \mathbf{M},
\]

where \( \kappa \) (linear elasticity bulk modulus), \( \alpha \), \( \hat{L} \), are the values of \( K, y, \mathbf{c} \), respectively, at zero strain.

Let us now assume that the linear elasticity tensor \( \mathbf{L} \) can be obtained independently of the above reasoning, from experimental tests. By left- and right-contraction of \( \mathbf{L} \) with \( I = K + M \), it can be shown that \( \mathbf{L} \) admits the purely algebraic decomposition

\[
\mathbf{L} = \mathbf{K} : \mathbf{L} + \mathbf{K} : \mathbf{L} = \mathbf{K} : \mathbf{L} + \mathbf{K} + \mathbf{M} : \mathbf{L} : \mathbf{M}.
\]

In order to be consistent with the experimentally determined linear elasticity tensor (14), the linear elasticity tensor (13), obtained from the linearisation of the spatial elasticity tensor, must coincide with (14). It is easy to show that this implies that the four terms in (14) correspond one-by-one to the four terms in (13), e.g., \( \kappa \mathbf{K} = \mathbf{K} : \mathbf{L} \), etc.

5. Fully Decoupled Elastic Potential

For a fully decoupled elastic potential \( W(E) = \Psi(J, \mathbf{E}) = U(J) + W(E) \), the mixed derivatives \( \partial^2 \Psi / \partial J \partial E = \partial^2 W / \partial J \partial E \), corresponding to the material interaction tensor \( Y \) in Eq. (11), to the spatial interaction tensor \( y \) in Eq. (12), and to the linear elasticity interaction tensor \( \alpha \) in Eq. (13), vanish identically. This leads to a linear elasticity tensor in the form

\[
\mathbf{L} = \kappa \mathbf{K} + \mathbf{M} : \mathbf{L} = \mathbf{K} + \mathbf{M} : \mathbf{L}.
\]

This linear elasticity tensor, obtained from the linearisation of the spatial elasticity tensor corresponding to the non-linear potential (15), is compatible with the linear elasticity tensor (14), in principle obtainable from experimental tests, if, and only if, \( \mathbf{K} : \mathbf{L} = \mathbf{M} : \mathbf{L} = \mathbf{0} \).

Therefore, the fully decoupled potential (15) is not compatible with its linearised counterpart, unless conditions (17) are satisfied.

The compatibility conditions (17) provide the necessary and sufficient conditions for a linear elastic material, described by the quadratic potential \( W^{(\alpha)}(\epsilon) = \frac{1}{2} \mathbf{e} : \mathbf{L} : \mathbf{e} \), to admit a purely volumetric strain under a hydrostatic stress. This problem has been approached by Ting\(^9\) based on the Voigt-Piola formalism, in which the elasticity tensor is represented by a 6 x 6 matrix. The solution proposed here is coordinate-free and based on the algebraic properties of idempotence and orthogonality of the spherical operator \( \mathbf{K} \) and the deviatoric operator \( \mathbf{M} \).

The proof of the necessary condition follows from the stress strain relationship \( \sigma = \mathbf{L} : \mathbf{e} \), Equation (16a), and the fact that the stress is hydrostatic, i.e., \( \mathbf{M} : \sigma = \mathbf{0} \).

The proof of the sufficient condition follows from the fact that the strain is spherical, i.e., \( \mathbf{e} = \hat{K} : \mathbf{e} \), and so is the stress, i.e., \( \mathbf{M} : \sigma = \mathbf{M} = \mathbf{0} \), and by applying \( \mathbf{M} \) on the left of \( \sigma = \mathbf{L} : \mathbf{e} \) and on the right of \( \sigma = \mathbf{e} : \mathbf{L} \), with use of the identity (14).

6. Discussion

Based on the volumetric-distortional decomposition of the deformation, hyperelastic materials\(^{10-11}\), including biological soft tissues\(^4\) have often been described by assuming a simplified, fully decoupled elastic potential, in the form (15).

The material, spatial, and linearised elasticity tensors for a general (non-decoupled) elastic potential, expressed as \( W(E) = \Psi(J, \mathbf{E}) \), were found by explicitly using the spherical and deviatoric operators. In particular, the explicit definition of the spatial operators \( \mathbf{K} \), \( \kappa \), and \( \mathbf{\hat{K}} \), clarifies the physical meaning of the term in \( \mathbf{K} \) in the material elasticity tensor (11) and the term in \( \mathbf{\hat{K}} \) in the spatial elasticity tensor (12). Indeed, these terms refer to purely volumetric strains and purely hydrostatic stresses.

Furthermore, it has been shown that a fully decoupled potential is in general not compatible with its linearisation. The compatibility of a non-linear elastic potential and its linearised counterpart is deemed to be particularly important\(^9\); indeed, at least \( N \) of the material constants featuring in the elastic potential must be a function of the \( N \) independent linear elastic moduli (\( N \) depends on the material symmetry, and is 2 for isotropy, 5 for transverse isotropy, 9 for orthotropy, etc). For a decoupled potential in the form (15), some of the elastic constants are identically zero, which leads to incompatibility with the linearised potential \( W^{(\alpha)}(\epsilon) = \frac{1}{2} \mathbf{e} : \mathbf{L} : \mathbf{e} \), unless its elasticity tensor \( \mathbf{L} \) satisfies the decoupling conditions (17).

Finally, the decoupling conditions (17) were found to be necessary and sufficient for a hydrostatic stress to cause a purely volumetric strain in a linearly elastic material. As for the case of the solution proposed by Ting\(^9\), the proposed approach is completely independent of the material symmetry.

References