As the PDE model of MRE problem, we take the dynamical isotropic viscoelasticity model. By the exponential decay property in time of the dynamical wave displacement field, the model can be reduced to a mixed boundary value problem for the stationary isotropic viscoelasticity equation. The incompressibility assumption of the medium used by the other MRE research teams can be recovered in our model to show the applicability of the corresponding reduced models. That is, by an asymptotic analysis, we can reduce the stationary isotropic viscoelasticity model to the modified modified Stokes model under some reasonable assumption for the soft tissues of a living body. Due to the MRE experiment, the boundary condition for the PDE model must be of mixed type. We prove the well-posedness of the boundary value problem for the PDE model and numerically simulate the solution to this boundary value problem.

1 Mathematical Models

Let $\Omega \subset \mathbb{R}^3$ which is a bounded domain with Lipschitz continuous boundary $\partial \Omega$ be the reference domain we considered (i.e. a part of the living body). Based on the setup of MRE experiment done by M. Suga in Chiba University, Japan, we have to assume that the input has to be time harmonic and it is given on a part $\Gamma_D$ of surface $\partial \Omega$ of the living body while the rest of the part $\Gamma_N$ is traction free. Here $\Gamma_D, \Gamma_N \subset \partial \Omega$ are open sets such that $\partial \Omega = \overline{\Gamma_D} \cup \partial \Omega_N$, $\Gamma_D \neq \emptyset$, $\Gamma_D \cap \Gamma_N = \emptyset$ and $\partial \Gamma_D$, $\partial \Gamma_N$ are of piecewise $C^{1,1}$ class. For simplicity, we assume that our living body of soft tissues is isotropic viscoelastic, then the wave displacement field $U(t, x)$ in the medium can be described by the following mixed problem:

$$\rho \partial_t^2 U = \left\{ \begin{array}{l} \nabla [\lambda \nabla \cdot U] + \nabla \cdot [2 \mu \varepsilon(U)] \\ \nabla [\zeta \nabla \cdot \partial_t U] + \nabla \cdot [2 \eta \varepsilon(\partial_t U)] \end{array} \right\} \quad \text{in} \ (0, +\infty) \times \Omega,$$

$$U(t, x) = \chi(t)e^{i\omega t} f(x) \quad \text{on} \ (0, +\infty) \times \Gamma_D,$$

$$\partial_\nu U(t, x) = 0 \quad \text{on} \ (0, +\infty) \times \Gamma_N,$$

$$U = \partial_t U = 0 \quad \text{on} \ \{0\} \times \Omega,$$

Here $\omega$ is a given angular motion frequency (L.F., $\sim 50$–$1000$ Hz), $\chi(t) \in C^\infty([0, \infty))$, $\chi(t) = 0$ ($0 \leq t \leq 1/2$), $1 (t \geq 1)$ is a cutoff function and the conormal derivative along the outward unit normal the $\nu$ to $\partial \Omega$ is defined as

$$\partial_\nu U(t, x) := \left\{ \begin{array}{l} [\lambda (\nabla \cdot U) + 2 \mu \varepsilon(U)] \\ + [\zeta (\nabla \cdot \partial_t U) + 2 \eta \varepsilon(\partial_t U)] \end{array} \right\} \nu.$$

$\lambda(x)$ and $\mu(x)$ are the Lamé modulus, while $\zeta(x)$ and $\eta(x)$ are the viscosity coefficients. Especially, $\mu(x)$ and $\eta(x)$ are called shear modulus and shear viscosity respectively. Physically well known Possion’s ratio $\nu$ is given by $\nu = \lambda / (2(\lambda + \mu))$. Moreover we have the strong convexity:

$$\left\{ \begin{array}{l} \mu > \delta, \ \eta > \delta, \\ 3\lambda + 2\mu > \delta, \ 3\zeta + 2\eta > \delta \quad (\text{a.e. in } \Omega) \end{array} \right.$$ for some constant $\delta > 0$.

We can prove the exponential decay property in time of this dynamical wave displacement field $U(t, x)$, and then the governing model becomes the following mixed boundary value problem:

$$\left\{ \begin{array}{l} [\nabla (\lambda \nabla \cdot u) + \nabla \cdot (2 \mu \varepsilon(u))] + \rho \omega^2 u \\ + i \omega [\zeta \nabla \cdot u + \nabla \cdot (2 \eta \varepsilon(u))] = 0 \quad \text{in} \ \Omega, \\ u = f \quad \text{on} \ \Gamma_D, \\ \partial_\nu u = 0 \quad \text{on} \ \Gamma_N. \end{array} \right.$$ (1)

with a normal derivative

$$\partial_\nu u := [\lambda \nabla \cdot u + 2 \mu \varepsilon(u)] \nu + i \omega [\zeta \nabla \cdot u + 2 \eta \varepsilon(u)] \nu.$$
In a living body of soft tissues, usually the Poisson’s ratio is near 0.5 which means λ is much more larger that μ. A typical order of value for λ is GPa while μ is of the order of kPa which is the case for nearly incompressible material. Although, we can use the model (1), a very large λ and too many viscoelasticity coefficients may cause noise in MRE measurement and make the inversion analysis complex. We still desire to find an approximate model which doesn’t contain any large λ.

By letting

$$\lambda(x) = \alpha \lambda(x), \quad \mu(x) = \beta \mu(x), \quad \kappa = \alpha/\beta, \quad |\kappa| > > 1;$$

$$\tilde{\zeta} := \beta^{-1} \zeta, \quad \tilde{\eta} := \beta^{-1} \eta, \quad \tilde{\rho} := \beta^{-1} \rho, \quad \tilde{h} := \beta^{-1} h$$

with some constants α, β such that $|\tilde{\lambda}|/|\tilde{\mu}| = O(1)$ and

$$p := -\beta^{-1} \lambda \nabla \cdot \mathbf{u} = -\kappa \tilde{\lambda} \nabla \cdot \mathbf{u},$$

we rewrite (1) into the following mixed boundary value problem for the modified Stokes system:

$$\begin{align*}
\nabla \cdot (2\tilde{\mu} \xi(\mathbf{u})) - \nabla p + i \omega \tilde{\rho} \omega^2 \mathbf{u} &= 0 \quad \text{in } \Omega, \\
\nabla \cdot \mathbf{u} + (\kappa \tilde{\lambda})^{-1} p &= 0 \quad \text{in } \Omega, \\
\mathbf{u} &= \mathbf{f} \quad \text{on } \Gamma_D, \\
|2\tilde{\mu} \xi(\mathbf{u}) - p| \nu + i \omega [\tilde{\zeta} \nabla \cdot \mathbf{u} + 2\tilde{\eta} \xi(\mathbf{u})] \nu &= 0 \quad \text{on } \Gamma_N,
\end{align*}$$

We apply the asymptotic analysis to the mixed boundary value problem (??). By representing $(\mathbf{u}, p)$ in the formal series:

$$\mathbf{u} = \sum_{j=0}^{\infty} \kappa^{-j} \mathbf{u}_{-j}, \quad p = \sum_{j=0}^{\infty} \kappa^{-j} p_{-j}$$

and a simple computation, we have for $(\mathbf{u}_0, p_0)$:

$$\begin{align*}
\nabla \cdot \left[2(\mu + i \omega \eta) \xi(\mathbf{u}_0)\right] - \nabla p_0 + \tilde{\rho} \omega^2 \mathbf{u}_0 &= 0 \quad \text{in } \Omega, \\
\nabla \cdot \mathbf{u}_0 &= 0 \quad \text{in } \Omega, \\
\mathbf{u}_0 &= \mathbf{f} \quad \text{on } \Gamma_D, \\
\left[2(\mu + i \omega \eta) \xi(\mathbf{u}_0) - p_0\right] \nu &= 0 \quad \text{on } \Gamma_N.
\end{align*}$$

And for $(\mathbf{u}_{-j}, p_{-j})$ ($j \geq 1$):

$$\begin{align*}
\nabla \cdot \left[2(\mu + i \omega \eta) \xi(\mathbf{u}_{-j})\right] - \nabla p_{-j} + \tilde{\rho} \omega^2 \mathbf{u}_{-j} &= -\nabla (\lambda^{-1} \tilde{\zeta} p_{-j+1}) \quad \text{in } \Omega, \\
\nabla \cdot \mathbf{u}_{-j} &= -\lambda^{-1} \tilde{\lambda} p_{-j+1} \quad \text{in } \Omega, \\
\mathbf{u}_{-j} &= 0 \quad \text{on } \Gamma_D, \\
\left[2(\mu + i \omega \eta) \xi(\mathbf{u}_{-j}) - p_{-j}\right] \nu &= -\lambda^{-1} \tilde{\lambda} p_{-j+1} \nu \quad \text{on } \Gamma_N.
\end{align*}$$

Remark. $\|\mathbf{u} - \mathbf{u}_0\| = O(\kappa^{-1})$.

By resetting $p = \beta p$, we have the new approximate model:

$$\begin{align*}
\nabla \cdot \left[2(\mu + i \omega \eta) \xi(\mathbf{u})\right] - \nabla p + \rho \omega^2 \mathbf{u} &= 0 \quad \text{in } \Omega, \\
\nabla \cdot \mathbf{u} &= 0 \quad \text{in } \Omega, \\
\mathbf{u} &= \mathbf{f} \quad \text{on } \Gamma_D, \\
\partial_{\nu} \mathbf{u} &:= \left[2(\mu + i \omega \eta) \xi(\mathbf{u}) - p\right] \nu = 0 \quad \text{on } \Gamma_N.
\end{align*}$$

2 Numerical Simulations

We present some numerical simulations of model (1) and (2) in two dimensions. All of the models are transformed to corresponding ones in two dimensions by plane strain assumption. We setup the parameters for numerical computation corresponding to the experiment by using 3 T MRE system with longitudinal vibration system. By using FreeFem++, we have several sets of simulated data on a $65 \times 65$ grids of 128 mm $\times$ 128 mm rectangle. Here we set $\mathbf{u} = \Phi - i \Psi$ ($\Phi := (\phi_1, \phi_2), \Psi := (\psi_1, \psi_2)$). Every value is in mm unit. Figure 1 shows the real part of $x_1$ component of each simulated data ($\phi_1$). We shall find that when $\lambda$ is large, the effect of the term $\lambda \nabla \cdot \mathbf{u}$ is sensitive and incompressible stationary viscoelasticity model is not suitable in this case. The stationary viscoelasticity model can well simulate the wave displacement field measured by MRE (see Figure 1), and also the solution to the Stokes model can well approximate the stationary viscoelasticity model.

References

