2 Introduction

The nonlinear Schrödinger equation shows the dynamics of various phenomena in quantum physics, plasma physics, hydrodynamics, fiber-optics, and so on[1][2]. The fact that the solution for the nonlinear Schrödinger equation can be a soliton(envelope soliton) is known and of interest. The soliton is the typical nonlinear phenomenon and many studies have been carried out[3][4][5][6][12].

In this paper, we study the following, one dimensional nonlinear Schrödinger equation with double powers:

\[ iA_t + pA_{xx} + q|A|^2A + \mu |A|^nA = 0, \]

where \( A = A(x,t) : \mathcal{R} \times \mathcal{R} \rightarrow \mathbb{C}, -\infty < x < \infty, t \geq 0, \]
\( p,q \in \mathcal{R}, i = \sqrt{-1} \) and \( n = 3,4,5,... \). Also \( \mu > 0 \) is a small parameter. The subscript means the partial derivative. This equation is based on the cubic nonlinear Schrödinger equation, that is, \( \mu = 0 \) in Eq. (1.1).

There exist some studies concerned Eq.(1.1) from physical view points[8] as well as mathematical sense[11]. The cubic nonlinear Schrödinger equation with quintic nonlinear terms has been formulated in the vibration of elastic plates with cubic characteristics of spring[7]. Moreover, a sufficient condition is suggested to have soliton solutions for the Schrödinger equation with the perturbed term of the general degree \((n+1)[10]\) and the approximation formula of the perturbed soliton solution is shown[9].

The aim of this paper is to describe the features of the amplitude equation induced from the cubic nonlinear Schrödinger equation with the higher-order perturbed term. The one-dimensional Schrödinger equation with double powers is written by \( iA_t + pA_{xx} + q|A|^2A + \mu |A|^nA = 0 \), where \( i = \sqrt{-1} \), and \( \mu > 0 \) is a fixed coefficient. The amplitude equation is obtained based on the standing wave solutions of the form \( A(x,t) = \varphi(x)e^{-i\Omega t} \) for \( \Omega > 0 \). We show that the approximated equation of the amplitude equation is equivalent to the the Legendre associated differential equation.

\[ A(x,t) = \varphi(x)e^{-i\Omega t}, \]

where \( \Omega > 0 \) denotes the angular frequency. We call the equation which \( \varphi(x) \) satisfies the amplitude equation. By virtue of the above form of the solution, Eq. (1.1) is reduced to the second order ordinary differential equation.

2 Main results

We have the amplitude equations by substituting Eq.(1.2) into Eq.(1.1) and consider the following two interesting cases:

\[ \ddot{\varphi} - \varphi + \varphi^3 + \varepsilon \varphi^{n+1} = 0, \]  
\[ \ddot{\varphi} + \varphi - \varphi^3 + \varepsilon \varphi^{n+1} = 0, \]

where \( \cdot \) indicates the derivative with respect to \( x, \varepsilon \) is a positive, small number and \( n = 3,4,5,... \). The coefficients of Eqs.(2.1) and (2.2) are normalized without the loss of generality.

2.1 Analysis of Eq. (2.1)

We analyze those equations using the perturbation method. Substituting the naïve expansion of \( \varphi \)

\[ \varphi = \varphi_0 + \varepsilon \varphi_1 + \varepsilon^2 \varphi_2 + \ldots, \]

into Eq.(2.1) yields the following equations by equating the same order of \( \varepsilon \).

\[ \varepsilon^0 : \ddot{\varphi}_0 - \dot{\varphi}_0 + \varphi_0^3 = 0, \]
\[ \varepsilon^1 : \ddot{\varphi}_1 - (1 - 3\varphi_0^2)\dot{\varphi}_1 = -\varphi_0^{n+1}, \]
\[ \varepsilon^2 : \ddot{\varphi}_2 - (1 - 3\varphi_0^2)\dot{\varphi}_2 = 3\varphi_0\varphi_1^2 - (n+1)\varphi_0^n \varphi_1, \]
\[ \cdots \]
\[ \varepsilon^k : \ddot{\varphi}_k - (1 - 3\varphi_0^2)\dot{\varphi}_k = f(\varphi_0, \varphi_1, \ldots, \varphi_{k-1}). \]

\[ 1 \text{In the other cases of the sign of } p,q, \text{ the fact that there does not exist soliton solutions in Eq.(1.1) is known.} [10] \]
We call Eq.(2.7) the $k$-th order deformation equation. Note that the right hand side of the $k$-th order deformation equation consists of all known functions obtained from the $(k - 1)$-th order deformation equation.

The homogeneous equations of the $k$-th $(k \geq 1)$ order deformation equation are the same formula so that we write this

$$\ddot{\Psi} - (1 - 3\varphi_0^2)\ddot{\Psi} = 0, \quad (2.8)$$

where

$$\ddot{\Psi} = \varphi_1, \quad i = 1, 2, 3, \ldots. \quad (2.9)$$

We call Eq.(2.8) the $\varepsilon$-approximated equation.

**Theorem 2.1.** (1) Let $\varphi_0$ be a solution of the ordinary differential equation $\ddot{\varphi} - \varphi + \varphi^3 = 0$, which satisfies $\ddot{\varphi} = \frac{1}{2}(2 - \varphi^2)^2$. The $\varepsilon$-approximated equation of the equation with a higher order term $\ddot{\varphi} - \varphi + \varphi^3 + \varepsilon\varphi^{n+1} = 0$, where $\varepsilon > 0$ is sufficiently small and $n = 3, 4, 5, \ldots$ is presented by $\ddot{\Psi} - (1 - 3\varphi_0^2)\ddot{\Psi} = 0$, where $\ddot{\Psi} = \varphi_1$, $i = 1, 2, 3, \ldots$ and $\varphi = \varphi_0 + \varepsilon\varphi_1 + \varepsilon^2\varphi_2 + \ldots$.

(2) We consider the function $\varphi_0 = \varphi_0(x)$ that satisfies the equation

$$\ddot{\varphi}_0 = \frac{1}{2}(2 - \varphi_0^2)\varphi_0^3 + E, \quad (2.10)$$

where $E$ is a constant. From this, we have easily $\ddot{\varphi}_0 = \varphi_0(1 - \varphi_0^2)$ so that $\varphi_0$ is a solution of the non-perturbed equation $\ddot{\varphi} - \varphi + \varphi^3 = 0$.

Now let

$$\xi^2 = \frac{2 - \varphi_0^2(x)}{2}, \quad (2.11)$$

and $\eta = \eta(\xi)$, where $\eta \circ \xi(x) = \ddot{\Psi}(x)$, then the $\varepsilon$-approximated equation is rewritten as

$$\left(1 - \xi^2 + \frac{E}{2\xi^2}\right)\frac{d^2 \eta}{d\xi^2} + \left(\frac{\xi}{2\sqrt{2}E^3}\frac{1}{1 - \xi^2} - \sqrt{2}\right)\frac{d\eta}{d\xi} + \left(6 - \frac{1}{1 - \xi^2}\right)\eta = 0. \quad (2.12)$$

Substituting $E = 0$ into Eq.(2.12) yields

$$\left(1 - \xi^2\right)\frac{d^2 \eta}{d\xi^2} - 2\sqrt{2}\frac{d\eta}{d\xi} + \left(6 - \frac{1}{1 - \xi^2}\right)\eta = 0, \quad (2.13)$$

which is the Legendre associated differential equation

$$\left(1 - \xi^2\right)\frac{d^2 \eta}{d\xi^2} - 2\sqrt{2}\frac{d\eta}{d\xi} + \left(m(m+1) - \frac{j^2}{1 - \xi^2}\right)\eta = 0, \quad (2.14)$$

with $j = 1, m = 2$.

### 2.2 Analysis of Eq. (2.2)

We obtain the following theorem for Eq. (2.2).

**Theorem 2.2.** (1) Let $\varphi_0$ be a solution of the ordinary differential equation $\ddot{\varphi} - \varphi + \varphi^3 = 0$, which satisfies $\ddot{\varphi} = \frac{1}{2}(\varphi^2 - 1)^2$. The $\varepsilon$-approximated equation of the equation with a higher order term $\ddot{\varphi} - \varphi + \varphi^3 + \varepsilon\varphi^{n+1} = 0$, where $\varepsilon > 0$ is sufficiently small and $n = 3, 4, 5, \ldots$, is presented by $\ddot{\Psi}(1 - 3\varphi_0^2)\ddot{\Psi} = 0$, where $\ddot{\Psi} = \varphi_1$, $i = 1, 2, 3, \ldots$ and $\varphi = \varphi_0 + \varepsilon\varphi_1 + \varepsilon^2\varphi_2 + \ldots$.

(2) Let $\eta = \eta(\xi)$, where $\eta \circ \xi(x) = \ddot{\Psi}(x)$ and $\xi = \varphi_0(x)$. Then the $\varepsilon$-approximated equation is equivalent to the Legendre associated differential equation $(1 - \xi^2)\frac{d^2 \eta}{d\xi^2} - 2\sqrt{2}\frac{d\eta}{d\xi} + \left(6 - \frac{4}{1 - \xi^2}\right)\eta = 0$.

### References


