Electroelastic Fracture Mechanics Analysis of Piezoelectric Ceramic Strip

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The effects of crack face boundary conditions of the piezoelectric fracture mechanics are discussed by analyzing the plane strain electroelastic problem of an orthotropic piezoelectric ceramic strip with a central permeable or impermeable crack. The problem of a long strip is formulated by means of integral transforms and reduced to the solution of a system of Fredholm integral equations of the second kind. Fracture mechanics parameters such as stress intensity factor, energy release rate and energy density factor, etc. based on both permeable model and impermeable model are compared. A finite element method is also used to calculate these fracture mechanics parameters, and the results are compared with the exact solutions. The numerical results illustrate that the impermeable assumption can lead to significant errors regarding the effects of the electric fields on crack propagation.

1. Introduction

The problem of permeable crack in piezoelectric ceramics under mode I loading was first investigated by Shindo et al., 1) who performed an analysis of the singular stress and electric field using integral transform techniques, and found that crack would propagate easily normal to the poling direction. Hao and Shen 2) proposed a new electrical boundary condition by considering the electric permeability of air in a crack gap. Sosa and Kulikovsky 3) showed that invoking the condition of electric impermeability at the boundary of the sharp crack may result in erroneous conclusions. On the other hand, Pak 4) proposed the impermeable model for the antiplane shear piezoelectric crack. However, the impermeability condition resulted in erroneous conclusions 5). Although the impermeable model may provide the mathematical solution of the piezoelectric crack, it is impractical to search for fracture design parameters characterizing the electric failure.

This paper constitutes a continuing study of the previous work 6) on piezoelectric fracture mechanics. A mode I permeable crack problem revealed that, under applied uniform strain, positive electrical fields (electrical fields in poling direction) normal to the crack surface impede crack propagation while negative electrical fields (electrical fields applied opposite to the poling direction) aid crack propagation. In this work we will see whether the same phenomena can be observed for the impermeable crack problem. By the use of Fourier transforms, the mixed boundary value problem is reduced to a system of Fredholm integral equations of the second kind which is solved numerically to determine the fracture mechanics parameters such as stress intensity factor, energy release rate and energy density factor, etc. A finite element analysis is also employed to calculate these fracture mechanics parameters. The results for the permeable and impermeable boundary conditions are presented in graphical form and compared for a piezoelectric ceramic PZT.

2. Problem Statement and Method of Solution

Consider a piezoelectric ceramic strip of width $2b$ and length $2L$ which contains a centrally plane strain crack of length $2c$ aligned with its plane normal to the free edges $x = \pm h$ as shown in Fig. 1. A set of Cartesian coordinates $(x, y, z)$ is attached to the center of the crack normal to the $z$-axis. The piezoceramic strip has symmetry properties of hexagonal crystal of class 6mm with respect to the $x, y, z$-axes. The $x$-axis is directed along the line of the crack and $z$-axis along the direction of the perpendicular bisector of the crack. The edges of the strip are therefore the lines with equations $x = \pm h$, while the crack occupies the segment $-c < x < c, z = 0$. Consider two possible cases of loading conditions at $z = \pm h$. The first case is a uniform normal stress, $\sigma_z = \sigma_{\infty}$, applied with a uniform electric displacement, $D_z = D_0$; and the second is a uniform normal stress, $\sigma_z = \sigma_{\infty}$, with a uniform electric field, $E_z = E_0$. Only the first quadrant with appropriate boundary conditions need to be analyzed owing to symmetry.

The constitutive equations for piezoceramics poled in the $z$-direction can be written as

$$
\begin{align*}
\sigma_{zz} &= \epsilon_{11}u_x + \epsilon_{33}u_z - \epsilon_{33}E_z \\
\sigma_{xz} &= \epsilon_{11}(u_x + u_z) - \epsilon_{11}E_x \\
\sigma_{zx} &= \epsilon_{11}(u_x + u_z) - \epsilon_{33}E_z \\
D_z &= \epsilon_{33}(u_x + u_z) + \epsilon_{33}E_x \\
D_x &= \epsilon_{11}u_x + \epsilon_{33}u_z + \epsilon_{33}E_z
\end{align*}
$$

where $(\sigma_{xx}, \sigma_{yy}, \sigma_{zz})$ and $(D_x, D_z)$ are the components of stress tensor and electric displacement vector, $(u_x, u_z)$ and

![Fig. 1. Cracked piezoelectric ceramic strip](image)
\( (E_x, E_z) \) are the components of displacement vector and electric field vector, \( c_{11}, c_{12}, c_{33}, c_{44} \) are the elastic stiffness constants measured in a constant electric field, \( c_{11}, c_{22} \) are the dielectric constants measured at constant strain, \( e_{15}, e_{31}, e_{33} \) are the piezoelectric constants, and the subscript comma denotes a partial derivative with respect to the coordinates. The electric field components can be written in terms of an electric potential \( \phi(x, z) \) as

\[
E_x = -\phi_x, \quad E_z = -\phi_z
\]

(3)

The governing equations are obtained as

\[
\begin{align*}
&c_{11}u_{xx} + c_{44}u_{zz} + (c_{13} + c_{44})u_{x,z} + (c_{33} + c_{15})\phi_x = 0 \\
&c_{44}u_{zz} + c_{33}u_{x,z} + (c_{13} + c_{44})u_{z,x} + (c_{33} + c_{15})\phi_z = 0 \\
&(c_{31} + e_{31})u_{z,x} + e_{15}u_{x,z} + e_{33}u_{z,z} - c_{13}\phi_z - c_{32}\phi_x = 0
\end{align*}
\]

(4)

(5)

The boundary conditions can be written as

\[
\begin{align*}
\sigma_{zz}(x,0) &= 0 \quad (0 \leq x \leq h) \\
\sigma_{zz}(x,c) &= 0 \quad (c \leq x \leq h)
\end{align*}
\]

(6)

\[
\begin{align*}
u_x(x,0) &= 0 \quad (0 \leq x \leq c) \\
u_x(x,c) &= 0 \quad (c \leq x \leq h)
\end{align*}
\]

(7)

\[
\begin{align*}
E_x(x,0) &= E_x' \quad (0 \leq x < c) \\
\phi(x,0) &= 0 \quad (c \leq x \leq h)
\end{align*}
\]

(8)

\[
\begin{align*}
D_z(x,0) &= D_z' \quad (0 \leq x < c) \\
\sigma_{zz}(h,z) &= 0
\end{align*}
\]

(9)

\[
\begin{align*}
\sigma_{zz}(h,z) &= 0 \\
D_z(h,z) &= 0
\end{align*}
\]

(10)

(11)

(12)

Case I: \( \sigma_{zz}(x,l) = \sigma_\infty, \quad D_z(x,l) = D_0 \quad (0 \leq x \leq h) \)

(13)

Case II: \( \sigma_{zz}(x,l) = \sigma_\infty, \quad E_x(x,l) = E_0 \quad (0 \leq x \leq h) \)

(14)

where the superscript \( c \) stands for the electric quantities in the void inside the crack. The normal stress \( \sigma_\infty \) is expressed as

\[
\sigma_\infty = \begin{cases} 
  c_1\sigma_0 - c_1D_0, & \text{Case I} \\
  \sigma_0 - c_2E_0, & \text{Case II}
\end{cases}
\]

(15)

where \( c_1, c_1, c_2 \) are constants reflecting the material properties. Note that \( \sigma_0 \) is a uniform normal stress for a closed-circuit condition with the potential forced to remain zero (grounded). Equations (8) and (9) are the permeable boundary conditions. On the other hand, the impermeable boundary condition is

\[
\begin{align*}
D_z(x,0) &= 0 \quad (0 \leq x < c) \\
\phi(x,0) &= 0 \quad (c < x < \infty)
\end{align*}
\]

(16)

3. Theoretical analysis of a long cracked piezoceramic strip

In this section, consider the problem of a long piezoceramic strip with a central crack for \( l \to \infty \). Fourier transforms are used to reduce the problem to the solution of a pair of dual integral equations. The solution of the dual integral equations is then expressed in terms of a system of Fredholm integral equations of the second kind.\(^{4}\)

The stress intensity factor \( K_I \), electric displacement intensity factor \( K_D \) and energy release rate \( G \) for the permeable crack were obtained by Shindo et al.\(^{5}\). The energy desity factor for the permeable crack is

\[
S = (a_M + a_E)K_I^2
\]

(17)

where \( a_M \) and \( a_E \) are constants reflecting the material properties. The stress intensity factor, energy release rate and energy density factor for the impermeable crack are also obtained.

4. Finite element analysis of a cracked piezoceramic strip

In this section, the finite element computer program ANSYS is selected for the analysis of the configuration considered here. For electrical loads, a negative or positive electric potential \( \phi_0 \) was applied at the edge \( 0 \leq x \leq h, \ z = l \) (Fig. 1). The edge \( 0 \leq x \leq h, \ z = 0 \) produces an electric field that is parallel to the poling direction. This type of electric field is positive. A negative electric field, which was opposite to poling direction, was produced by the application of a positive electric potential at the edge \( 0 \leq x \leq h, \ z = l \). Four-node plane elements were used in the analysis. The total number of nodes and elements in the finite element analysis are 1330 and 1263, respectively.

5. Numerical Results and Discussion

Fig. 2 shows the total energy release rate \( G \) for the permeable crack under the normalized electric field \( c_2E_0/\sigma_0 \) (Case II) and \( c/h = 0.2 \), where the result has been normalized by the energy release rate \( G_{E0} \) for \( E_0 = 0 \). For comparison, the mechanical strain energy release rate \( G_M \) for the permeable crack, total energy release rate \( G_T \) and mechanical strain energy release rate \( G_{M_T} \) for the impermeable crack are included in the figure. Energy release rates \( G_M, G_T \) and \( G_{M_T} \) are normalized by the mechanical strain energy release rate \( G_{M_{E0}} \), total energy release rate \( G_{E0} \) and mechanical strain energy release rate \( G_{M_{E0}} \) for \( E_0 = 0 \), respectively. Also shown in Fig. 2 are the finite element solutions for \( l/h = 5.0 \).

Fig. 2. Energy release rate versus applied electric field (Case II)

References