Deflated Conjugate Gradient Method for Magnetostatic Analysis

Kota Watanabe¹, Hajime Igarashi²

¹ Graduate School of Engineering, Muroran Institute of Technology
² Hokkaido University

*k-wata@mmm.muroran-it.ac.jp

Received: March 2, 2014; Accepted: November 13, 2014; Published: February 20, 2015

Abstract. The deflated conjugate gradient (CG) method which can improve the convergence of CG is discussed. The distorted finite element mesh produces the system matrix with a large condition number, which results in poor convergence of CG method. The deflation technique replaces small eigenvalues with zeros in the system matrix. Therefore the deflated CG method is a useful solver for such ill-conditioned finite element analyses. However, the computational cost to obtain the eigenvectors is significantly high. To solve this problem, quasi eigenvectors which can be obtained with low cost are used in this paper. Moreover, the robustness of deflated CG method against distorted finite mesh with flat elements which bring ill-condition is presented.

Keywords: Conjugate gradient method, Deflation method, Eigenvalue, Finite element method

1. Introduction

In finite element (FE) analyses, fast linear solvers such as a preconditioned conjugate gradient (PCG) method are an important factor to reduce computation time. The FE mesh with distorted or flat elements produces the system matrix with a large condition number defined by the ratio of the largest eigenvalue to the smallest non-zero eigenvalue. This results in poor convergence of PCG method. Recently, some numerical techniques to improve the convergence of PCG method such as explicit error correction (EEC) and implicit error correction (IEC) has been introduced [1].

The deflation technique, which replaces small eigenvalues with zeros in the system matrix, can improve the convergence of CG method [2-7]. Therefore the deflated CG method is a useful solver for such ill-conditioned finite element analyses. The relationship between the deflation technique and other methods such as IEC and EEC method is discussed in [2]. The deflated technique has been applied to diffusion problem with extreme contrasts in the coefficient matrix [3], and to boundary value problems [4]. Reference [5] shows the effect of the deflated technique in magnetostatic problems with large jumps in the magnetic permeability.
We applied the deflated technique to FE analysis with infinite element which affect convergence of CG method [6], and showed improvement of convergence. Moreover we showed convergence property of deflated CG method based on [7] in a magnetostatic problem [2]. In [2], the numerical results show that the deflated PCG can reduce the iteration in FE analysis with distorted mesh. However, the computational cost to obtain the eigenvectors is significantly high. Therefore, approximate eigenvectors which can be obtained with low cost are used in this paper. Moreover, the robustness of deflated CG method against distorted finite mesh with flat elements which bring ill-condition is presented.

2. Formulation

Let us consider a system of linear equations obtained by magnetostatic FE analysis.

\[ Ax = b \]  

where \( A \) is coefficient matrix, \( x \) and \( b \) are solution and right hand side vector respectively. The solution \( x \) is decomposed into slowly and fast components as,

\[ x = Wy + (x - Wy) \]  

where \( W = [w_1, w_2, ..., w_k] \in \mathbb{R}^{n \times k}, \ w_k \in \mathbb{R}^n \). A-orthogonality is imposed on the vectors \( w_i \) to \( x - Wy \) results in

\[ W^t AWy = W^t Ax. \]  

The slowly converging component \( Wy \) can be expressed as

\[ Wy = W(W^t AW)^{-1} W^t Ax = Qx \]  

where \( Q \in \mathbb{R}^{n \times n} \). Moreover, let us introduce the matrix \( P \) given by,

\[ P = I - Q = I - W(W^t AW)^{-1} W^t A. \]  

Consequently, the solution \( x \) can be expressed as,

\[ x = Px + Qx. \]  

The fast converging component \( Px \) can be obtained by solving \( APx = P^t Ax = Pb \). The slowly component \( Qx \) is obtained from,

\[ Qx = W(W^t AW)^{-1} Wb. \]  

The algorithm of deflated incomplete Cholesky (IC) CG method is shown in Algorithm 1. In the main iteration loop, there is a solving process corresponded to (7) shown in Algorithm 1 (*). To solve (7), direct solvers or iterative solvers can be applied. In the former case, the inverse matrix of \( WAW \) is computed before entering the main iteration loop. If the dimension of \( WAW \) is large, the computational cost significantly increases. In the latter case, a suitable stop criterion for iterative solver must be set. The tighter criterion results in increase of iteration counts. If \( W \) is constructed from eigenvectors of \( A \), the matrix \( WAW \) is dense. Therefore,
ICCG method is not suitable because the IC decomposition becomes complete Cholesky decomposition which is equivalent to computation of inverse matrix.

Algorithm 1. Algorithm of deflated ICCG method

Choose initial guess $x_0$ such that $W(b - Ax_0) = 0$;
$r_0 = b - Ax_0$;
Solve $M u_0 = r_0$;

(where $M$ is IC decomposed matrix of $A$)

Solve $WAW \hat{u}_0 = WA u_0$;
$p_0 = u_0 - W \hat{u}_0$;
for $i = 0, 1, 2, ...$
\[
\alpha = (r_i, u_i)/(p_i, Ap_i);
\]
$x_{i+1} = x_i + \alpha p_i$;
$r_{i+1} = r_i - Ap_i$;
if converged, stop iteration;
Solve $M u_{i+1} = r_{i+1}$;
\[
\beta = (r_{i+1}, u_{i+1})/(r_i, u_i);
\]
Solve $WAW \hat{u}_{i+1} = WAW u_{i+1}$; \((*)\)
\[
p_{i+1} = u_{i+1} + \beta p_i - W \hat{u}_{i+1};
\]
end for;

3. III. NUMERICAL RESULTS

3.1. A simple magnetostatic model

We analyzed a magnetostatic model which consists of very thin FE shown in Fig. 1. 20×20×20 hexahedral elements are used and the number of unknowns is 26460. The constant 1T external flux density whose direction is parallel to $z$ is applied to the whole region.

In the standard deflation method, the vectors $w_i$, $i = 1, 2, ..., n_w$ are chosen as eigenvectors corresponding to the smallest $n_w$ non-zero eigenvalues. In the present study, to reduce the computational cost of obtaining eigenvectors, we chose $w_i$ as constant vectors corresponding to the edge whose direction is parallel to $x$ or $y$ in magnetic material and air regions, respectively, i.e. $n_w = 4$. The value of component of $w_i$ was set $w$. The acceleration factor for IC decomposition is 1.09.

The convergence history of deflated ICCG shown in Fig. 2 suggests that the value of $w$ affects the convergence. In particular, the minimum value of $r$ clearly depends on value of $w$. Thus the sufficient convergence requires the small $w$. In this reason, $10^{-4}$ is adopted for the following numerical results. Table I shows that the stop criterion for inner ICCG iteration also influences the convergence of outer loop, where the outer iteration is stop over $|r|/|b| < 10^{-7}$. These results suggest that the criterion for inner loop should be smaller than that for outer loop. However, the exceed criterion such as $10^{-10}$ results in divergence of iteration. In this case, the ICCG of inner loop is diverged due to tight criterion.
Figure 1: Simple magnetostatic model

Table 1: Relationship between convergence of outer loop and inner stop criterion in simple model

<table>
<thead>
<tr>
<th>Stop criterion for inner ICCG</th>
<th>$&gt;10^{-7}$</th>
<th>$10^{-8}$</th>
<th>$10^{-9}$</th>
<th>$10^{-10}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of iterations</td>
<td>diverge</td>
<td>1377</td>
<td>1354</td>
<td>diverge</td>
</tr>
</tbody>
</table>

Figure 2: Convergence history

3.2. Thin iron model

Next, we analyzed another magnetostatic model which consists of thin iron surrounded by air shown in Fig. 3. A coarse $20 \times 20 \times 20$ FE mesh and a fine $40 \times 40 \times 40$ FE meshes are prepared to investigate relationship between computational cost and scale of problem. The number of unknowns in the coarse and fine mesh is 26460 and 201720, respectively. The vectors $w_i$ are chosen by two methods. One is that $w_i$ are chosen as a constant vector using same procedure in the previous section ($n_w = 4$). The other is that $w_i$ are chosen as a constant vector whose non-zero component is corresponding to edge group which has same $x$ and $y$ coordinates ($n_w = 760$ and 3120) [8]. The constant $w$ is set as $10^{-4}$ and the stop criterion of outer and inner loops are $|\mathbf{r}|/|\mathbf{b}| < 10^{-6}$ and $|\mathbf{r}|/|\mathbf{b}| < 10^{-9}$, respectively [9]. The computations was performed on a PC with Intel Xeon 5160 (Dual Core 3GHz) and Linux OS. Open MP was used for parallel computation of matrix-vector multiplication. Other settings are same as the previous section.

Table II and III shows the relationship between the number of iterations and flatness of elements. The results of standard ICCG method are also listed in these tables. The iteration
counts in the deflated ICCG as well as the ICCG method increase with decreasing $dz$. However, the rate of increase in the deflated ICCG is very low. Therefore, the deflated ICCG seems to be robust against the distorted meshes. In particular the iteration counts of present method in $dz = 0.005$ is significantly less than that of ICCG method. The CPU time of deflated ICCG is longer than that of the conventional ICCG in non-flat mesh ($dz = 1.0$ and $0.1$). That relationship is reversed in flat mesh. Moreover, the CPU time of deflated ICCG with $n_w = 760$ in Table II and $n_w = 3120$ in Table III is almost unchanged though the number of iterations in $dz = 0.005$ is approximately twice as many as that in $dz = 1.0$. This is because the almost all CPU time is consumed in matrix-matrix multiplication of $WAW$. It can be reduced by improvement of algorithm for matrix-matrix multiplication.

The vectors $w_i$ should be chosen as eigenvectors corresponding to the smallest $n_w$ non-zero eigenvalues in the theoretical reason [2]. However, the computation of eigenvectors is significantly high computational cost. For example, the computation of eigenvectors of system matrix in the coarse mesh with $dz = 1.0$ takes 131 minutes by using Intel MKL library. Thus the conventional deflated method in which the $w_i$ is constructed using eigenvectors is not practical. On the other hand, the $w_i$ constructed by the above-mentioned procedure has negligible computational cost.

![Figure 3: Thin iron model](image)

**Table 2:** Iterations and computation time in coarse mesh of thin iron model, DoF = 26460

<table>
<thead>
<tr>
<th>$dz$</th>
<th>ICCG</th>
<th>Deflated ICCG $n_w = 4$</th>
<th>Deflated ICCG $n_w = 760$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>33(0.87)</td>
<td>27(1.26)</td>
<td>25(9.19)</td>
</tr>
<tr>
<td>0.1</td>
<td>196(5.11)</td>
<td>138(4.35)</td>
<td>37(9.24)</td>
</tr>
<tr>
<td>0.01</td>
<td>1438(37.00)</td>
<td>1047(29.52)</td>
<td>51(9.90)</td>
</tr>
<tr>
<td>0.005</td>
<td>1960(50.92)</td>
<td>1417(40.19)</td>
<td>53(9.77)</td>
</tr>
</tbody>
</table>

**Table 3:** Iterations and computation time in fine mesh of thin iron model, DoF = 201720

<table>
<thead>
<tr>
<th>$dz$</th>
<th>ICCG</th>
<th>Deflated ICCG $n_w = 4$</th>
<th>Deflated ICCG $n_w = 3120$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>70(14.48)</td>
<td>57(41.16)</td>
<td>51(372.49)</td>
</tr>
<tr>
<td>0.1</td>
<td>453(89.26)</td>
<td>312(97.81)</td>
<td>71(370.52)</td>
</tr>
<tr>
<td>0.01</td>
<td>4062(833.74)</td>
<td>2978(667.29)</td>
<td>100(365.32)</td>
</tr>
<tr>
<td>0.005</td>
<td>6423(1317.4)</td>
<td>4708(1039.8)</td>
<td>109(367.40)</td>
</tr>
</tbody>
</table>
4. IV. CONCLUSION

We evaluate the deflated ICCG method in the finite element analysis for magnetostatic field problems. The deflation technique which replaces small eigenvalues with zeros in the system matrix, can improve the convergence of ICCG method. Thus the present method improves the convergence for distorted finite element mesh. The numerical investigation shows that the deflated ICCG method with appropriate selection of quasi eigenvectors has good robustness and less computational cost against distorted finite mesh with flat elements. The flat elements often appear in finite element analyses (e.g., modeling of laminated steel). Therefore the present method is useful for such analyses. In the future work, we plan to apply the deflated method to other solvers.

Acknowledgement

This work was supported by Grants-in-Aid for Scientific Research 24560319.

References


