DESIGN OF FIXED-GAIN FEEDBACK CONTROL SYSTEM WITH REQUIRED TOLERANCE

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A design method was proposed for the fixed-gain feedback controller which is sufficiently robust under the variations of process dynamics. The variations or uncertainties in the process dynamics are represented by variations of uncertain parameters in the process model. For each typical operational condition the region of possible variation is approximated by an orthotope. This orthotope is the required region of tolerance. On the other hand, the region of uncertain parameters, in which the control system has admissible system performance, is the inherent region of system tolerance. The controller is designed so that the inherent region of tolerance contains the required region of tolerance. The design algorithm is composed of the iteration of two steps: a robustness test for the proposed design and a controller parameter retuning for the required tolerance. The method was applied successfully to processes for which the dynamics depends on the throughput.

Introduction

Industrial processes have come to be operated under a variety of conditions. In some cases on-line optimization is performed¹⁾ to operate the plant efficiently. In consequence, the plant dynamics changes

with operational conditions.

The fixed-gain feedback controller is widely used for these time-varying processes as well as for time-invariant processes. Usually, a controller is tuned at a specified operating condition. As a result, a changes in the operating condition may cause a large deviation in process dynamics and degrade system performance. To cope with these phenomena, operators retune the

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controller parameters manually. Recently, adaptive control strategy is widely studied for the control of time-varying processes, especially for processes that are poorly understood. But there are still many problems in practice, such as the parameter bias problem in on-line identification. Under these circumstances, if possible, a controller robust enough for the specified changes should be designed.

In this paper, uncertainties are represented by parameter variations in the fixed model structure. The tolerance analysis approach²⁾ in the uncertain parameter space is used to analyze the robustness of the control system. An iterative method of robust controller design for the time-varying process is discussed.

1. Process Variations and Tolerance of Control System

1.1 Uncertainties

Although considerable progress has been made in our understanding of process dynamics, mismatch between the actual process and its model cannot be avoided in process modeling. The uncertainties in modeling are characterized by parameter variations in the mathematical model. In many cases, the possible region of parameter variations can be predicted approximately on the basis of physical considerations.

Consider the fixed-gain feedback control system shown in Fig. 1. The plant, which is an SISO process, is represented by a transfer function in the s -domain as follows:

$$G = G(s, \mathbf{p}) \quad (1)$$

where $\mathbf{p} = [p_1, p_2, \dots, p_r]$, and the p_i s are the uncertain parameters. On the other hand, the fixed-gain linear controller is generally described as follows:

$$C = C(s, \mathbf{q}) \quad (2)$$

where $\mathbf{q} = [q_1, q_2, \dots, q_l]$ and the q_i s are the tuning parameters.

The control system performance factors such as steady-state characteristics, stability margin and response speed depend on both the process and controller parameters. For example, the damping factor and settling time are useful criteria in the time domain, and the gain and phase margins are important measures in the frequency domain. When the process dynamics varies under the fixed controller gains, the control system performance will change.

1.2 Inherent region of tolerance

The inherent region of tolerance S_i of the control system for i -th system performance is defined in the uncertain parameter space as follows:

$$S_i(\mathbf{q}) = [\mathbf{p} | f_i(\mathbf{p}, \mathbf{q}) \in F_i] \quad (3)$$

where f_i is i -th system performance, and F_i is its

Fig. 1. Fixed-gain feedback control system

admissible region. Equation (3) means that S_i is the set of uncertain parameter vectors for which control system performance is within the admissible region. If there are L performance indices, the inherent region of system tolerance S is the intersection of inherent regions of tolerance for all performance indices.

$$S(\mathbf{q}) = \bigcap_{i=1}^L S_i(\mathbf{q}) = \bigcap_{i=1}^L [\mathbf{p} | f_i(\mathbf{p}, \mathbf{q}) \in F_i] \quad (4)$$

1.3 Required region of tolerance

The nominal value p_j^o of an uncertain parameter p_j changes with time. The change is described by a trajectory in the uncertain parameter space. Although it is hard to obtain the trajectory of nominal value of the uncertain parameter vector $\mathbf{p}^o$, several typical points on the trajectory can be estimated by using process analysis or identification. In this paper it is assumed that a set of M nominal points are given, i.e.,

$$\mathbf{P}_0 = [p^{o1}, \dots, p^{oM}]$$

On the other hand, statistical variation around a nominal value will also occur and it can be described as follows:

$$|\Delta p_j| \leq \mu_j \quad (j=1, \dots, r) \quad (5)$$

where $\Delta p_j = p_j - p_j^o$ ($j=1, \dots, r$), and μ_j is the maximum magnitude of statistical variation.

To design a control system which adapts the specified variations, it is necessary to determine the required region of tolerance R_i in the uncertain parameter space.

At i -th nominal point p^{oi} , the required region of tolerance is defined as follows:

$$R_i = [\mathbf{p} | |p_j - p_j^{oi}| \leq \mu_j^{oi}, \quad j=(1, \dots, r)] \quad (6)$$

where μ_j^{oi} is the maximum magnitude of statistical variation of j -th element around nominal point p^{oi} . Therefore, R_i becomes an orthotope, and it has 2^r vertices.

The design problem is stated as the problem of finding a controller parameter vector $\mathbf{q}$ so that $S(\mathbf{q})$ contains all the R_i s. This problem is difficult to solve analytically. An iterative design method is used to solve the problem.

2. Iterative Design Method

2.1 Robustness test

If all the R_i s are contained within the inherent region of system tolerance $S(\mathbf{q})$, the corresponding

control system will be robust enough for the predicted uncertainties. To test whether $S(\mathbf{q})$ contains all the R_i s or not is called the robustness test. The test can be achieved as follows:

First, a test coordinate axis p_k is chosen among all axes $(p_1, \dots, p_r)$. It is denoted by v . If S is one-dimensional convex and bounded, the one-dimensional manifold passing through a vertex and parallel to the test axis makes an interval S_v^i on the manifold in S as shown in Fig. 2.

$$S_v^i = [\underline{v}^i, \bar{v}^i] \quad (7)$$

where

$$p_j = p_j^i \quad (j = 1, \dots, k-1, k+1, \dots, r).$$

If vertex p^i is within the inherent region of tolerance S , then k -th element of the point p^i , i.e., v^i , is in the interval S_v^i . The reverse is also true. This property is derived in Appendix.

The length between vertex p^i and the upper and lower boundaries of the inherent region of system tolerance S are calculated respectively as follows:

$$\Delta \bar{v}^i(\mathbf{q}) \equiv \bar{v}^i - \bar{v}^i(\mathbf{q}), \quad (8)$$

$$\Delta \underline{v}^i(\mathbf{q}) \equiv \underline{v}^i - \underline{v}^i(\mathbf{q}), \quad (9)$$

If either $\Delta \bar{v}^i(\mathbf{q})$ or $\Delta \underline{v}^i(\mathbf{q})$ is positive, the point is outside of S . The farthest points from both upper and lower boundaries, indices of which are denoted by N_1 and N_2 respectively, give us important information for retuning of $\mathbf{q}$.

$$N_1 = [i \mid \Delta \bar{v}^i(\mathbf{q}) = \max(\Delta \bar{v}^j(\mathbf{q}), j = 1, \dots, m \times 2^r)] \quad (10)$$

$$N_2 = [i \mid \Delta \underline{v}^i(\mathbf{q}) = \max(\Delta \underline{v}^j(\mathbf{q}), j = 1, \dots, m \times 2^r)] \quad (11)$$

2.2 Retuning of the controller parameters

The instruction for retuning is that of adjusting the controller parameter vector $\mathbf{q}$ so that the two points p^{N_1} and p^{N_2} are contained in the inherent region of system tolerance S . Consider the situation shown in Fig. 3. Points p^{N_1} and p^{N_2} are outside of S . Then both $\Delta \bar{v}^{N_1}$ and $\Delta \underline{v}^{N_2}$ are positive. If two controller parameters q_1 and q_2 are adjusted simultaneously, the Newton-Raphson method gives the following retuning algorithm.

$$\begin{bmatrix} q_1(k+1) \\ q_2(k+1) \end{bmatrix} = \begin{bmatrix} q_1(k) \\ q_2(k) \end{bmatrix} + (\mathbf{G}_r)^{-1} \begin{bmatrix} \Delta \bar{v}^{N_1}(\mathbf{q}) \\ -\Delta \underline{v}^{N_2}(\mathbf{q}) \end{bmatrix} \quad (12)$$

where

$$\mathbf{G}_r = \begin{bmatrix} \frac{\partial \bar{v}^{N_1}}{\partial q_1} & \frac{\partial \bar{v}^{N_1}}{\partial q_2} \\ \frac{\partial \underline{v}^{N_2}}{\partial q_1} & \frac{\partial \underline{v}^{N_2}}{\partial q_2} \end{bmatrix} \quad (13)$$

Fig. 2. Robustness test for vertex p^i

Fig. 3. Geometrical meaning of retuning rule
Test variable

where $\mathbf{G}_r$ is a kind of sensitivity matrix. Each element of the matrix is calculated by numerical differentiation. Usually, $\bar{v}(\mathbf{q})$ and $\underline{v}(\mathbf{q})$ are nonlinear functions of q_1 and q_2 . The strong nonlinearity may lead to unstable design calculation. In this case, the following retuning rule is used to stabilize the calculation and to adjust the convergence speed.

$$\begin{bmatrix} q_1(k+1) \\ q_2(k+1) \end{bmatrix} = \begin{bmatrix} q_1(k) \\ q_2(k) \end{bmatrix} + \mathbf{D}(\mathbf{G}_r)^{-1} \begin{bmatrix} \lambda_1 \Delta \bar{v}^{N_1}(\mathbf{q}) \\ -\lambda_2 \Delta \underline{v}^{N_2}(\mathbf{q}) \end{bmatrix} \quad (14)$$

where $\mathbf{D} = \text{diag}(\delta_1, \delta_2)$ is the weighting matrix to avoid the effect of the nonlinearity of the functions $\bar{v}(\mathbf{q})$ and $\underline{v}(\mathbf{q})$ and λ_1 and λ_2 are the weighting factors to accelerate the convergence of the design calculation. If $\mathbf{G}_r$ is singular or illconditioned, then $(\mathbf{G}_r)^{-1}$ is replaced by the following matrix.

$$(G_r)^{-1} = \frac{1}{2} \begin{bmatrix} \left(\frac{\partial \bar{v}^{N1}}{\partial q_1} \right)^{-1} & \left(\frac{\partial \bar{v}^{N2}}{\partial q_1} \right)^{-1} \\ \left(\frac{\partial \bar{v}^{N1}}{\partial q_2} \right)^{-1} & \left(\frac{\partial \bar{v}^{N2}}{\partial q_2} \right)^{-1} \end{bmatrix} \quad (15)$$

If a controller has more than two tuning parameters, two effective elements for the robustness are chosen among all parameters.

3. Design Algorithm

The flowchart of design algorithm is shown in Fig. 4. The design procedure is composed of four steps as follows:

Step 1 (Design specifications) Choose some typical operational conditions and determine the set of M nominal points.

$$P_0 = [p^{o1}, \dots, p^{oM}]$$

Then, determine the required region of tolerance R_i for each nominal point.

Step 2 (Initial design) Design a controller by any method under a nominal point. An initial set of values of controller parameter $q(0)$ is obtained.

Step 3 (Robustness test): Choose a test variable among all the uncertain parameters. Determine the upper and lower limits $\bar{v}(q(k))$, $\underline{v}(q(k))$ of the inherent region of system tolerance for each vertex. Judge if v^i is contained in the interval $S_v^i = [\underline{v}^i, \bar{v}^i]$ or not. If all of the vertices are contained in S , then a robust controller is obtained. Otherwise, go to step 4.

Step 1 (Retuning): Calculate new controller parameter values $q(k+1)$ by using Eq. (12) or Eq. (14). Then go to Step 3 until the robust controller is obtained.

4. Illustrative Examples

4.1 Example 1: flow-rate control system

A valve lift-flow rate process is described by the following transfer function.

$$G(s) = \frac{K_p}{(T_1 s + 1)(T_2 s + 1)} \quad (16)$$

where K_p is the process static gain and T_1 and T_2 ($T_1 < T_2$) are time constants. From the nonlinearity of the static valve characteristics, the static gain K_p varies considerably by the flow rate. The parameters in Eq. (16) were estimated under two different typical flow rates as shown in Table 1. It is assumed that two nominal points have no statistical variation in this example.

The problem is to design a PI controller which is robust enough for the variations.

$$C = K_c(1 + 1/T_I s) \quad (17)$$

where K_c is the gain and T_I is the integral time of the

Fig. 4. Flowchart of design algorithm

Table 1. Parameter variations under two different loads (valve lift-flow rate process)

Operating conditions			Process parameters		
Nominal point	Valve lift [%]	Feed flow rate [kg·s ⁻¹]	K_p [kg·s ⁻¹ ·% ⁻¹]	T_1 [s]	T_2 [s]
1	36.489	27.376	1.717	1.505	3.572
2	44.543	57.932	7.533	1.505	3.572

controller. The admissible region of control system performance is given with respect to the damping factor ζ as follows:

$$0.6 \leq \zeta \leq 0.8 \quad (18)$$

The larger time constant T_2 is chosen as the test variable. First, the retuning rule Eq. (12) was used. But due to strong nonlinearity, the calculation was not stable. Thus, the revised retuning rule Eq. (14) with the following weighting factors were used.

$$\lambda_1 = \lambda_2 = 1, \quad \delta_1 = 10^{-1}, \quad \delta_2 = 10^{-4}$$

After four iterations, a robust controller was obtained as shown in Table 2-(a). When new weighting factors ($\lambda_1 = 4.0$, $\lambda_2 = 2.0$, $\delta_1 = 10^{-1}$, $\delta_2 = 10^{-4}$) are chosen to accelerate the calculation, the result in Table 2-(b) was obtained by only one iteration.

4.2 Example 2: catalytic reactor system²⁾

The dynamic behavior of a catalytic reactor system was represented by three nonlinear ordinary differential equations under some assumptions.³⁾ By linearizing the nonlinear dynamic model around the corresponding steady state, a set of linear ODEs was obtained. The transfer function between process input (manipulated signal to the heater wire) and process

output (the temperature of the catalyst bed) was represented as follows:³⁾

$$G(s) = -\frac{K_p(\tau_3 s + 1)e^{-\tau_4 s}}{(\tau_1 s + 1)(\tau_2 s + 1)} \quad (19)$$

where $\tau_1 > \tau_2$.

Usually, the feed flow rate is decreased or increased according to the production schedule. When the flow rate of reactants is changed, the reaction temperature must be changed to maintain constant conversion in the reaction. When the feed flow rate is changed, the parameters in the linear model vary considerably because the process dynamics is nonlinear. Under three typical loads, the process parameters in the transfer function deviate as shown in **Table 3**. All parameters in Eq. (16) except τ_4 vary with load. Since the dead time comes from measurement it is considered as a constant parameter. Therefore, the uncertain parameters in this problem are as follows:

$$p^T = [K_p, \tau_1, \tau_2, \tau_3]$$

Here the set of parameter values at three operating conditions are chosen as the nominal points. Under the assumption that the catalyst bed temperature perturbs in the range of ± 10 K at each nominal operating point, the maximum magnitude of the statistical variation of each plant parameter was determined by using the nonlinear dynamic model³⁾

$$\begin{aligned} \Delta K_p &= 0.2 \text{ K} \cdot \text{V}^{-1}, \quad \Delta \tau_1 = 7.5 \text{ s}, \\ \Delta \tau_2 &= 6.0 \text{ s}, \quad \Delta \tau_3 = 1.0 \text{ s} \end{aligned}$$

There are $2^4 = 16$ vertices for each nominal point. Therefore, there are $3 \times 2^4 = 48$ vertices for the robustness test.

A fixed-gain PI controller is used for temperature control under load changes. The admissible region of system performance is given with respect to gain and phase margin (*GM* and *PM*) as follows:

$$6.0 \leq GM \leq 12.0 \text{ dB}, \quad 25.0 \leq PM \leq 34.0 \text{ deg} \quad (20)$$

The algorithm proposed in section 3 was used to design a robust controller under the specified load changes. The parameter τ_1 was chosen as the test variable because τ_1 is closely related to both the gain and phase margins.

First, the initial values of controller parameters were determined so that the control system satisfies the admissible system performance at nominal point 1.

$$K_c = 0.4263 \text{ V} \cdot \text{K}^{-1}, \quad T_I = 50.3481 \text{ s}$$

The simulation results under these PI parameters are shown in **Fig. 5-(a)**. At nominal point 1 the step response is satisfactory, but at nominal point 3 the

Table 2. Iterations of robust controller design
(a) Case 1 ($\lambda_1 = \lambda_2 = 1.0$, $\delta_1 = 10^{-1}$, $\delta_2 = 10^{-4}$)

Iteration number	K_c [% · s · kg ⁻¹]	T_I [s]	$\Delta \bar{v}^{N_1}$ [s]	$\Delta \bar{v}^{N_2}$ [s]
1	0.800	45.000	0.0764	-0.1048
2	0.785	28.071	0.0328	-0.0848
3	0.780	23.081	0.1218	-0.0747
4	0.781	20.635	0.1243	-0.0747
5	0.782	18.799	-0.0136	-0.0747

(b) Case 2 ($\lambda_1 = 4.0$, $\lambda_2 = 2.0$, $\delta_1 = 10^{-1}$, $\delta_2 = 10^{-4}$)

Iteration number	K_c [% · s · kg ⁻¹]	T_I [s]	$\Delta \bar{v}^{N_1}$ [s]	$\Delta \bar{v}^{N_2}$ [s]
1	0.800	45.000	0.0764	-0.1048
2	0.747	28.025	-0.0243	-0.0347

Fig. 5-1. Step responses under two operating conditions (initial design)

Table 3. Parameter variations under three loads

Operating conditions			Process parameters				
Nominal point	Feed flow rate [cm ³ · s ⁻¹]	Operating temperature [K]	K_p [K · V ⁻¹]	τ_1 [s]	τ_2 [s]	τ_3 [s]	τ_4 [s]
1	3.33	940.0	258.339	3099.073	516.598	546.994	15.0
2	6.67	973.2	200.277	1849.631	497.644	564.788	15.0
3	10.00	994.5	169.620	1360.734	477.954	573.165	15.0

Fig. 5-2. Step responses under two operating conditions (final design)

response is oscillatory.

Since the matrix G_r for retuning was illconditioned, Eqs. (12) and (15) were adopted for the retuning. After two iterations of retuning with $\lambda_1 = \lambda_2 = \delta_1 = \delta_2 = 1$, the following result was obtained.

$$K_c = 0.2733 \text{ V} \cdot \text{K}^{-1}, \quad T_I = 50.3481 \text{ s}$$

The retuning process is summarized in Table 4. The resulting PI controller shows satisfactory responses at both nominal points 1 and 3 as shown in Fig. 5-(b).

Conclusion

An iterative design method for the fixed-gain feedback controller was developed. Characteristic of this method are a simple robustness test by one-dimensional search in the uncertain parameter space, and an intelligible retuning by the modified convergence method. Using this method, engineers can design a fixed-gain feedback control system which will always provide satisfactory system performance under any variation in the specified region.

Appendix: Property for Robustness Test

In the p -space the following sets can be defined with respect to vertex p^i .

$$S_{p1} = [p | p_1 = p_1^i]$$

$$S_{p2} = [p | p_2 = p_2^i]$$

.

$$S_{pk-1} = [p | p_{k-1} = p_{k-1}^i]$$

$$S_{pk+1} = [p | p_{k+1} = p_{k+1}^i]$$

.

$$S_{pr} = [p | p_r = p_r^i]$$

These sets are planes which are orthogonal to the coordinate axes.⁴⁾ The intersection of these sets defines a one-dimensional manifold (line).

$$L_i = S_{p1} \cap S_{p2} \cap \cdots \cap S_{pk-1} \cap S_{pk+1} \cap \cdots \cap S_{pr}$$

Table 4. Iterations of robust controller design ($\lambda_1 = \lambda_2 = \delta_1 = \delta_2 = 1.0$)

Iteration number	K_c [V · K ⁻¹]	T_I [s]	$\Delta \bar{v}^{N_1}$ [s]	$\Delta \bar{v}^{N_2}$ [s]
1	0.426	50.348	-2064.041	625.604
2	0.228	50.348	343.645	-298.660
3	0.273	50.348	-186.088	-95.308

Since p^i is on the line L_i , $p^i \in L_i$.

If S is convex, the intersection S and L_i , $S_v^i = S \cap L_i$ is convex according to the following theorem.

Theorem⁴⁾: If F_i , $i \in I$ is a family (finite or infinite) of convex sets in R^n , then their intersection $\bigcap_{i \in I} F_i$ is a convex set.

If $p^i \in S$, then $p^i \in S_v^i$. This means that $v^i \in S_v^i$. Inversely, the fact $v^i \in S_v^i$ implies that $p^i \in S_v^i$. Therefore, $p^i \in S$.

Nomenclature

C	= transfer function of controller	[—]
G	= transfer function of process	[—]
G_r	= sensitivity matrix of controller parameters	[—]
F_i	= admissible region of i -th system performance	[—]
f_i	= i -th system performance	[—]
K_c	= gain of PI controller	[V · K ⁻¹] or [% · s · kg ⁻¹]
K_p	= process static gain	[K · V ⁻¹] or [kg · s ⁻¹ · % ⁻¹]
L	= number of uncertain parameters	[—]
M	= number of nominal operating points	[—]
N_1, N_2	= indices of farthest points from S	[—]
p	= uncertain parameter vector	[—]
p_j	= j -th element of p	[—]
p_j^0	= nominal value of j -th element	[—]
p^{oi}	= i -th nominal operating point	[—]
q	= parameter vector of controller	[—]
R_i	= required region of tolerance around i -th nominal operating point	[—]
S	= inherent region of tolerance	[—]
S_i	= inherent region of tolerance for i -th performance index	[—]
s	= Laplace transformation variable	[—]
S_v^i	= interval of test variable for i -th vertex	[—]
T_I	= integral time of PI controller	[s]
T_1, T_2	= time constant in Eq. (16)	[s]
Δp_i	= statistical variation	[—]
$\Delta \bar{v}^i$	= length from p_k^i to upper bound of S_v^i	[—]
$\Delta \underline{v}^i$	= length from p_k^i to lower bound of S_v^i	[—]
v	= tests variable	[—]
λ_i, δ_i	= weighting factors	[—]
τ_i	= parameter in transfer function of catalytic reactor	[s]
μ_j^i	= maximum magnitude of statistical variation of j -th element of vertex p^i	[—]
ζ	= damping factor	[—]

Literature Cited

- 1) Hashimoto, I. and T. Takamatsu: Proceedings of Chemical Process Control II, United Engineering Trustees, Inc., 147 (1981).

- 2) Kunugita, E., T. Hashitsume, X. W. Tao and Y. C. Wan: 3rd World Congress of Chem. Eng., Tokyo, Japan, I, 346, Sept. (1986).
- 3) Kunugita, E., H. Nishitani and X. W. Tao: *J. Chem. Eng. Japan* **22**, 258 (1988).
- 4) Mangasarian, O. L.: *Nonlinear Programming*, p. 39, McGraw-Hill Book Company, New York (1969).
- 5) Nishitani, H., X. W. Tao and E. Kunugita: Proceedings of 26th SICE Annual Conference, Vol. II, Intern. Session, Hiroshima, Japan, July 15–17 (1987).