DEVELOPING COMBINED FREE AND FORCED LAMINAR CONVECTION BETWEEN VERTICAL PARALLEL PLATES WITH CONSTANT WALL TEMPERATURE

MASAMOTO NAKAJIMA, KEISUKE FUKUI,
HIROMASA UEDA AND TOKURO MIZUSHINA
*Department of Chemical Engineering,
Himeji Institute of Technology, Himeji 671-22*

A theoretical study is presented on the transfer problem subject to developing flow of mixed convection between vertical parallel plates. Present analysis aims to make clear the transfer mechanism for aiding and opposing flow, based upon and developing Leveque's solution.

Average mass transfer measurements were obtained for both aiding and opposing flow in a vertical annulus by using the electrochemical method, i.e. the reduction of ferricyanide at the cathode in a solution of $K_4Fe(CN)_6$ - $K_3Fe(CN)_6$ -NaOH.

Both theoretical and experimental results show that the parameter Gr_h/Re_h is able to express the effect of the buoyancy force in aiding and opposing flows. With increasing Gr_h/Re_h in the case of aiding flow, Nusselt numbers increase, being proportional to $(Re_h Pr D_h/L)^n$, and the exponent n decreases. As to opposing flow, the buoyancy effect is more remarkable than in the case of aiding flow. It reduces the velocity in the thermal-boundary layer and promotes rapid transition to turbulent state owing to flow instability.

Introduction

Buoyancy effects caused by density distribution are important in the laminar heat transfer processes with high heat flux. In this case, it is necessary to take into consideration both forced and free convection simultaneously. Recently, many authors have been interested in the subject of combined forced and free convection (mixed convection), which is related to heat transfer operations with high heat flux, e.g. in nuclear reactors.

Hallman¹⁾, Hanratty *et al.*²⁾ and Morton⁹⁾ presented exact solutions for combined free and forced convection for fully developed laminar flow in a vertical tube. Rokerya and Iqbal¹¹⁾, as well as Maitra and Subba⁶⁾, studied theoretically the fully developed mixed convection in an annulus. It is more difficult to analyze the problem of developing mixed convection. For developing mixed convection in a vertical tube, Pigford¹⁰⁾ presented an approximate analysis, and Marner and McMillan⁷⁾ investigated the problem theoretically by finite difference methods. Sherwin and Wallis¹³⁾ presented a finite difference solution for developing mixed flow in a vertical annulus with

uniform heat flux at the inner surface. For mixed laminar flow between two parallel plates, Savkar¹²⁾ obtained a linearized-approximate solution subject to uniform heat flux. In contrast to the large number of theoretical studies, the information from experimental work has been limited.

Present work is concerned with developing mixed convection between vertical parallel plates with constant wall temperature. Local and average Nusselt numbers were derived theoretically, based upon Leveque's solution⁵⁾ and extending the approximate analysis presented by Pigford. Experiments were carried out by the electrochemical method and compared with the predicted results.

1. Analysis

The differential equations associated with combined laminar free and forced convection between two parallel plates are the continuity equation, the momentum equation and the energy equation as given below:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (1)$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{dP}{dx} + \nu \frac{\partial^2 u}{\partial y^2} + g\beta(t-t_\infty) \quad (2)$$

$$u \frac{\partial t}{\partial x} + v \frac{\partial t}{\partial y} = \alpha \frac{\partial^2 t}{\partial y^2} \quad (3)$$

Received June 27, 1977. Correspondence concerning this article should be addressed to K. Fukui. H. Ueda is at National Institute for Environmental Studies, Yatabe, Ibaraki 300-21. T. Mizushina is at Dept. of Chem. Eng., Kyoto Univ., Kyoto 606.

The volume rate of flow is constant, so that

$$\int_0^D u dy = \langle u \rangle D = \text{const.} \quad (4)$$

where $\langle u \rangle$ and D are the average velocity and the distance between two parallel plates, respectively. The boundary conditions are

- I. at $y=0$, $u=v=0$
- II. at $y=D$, $u=v=0$
- III. at $y=0$, $t=t_w$ for all $x>0$
- IV. at $x \leq 0$, $t=t_\infty$ for all y

The heat transfer takes place far from the entrance to the conduit, where the laminar velocity profile is fully developed. The last term on the right side of Eq. (2) denotes the buoyancy force due to temperature difference. When the buoyancy term is positive or negative, the flow is aiding or opposing, respectively. Obviously, it is difficult to obtain the exact solution for these differential equations. Hence an approximate solution was derived under the following assumptions:

i) The velocity profile may be regarded as linear within the thermal-boundary layer, with a slope as a function of x which is related to the development of the thermal boundary, that is, $u=B(x)y$.

ii) Inertia terms in Navier-Stokes equation may be neglected when compared with the buoyancy term.

From the equation of continuity and the boundary condition(I) with the first assumption, the velocity components are given as follows:

$$u=B(x)y \quad (5)$$

$$v=-(1/2)y^2(dB/dx) \quad (6)$$

Substitution of Eqs. (5) and (6) into Eq. (3) and solving with boundary conditions (III) and (IV) gives

$$\frac{t-t_\infty}{\Delta t} = \int_\eta^\infty \exp(-\eta^3) d\eta / \Gamma(4/3) \quad (7)$$

where $\Delta t = t_w - t_\infty$, $\eta = y/\delta_c$ and y/δ_c approximates to infinite in the limit to $y \rightarrow D$. δ_c is the thermal boundary-layer thickness as a function of x , which must satisfy the following ordinary differential equation:

$$B \frac{d\delta_c}{dx} + \frac{1}{2} \frac{dB}{dx} \delta_c - \frac{3\alpha}{\delta_c^2} = 0 \quad (8)$$

where the value 3 is an arbitrary constant. Here it should be noted that the thermal boundary-layer thickness is determined not only by $B(x)$ but also by $dB(x)/dx$. Insertion of Eq. (7) into Eq. (2) and integration with respect to y by using the second assumption yields

$$\frac{\partial u}{\partial y} = \frac{1}{\mu} \frac{dP}{dx} y - (g\beta \Delta t / \nu) \delta_c I_1(\eta) + C_1 \quad (9)$$

$$u = \frac{1}{\mu} \frac{dP}{dx} \frac{y^2}{2} - (g\beta \Delta t / \nu) \delta_c^2 I_2(\eta) + C_1 y + C_2 \quad (10)$$

in which C_1 and C_2 are constants of integration and

$$I_n(\eta) = \int_0^\eta \cdots \int_0^\eta (d\eta)^n \int_\eta^\infty \exp(-\eta^3) d\eta / \Gamma(4/3) \quad (11)$$

$$\eta_D = D/\delta_c$$

C_1 and C_2 are evaluated by using the boundary conditions:

$$C_1 = \frac{1}{D} \left[-\frac{1}{\mu} \frac{dP}{dx} \frac{D^2}{2} + (g\beta \Delta t / \nu) \delta_c^2 I_2(\eta_D) \right] \quad (12)$$

$$C_2 = 0 \quad (13)$$

Substitution of Eq. (10) into Eq. (4) gives the Reynolds number:

$$Re = \frac{1}{6} \underline{P} + \frac{1}{2} Gr \delta_c^2 I_2(\eta_D) - Gr \delta_c^3 I_3(\eta_D) \quad (14)$$

in which

$$\underline{P} = -\frac{\rho}{2\mu^2} \frac{dP}{dx} D^3$$

$$Gr = g\beta \Delta t D^3 / \nu^2$$

$$\delta_c = \delta_c / D$$

The velocity gradient at the wall, $B(x)$, is obtained from Eq. (9) by introducing a dimensionless variable $\phi = BD^2/\nu$:

$$\phi = \underline{P} + Gr \delta_c^2 I_2(\eta_D) \quad (15)$$

By using Eqs. (14) and (15), the dimensionless slope ϕ is rewritten as

$$\phi = 6Re + Gr[-2\delta_c I_2(\eta_D) + 6\delta_c^2 I_3(\eta_D)] \delta_c \quad (16)$$

Here, according to Eq. (11), the behavior of $\delta_c I_2(\eta_D)$ and $\delta_c^2 I_3(\eta_D)$ at large η_D is as follows:

$$\delta_c I_2(\eta_D) \propto \left(\frac{\delta_c}{D}\right) \left(\frac{D}{\delta_c}\right) \rightarrow \text{const.}$$

$$\delta_c^2 I_3(\eta_D) \propto \left(\frac{\delta_c}{D}\right)^2 \left(\frac{D}{\delta_c}\right)^2 \rightarrow \text{const.}$$

hence

$$-2\delta_c I_2(\eta_D) + 6\delta_c^2 I_3(\eta_D) = \text{const.} = K_0$$

It is more convenient to work with the terms of dimensionless variables:

$$\xi = x / D Re Pr$$

$$\phi = \phi / Re$$

$$K = \frac{K_0}{2} \frac{Gr}{Re}$$

so that Eqs. (8) and (16) become

$$\phi = 6 + 2K\delta \quad (17)$$

$$\phi \frac{d\delta}{d\xi} + \frac{1}{2} \delta \frac{d\phi}{d\xi} - \frac{3}{\delta^2} = 0 \quad (18)$$

Finally, the problem reduces to solving the above set of equations simultaneously, and it is easy to get the solution. Inserting Eq. (17) into Eq. (18) and solving the resulting differential equation then gives

the following fourth-order algebraic equation for δ as a function of ξ :

$$\xi = \frac{2}{3}\delta^3 + \frac{K}{4}\delta^4 \quad (19)$$

Once ξ is given, then δ , ϕ , dP/dx and u are determined in series by using Eqs. (19), (16), (15) and (10), respectively. The local and average Nusselt numbers are given by

$$Nu_{loc.} = \left[\left(\frac{1}{9} \right)^{1/3} / \Gamma \left(\frac{4}{3} \right) \right] \phi^{1/2} / \left[\int_0^\xi \phi^{1/2} d\xi \right]^{1/3} \quad (20)$$

$$Nu_{ave.} = \frac{1}{\xi} \int_0^\xi Nu_{loc.} d\xi \quad (21)$$

2. Asymptotic Solution to the Free Convection

In the limit to $Re \rightarrow 0$, the above analysis gives a solution for the pure free convection. Here Eq. (17) approaches

$$\phi = 2K\delta \quad (22)$$

When this equation is inserted into Eq. (18), one gets

$$K\delta \frac{d\delta}{d\xi} = \frac{1}{\delta^2}$$

hence

$$\phi = (4K)^{3/4} \xi^{1/4} \quad (23)$$

Substitution of Eq. (23) into Eq. (20) and rewriting in terms of the original variable gives

$$Nu_{loc.} = [(2K_0)^{1/4} / \Gamma(4/3) / 8^{1/3}] (Gr \cdot Pr \cdot D/x)^{1/4} \quad (24)$$

Therefore, the local and average Nusselt numbers for the free convection are given by

$$Nu_{loc.,x} = [(2K_0)^{1/4} / \Gamma(4/3) / 8^{1/3}] (Gr_x \cdot Pr)^{1/4} \quad (25)$$

$$Nu_{ave.,L} = (4/3) [(2K_0)^{1/4} / \Gamma(4/3) / 8^{1/3}] (Gr_L \cdot Pr)^{1/4} \\ = 0.748 (Gr_L \cdot Pr)^{1/4} \quad (26)$$

Except for a small difference in the coefficient, Eq. (26) agrees with the previous experimental results for pure free convection, $Nu_{ave.,L} = 0.671 (Gr_L \cdot Pr)^{1/4}$, which Wilke *et al.*^{14,15)} and Ibl³⁾ obtained by using the electrochemical method.

3. Experiment

The experimental apparatus is schematically shown in Fig. 1. The electrochemical method⁸⁾ was used to measure the average mass transfer coefficients for both aiding and opposing flow in the vertical-annular region between two coaxial circular cylinders. It consists of an outer tube with inside diameter 2.2 cm and inner tube with outside diameter 1.0 cm. Both cylinders were nickel tubes of 100 cm long. The test section was located 60 cm downstream from the entrance. The inlet length was about 50 times the equivalent diameter of the annulus and the flow was presumed to be fully developed. The inner cylinder, the test electrode, was divided into 13 segments of about 2 cm in length which were insulated

Fig. 1 Diagrammatic sketch of experimental equipment

electrically from each other.

The limiting current was measured to obtain the mass transfer rate by the reduction of ferricyanide in a solution of $K_3Fe(CN)_6$ - $K_3Fe(CN)_6$ - $NaOH$. The inner cylinder, the test electrode, was used as the cathode and the outer cylinder as the anode, and so when the main flow was in the downwards or upwards direction, the free convection-mass transfer caused aiding or opposing flow, respectively.

The experiments were made at $Sc = 2300$ to 3000 , $Re_h = 40$ to 2000 , $Gr_h = 450$ to 8.4×10^4 . Physical properties of the solution were evaluated using the bulk concentration and temperature. The density difference between bulk solution and electrode surface was estimated by the method which Wilke *et al.*^{14,15)} adopted (Appendix).

4. Results and Discussion

From the analogy between heat and mass transfer, Nu and Pr were replaced by Sh and Sc , respectively, and the experimental results were compared with the theoretical ones. As a characteristic length, the hydraulic diameter for flow in an annulus was used. Figure 2 shows average Nusselt numbers in the range of low Gr_h/Re_h , i.e. those for forced convection, and also the solutions of the eigenvalue problem for flow between two parallel plates and in an annular passage at $r_i/r_o = 1.0$ cm/2.2 cm. At $Re_h Pr D_h/L > 10^3$, the solutions of the eigenvalue problem are found to approach Leveque's solution, which is also in fair agreement with the experiments for forced convection. The results of the present analysis for aiding flow are shown in Fig. 3, from which it is found that Nusselt numbers are, on the whole, proportional to $(Re_h Pr D_h/L)^n$ and the exponent n decreases with increasing the parameter Gr_h/Re_h . It is obvious that this solution, at $Gr_h/Re_h = 0$, agree with Leveque's solution. The average mass transfer rates measured in laminar aiding flows are plotted in Fig. 4(a), (b) and (c), where broken lines showed

Fig. 2 Comparison of theoretical and experimental results for forced convection

Fig. 3 Calculated local and average Nusselt numbers for aiding flow

Leveque's solutions. The theoretical results agree with the experimental ones. And it is clear that with increasing Gr_h/Re_h the mass transfer rates increase and the exponent n decreases.

Buoyancy effect in mixed convection is stressed remarkably on the local mass transfer rates. **Figure 5** shows the local mass transfer rates in mixed convection between two parallel plates measured by Itoh *et al.*⁴⁾ by means of the electrochemical method. The solid lines calculated by the present analysis are in good agreement with their data and show clearly both the increase of the local Nusselt numbers and the decrease of the exponent n with increasing Gr_h/Re_h .

Figure 6 shows the theoretical curves for opposing flow, i.e. when Gr_h/Re_h is negative. In the case of opposing flow, the buoyancy deters the flow in the thermal-boundary layer and so Nu decreases with increasing $|Gr_h/Re_h|$. Here it should be noted

Fig. 4 Comparison of predicted and measured average Nusselt numbers for aiding flow

that the buoyancy effect is remarkable even at small values of $|Gr_h/Re_h|$ when compared with the case of

aiding flow. In Fig. 6 the solid circles represent the points at which the velocity gradient at the wall becomes zero. In Fig. 7 the results of experiments for opposing flow are plotted. It is clear that mass transfer rates decrease with negative Gr_h/Re_h and that they are not so well in accord with the theoretical curves as in the case of aiding flow. Here, the linear-velocity profile assumption for opposing flow has come into question and flow-instability might induce turbulence in the thermal- or concentration-boundary layer and promote rapid transition to turbulent state even or small values of $Re_h Pr D_h/L$.

Conclusion

Theoretical analysis was made of heat or mass transfer in mixed forced and free convection under reasonable assumptions. The theoretical results were in good agreement with the experiments and the following results were obtained.

1) For aiding and opposing flows, the effect of the buoyancy force can be represented by the parameter Gr_h/Re_h .

2) For aiding flow, the Nusselt numbers are proportional to $(Re_h Pr D_h/L)^n$. The exponent n decreases and Nusselt numbers increase with increasing parameter Gr_h/Re_h .

3) For opposing flow, the Nusselt numbers decrease with increasing $|Gr_h/Re_h|$ and the buoyancy effect is remarkable even at small value of $|Gr_h/Re_h|$ when compared with the case of aiding flow.

Appendix

When the inner cylinder is used as cathode, the density difference between the bulk solution and the cathode-electrode surface is given by

$$\rho_\infty - \rho_i = \Delta C_{\text{ferri}} \frac{d\rho}{dC_{\text{ferri}}} - \Delta C_{\text{ferro}} \frac{d\rho}{dC_{\text{ferro}}} \quad (\text{A-1})$$

where ΔC_{ferro} is estimated as follows. The present experimental results can be correlated in the form

$$Sh \propto (Re_h Sc D_h/L)^n \quad (\text{A-2})$$

where $Sh = (i/n_e F \Delta C)(D_h/\mathcal{D})$. By using Eq. (A-2), the concentration difference ΔC_{ferro} is given by

$$\Delta C_{\text{ferro}} = \Delta C_{\text{ferri}} (\mathcal{D}_{\text{ferro}}/\mathcal{D}_{\text{ferri}})^{n-1} \quad (\text{A-3})$$

Substitution of Eq. (A-3) into Eq. (A-1) gives

$$\rho_\infty - \rho_i = \Delta C_{\text{ferri}} \left\{ \frac{d\rho}{dC_{\text{ferri}}} - \left(\frac{\mathcal{D}_{\text{ferro}}}{\mathcal{D}_{\text{ferri}}} \right)^{n-1} \frac{d\rho}{dC_{\text{ferro}}} \right\} \quad (\text{A-4})$$

n must be determined experimentally. Then, the density difference $(\rho_\infty - \rho_i)$ and so Grashof number can be calculated by use of Eq. (A-4).

Nomenclature

$B(x)$	= velocity gradient at wall	[1/sec]
C_1, C_2	= constant of integration	[—]
C	= concentration	[g-mol/cm ³]
ΔC	= $C_\infty - C_i$, bulk-to-wall concentration difference	[g-mol/cm ³]
D	= distance between two parallel plates	[cm]
D_h	= hydraulic diameter	[cm]

Fig. 5 Comparison of predicted and measured local Nusselt numbers for aiding flow

Fig. 6 Calculated local and average Nusselt numbers for opposing flow

Fig. 7 Comparison of predicted and measured average Nusselt numbers for opposing flow

$\mathcal{D}$	= molecular diffusivity	[cm ² /sec]
F	= Faraday constant, 9.652×10^4	[coulomb/g-equiv]
Gr	= $g \beta \Delta T D^3 / \nu^2$, Grashof number	[—]
Gr_h	= $g \beta \Delta T D_h^3 / \nu^2$, Grashof number	[—]
Gr_L	= $g \beta \Delta T L^3 / \nu^2$, Grashof number	[—]
Gr_x	= $g \beta \Delta T x^3 / \nu^2$, Grashof number	[—]

g	= gravitational acceleration	[cm/sec ²]	λ	= thermal conductivity	[cal/cm·sec°C]
h	= heat transfer coefficient	[cal/cm ² ·sec°C]	μ	= viscosity	[g/cm·sec]
I_n	= defined by Eq. (11)	[—]	ν	= kinematic viscosity	[cm ² /sec]
i	= electric current	[amp]	ξ	= $x/D \cdot Re \cdot Pr$	[—]
K	= $(K_0/2)(Gr/Re)$	[—]	ρ	= density	[g/cm ³]
K_0	= constant	[—]	ϕ	= BD^2/ν	[—]
L	= axial distance from the heated inlet (refer to average)	[cm]	ϕ	= ϕ/Re	[—]
$Nu_{ave.}$	= $h_{ave.}D/\lambda$, average Nusselt number	[—]	<Subscripts>		
$Nu_{ave.,h}$	= $h_{ave.}D_h/\lambda$, average Nusselt number	[—]	ave	= average	
$Nu_{ave.,L}$	= $h_{ave.}L/\lambda$, average Nusselt number	[—]	ferri	= ferricyanide ion	
$Nu_{loc.}$	= $h_{loc.}D/\lambda$, local Nusselt number	[—]	ferro	= ferrocyanide ion	
$Nu_{loc.,h}$	= $h_{loc.}D_h/\lambda$, local Nusselt number	[—]	i	= interface	
$Nu_{loc.,x}$	= $h_{loc.}x/\lambda$, local Nusselt number	[—]	loc	= local	
n	= exponent	[—]	W	= wall	
n_e	= valence charge of an ion	[—]	∞	= bulk	
P	= pressure	[g/cm·sec ²]	Literature Cited		
$\underline{P}$	= dimensionless pressure gradient	[—]	1)	Hallman, T. M.: <i>Trans. ASME</i> , 78 , 1831 (1956).	
Pr	= Prandtl number	[—]	2)	Hanratty, T. J., E. M. Rosen and R. L. Kabel: <i>Ind. Eng. Chem.</i> , 50 , 815 (1958).	
Re	= $\rho\langle u \rangle D/\mu$, Reynolds number	[—]	3)	Ibl, N.: <i>Chem. Ing. Tech.</i> , 35 , 353 (1963).	
Re_h	= $\rho\langle u \rangle D_h/\mu$, Reynolds number	[—]	4)	Itoh, R., Y. Hirata and Y. Inoue: <i>News of Heat Trans. Society Japan</i> , 12 , 18 (1973).	
r_i	= outside diameter of inner tube	[cm]	5)	Leveque, M. A.: <i>Ann. mines</i> , 13 , 201 (1928).	
r_o	= inside diameter of outer tube	[cm]	6)	Maitra, D. and R. K. Subba: <i>Trans. ASME, Ser.C</i> , 97 , 135 (1975).	
Sc	= Schmidt number	[—]	7)	Marnier, W. J. and H. K. McMillan: <i>ibid.</i> , 92 , 559 (1970).	
Sh	= Sherwood number	[—]	8)	Mizushima, T.: "Advances in Heat Transfer", Academic Press, New York, 7 , 87 (1970).	
t	= temperature	[°C]	9)	Morton, B. R.: <i>J. Fluid Mech.</i> , 8 , 227 (1960).	
Δt	= $t_w - t_{\infty}$, bulk-to-wall temperature difference	[°C]	10)	Pigford, R. L.: <i>Chem. Eng. Progr., Symp. Ser.</i> , 51 , 79 (1955).	
u	= velocity in x direction	[cm/sec]	11)	Rokerya, M. S. and M. Iqbal: <i>Int. J. Heat Mass Transfer</i> , 14 , 491 (1971).	
$\langle u \rangle$	= average velocity	[cm/sec]	12)	Savkar, S. D.: 4th IHTC, Paris, vol. IV, NC3.8 (1970).	
v	= velocity in y direction	[cm/sec]	13)	Sherwin, K. and J. D. Wallis: <i>ibid.</i> , vol. IV, NC3.9 (1970).	
x	= axial distance from the heated inlet (refer to local)	[cm]	14)	Wilke, C. R., M. Eisenberg and C. W. Tobias: <i>J. Electrochem. Soc.</i> , 100 , 513 (1953).	
y	= coordinate normal to wall	[cm]	15)	Wilke, C. R., C. W. Tobias and M. Eisenberg: <i>Chem. Eng. Progr.</i> , 49 , 663 (1953).	
α	= thermal diffusivity	[cm ² /sec]	(Presented at the 42nd Annual Meeting of The Soc. of Chem. Engrs., Japan, at Hiroshima, April 1977.)		
β	= coefficient of volumetric expansion	[1/°C]			
$\Gamma(x)$	= gamma function of x	[—]			
δ	= δ_C/D	[—]			
δ_C	= thermal boundary-layer thickness	[cm]			
η	= y/δ_C	[—]			
η_D	= D/δ_C	[—]			