THE ASYMPTOTIC ANALYSIS OF MUTUAL BOUNDARY VALUE PROBLEMS IN PARALLEL PLATE TRANSPORT PHENOMENA

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A compact analytical expression is obtained for the asymptotic overall mass or heat transfer coefficient in double flow mass or heat exchanger systems. The analysis is based on the integral equation for the trans-wall flux. The solution obtained in this paper is shown to be equivalent to one of the generalized eigenvalue problems proposed by Nunge, Gill and Blanco. Moreover the characteristic equation derived in the asymptotic analysis determines the eigenvalue spectrum of the mutual boundary value problem. This method can be similarly applied to co-current and counter-current systems. Various velocity profiles including non-Newtonian flows can be treated by the series expansion method.

Introduction

Industrially important transport phenomena can often be mathematically represented by mutual boundary value problems, that is convective diffusion equations mutually coupled by the boundary conditions. Heat exchangers and hemo-dialysers in laminar flows are typical examples. Nunge and Gill have treated for the first time mutual problems by generalized orthogonal expansion techniques as an extension of the classical Graetz method⁷⁾. Bentwitch has also proposed the integral equation approach, where coupled differential equations are combined and transformed to a Fredholm integral equation of the second kind by way of the boundary condition¹⁾. Exact solutions are expected in principle to be derived by these analytical methods. However, their solutions are composed of infinite series expansions and complicated numerical calculations are inevitable. Some limitations in practical application of the Nunge-Gill procedure are pointed out by Tien and Srinivasan¹⁰⁾.

In this paper, an analytical technique is proposed to derive a compact analytical expression for the asymptotic overall mass transfer coefficient in parallel plate mass exchangers where fully developed laminar flows are interrelated through the semi-permeable wall. The theory is for simplicity confined to the asymptotic state where the axial length of exchangers tends to infinity. The asymptotic behavior is important in practical applications to gain a high exchange performance. The expression derived by the

present method is shown to be equivalent to the asymptotic form of the Nunge-Gill-Blanco method²⁾. Moreover, the positive and negative eigenvalues of the Nunge-Gill-Blanco method in the composite (coupled) system are shown to be determined by the Green functions of the Sturm-Liouville problems in the component flows: the concentrated and dilute flows. These results confirm equivalence and validity of the generalized orthogonal expansion approach and the integral equation approach. Extension of the present method to the case of double-cylinder geometry is straightforward as in the Nunge-Gill method⁸⁾. Simple analytical expressions are not easily obtained in general cases including short and intermediate exchangers, where various approximate methods proposed by Tien and Srinivasan¹⁰⁾, and Seban *et al.*⁹⁾ are convenient for practical use.

Basic equations of the present analysis are the same as those of Nunge and Gill's original work⁷⁾ except for the existence of the finite wall resistance. To develop the theory, Bentwitch's integral equation approach is used. The key point of the analysis is the decomposition of the propagator (or integral kernel) into singular and secular terms. Arbitrary velocity profiles including non-Newtonian flows can be treated in the same framework in this method. In particular, the determining equation of the asymptotic overall mass transfer coefficient in the plug flow case is expressed by elementary functions. Co-current and counter-current cases are similarly treated in this asymptotic analysis. Although mass transfer alone is taken up in this paper, application of the analysis to the heat exchanger case is of course possible.

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1. Basic Formulation

Let C_1 and C_2 be the dimensionless concentrations of the concentrated and dilute flows, respectively. Both C_1 and C_2 are functions of the rectangular coordinate (x, z) in semi-infinite parallel plate geometry. The concentrated and dilute flows are separated by the plane of the semi-permeable wall: $x=a$, as shown in Fig. 1. For simplicity, the following conditions are assumed. 1) Both concentrated and dilute flows are fully developed laminar. 2) Axial (z -directional) diffusion in each flow is neglected compared to convection. 3) Mass capacity in the semi-permeable wall is neglected. 4) Fickian diffusional flux is dominant in the semi-permeable wall; pressure flow is neglected. 5) The solute partition coefficient between the flows is unity. 6) The initial concentration in the dilute flow is zero. 7) Physical properties of the fluids and the semi-permeable wall are constant.

Condition 2), which is widely accepted in the literature, e.g. Nunge and Gill⁷⁾ and Bentwich¹⁾, reduces the problem to a one-dimensional diffusion equation. Conditions 5) and 6) simplify the mathematical description with no loss of generality. Under these conditions, steady-state mass transfer in the parallel plate counter-current system is mathematically formulated in a dimensionless form as, for the convective diffusion equation in the concentrated flow

$$p_1(x) \frac{\partial C_1}{\partial z} = \frac{\partial^2 C_1}{\partial x^2}; \quad 0 < x < a \quad (1)$$

with the initial and boundary conditions

$$C_1(x, 0) = 1; \quad 0 < x < a \quad (2)$$

$$\frac{\partial C_1}{\partial x}(0, z) = 0 \quad (3)$$

for that in the dilute flow

$$p_2(x) \frac{\partial C_2}{\partial z} = -\frac{\partial^2 C_2}{\partial x^2}; \quad a < x < 1 \quad (4)$$

with

$$C_2(x, l) = 0; \quad a < x < 1 \quad (5)$$

$$\frac{\partial C_2}{\partial x}(1, z) = 0 \quad (6)$$

and for the mutual boundary condition

$$\begin{aligned} j(z) &= -Sh_1^{-1} \frac{\partial C_1}{\partial x}(a, z) = -Sh_2^{-1} \frac{\partial C_2}{\partial x}(a, z) \\ &= C_1(a, z) - C_2(a, z) \end{aligned} \quad (7)$$

The system of Eqs. (1)–(7) constitutes a mutual boundary value problem by way of Eq. (7). The system is therefore characterized by six dimensionless quantities: Sh_1 , Sh_2 , $p_1(x)$, $p_2(x)$, a and l . Sh_1 and Sh_2 are wall Sherwood numbers with respect to the concentrated and dilute fluids, respectively. The wall Sherwood number Sh_i is defined as $W/R_w/D_i$, where W

Fig. 1 Schematic diagram of counter-current mass or heat exchanger

is the total width of the exchanger normalized to unity in Fig. 1, R_w is the wall resistance, and D_i is the diffusion coefficient in i -th flow. The dimensionless velocity profiles, $p_i(x)$, $i=1, 2$, are positive definite in the defined regions, but otherwise arbitrary continuous functions. The average value of $p_i(x)$ is defined as the characteristic Peclet number, Pe_i . One-parameter velocity profiles such as n -th power flows can be represented by the Peclet numbers. When $p_i(x)$ approaches infinity, the present mutual problem is reduced to the model for hemo-dialysers with homogeneous dialysate flows^{3,4,6,11)}. The asymptotic case where the dimensionless axial length l tends to infinity is considered in a later section.

For co-current systems, similar formulation is possible by modifying Eqs. (4) and (5) as follows:

$$p_2(x) \frac{\partial C_2}{\partial z} = \frac{\partial^2 C_2}{\partial x^2}; \quad a < x < 1 \quad (8)$$

$$C_2(x, 0) = 0; \quad a < x < 1 \quad (9)$$

In the following analysis, the counter-current problem is mainly treated; the co-current system is similarly analyzed by the present method.

2. The Integral Equation Approach

Now the basic system of differential equations (1)–(7) is transformed to the integral equation for the trans-wall flux $j(z)$. The concentration $C_i(x, z)$ is derived by superposing $j(z)$ with the propagator $G_i(x, z)$ as a weighing function. The propagator $G_i(x, z-z_0)$ is defined by the following boundary value problem.

$$q_i(x) \frac{\partial G_i}{\partial z} = \frac{\partial^2 G_i}{\partial x^2}; \quad 0 < x < a_i \quad (10)$$

$$G_i(x, 0) = 0; \quad 0 < x < a_i \quad (11)$$

$$\frac{\partial G_i}{\partial x}(0, z) = 0 \quad (12)$$

$$\frac{\partial G_i}{\partial x}(a_i, z) = Sh_i \delta(z-z_0) \quad (13)$$

where

$$a_1 = a, \quad a_2 = 1 - a \quad (14)$$

$$q_1(x) = p_1(x), \quad q_2(x) = p_2(1-x) \quad (15)$$

In Eq. (13), $\delta(z)$ is the Dirac delta function. Then the concentration distributions in both flows are shown to be expressed as

$$C_1(x, z) = 1 - \int_0^z G_1(x, z - z_0) j(z_0) dz_0 \quad (16)$$

$$C_2(x, z) = \int_z^l G_2(1 - x, z_0 - z) j(z_0) dz_0 \quad (17)$$

Coupling of these expressions by Eq. (7) leads to the following closed equation for $j(z)$

$$j(z) = 1 - \int_0^z G_1(z - z_0) j(z_0) dz_0 - \int_z^l G_2(z_0 - z) j(z_0) dz_0 \quad (18)$$

where omission of the variable x in the propagator $G_i(x, z)$ means evaluation at the wall surface: $x = a_i$. An equivalent expression for the co-current system is as follows:

$$j(z) = 1 - \int_0^z G_1(z - z_0) j(z_0) dz_0 - \int_0^z G_2(z - z_0) j(z_0) dz_0 \quad (19)$$

Equation (18) determining $j(z)$ in the counter-current system is known as a Fredholm integral equation of the second kind, whereas Eq. (19) in the co-current system is reduced to a Volterra integral equation. In Eq. (19), $j(z)$ is determined by the lower values $j(z_0)$; $0 < z_0 < z$. Direct Laplace transformation is possible in this case, since the r.h.s. of Eq. (19) is composed of convolution integrals. The counter-current case in Eq. (18) is reduced to an essentially different problem; the lower and upper values $j(z_0)$ are necessary to determine $j(z)$. In the asymptotic behavior, however, it is possible to treat the co-current and counter-current systems in a similar manner.

Equation (18) or (19) can in principle be solved by successive iterations; thereby the mutual boundary value problem is solved by substitution of the solution $j(z)$ into Eq. (16) and (17) or equivalent equations for the co-current system. However, the integral equation approach does not simplify the problem substantially; difficulty in deriving the propagator $G_i(x, z)$ and obtaining the trans-wall flux $j(z)$ still remains.

3. The Asymptotic Solution

In this section, the asymptotic solution of Eq. (18) for an arbitrary velocity profile is derived. The analysis is based upon the decomposition of the integral kernel $G_i(z)$ into secular and singular terms. Let $\{\lambda_{n,i}\}$ and $\{\phi_{n,i}(x)\}$, $n=0, 1, 2, \dots$, be the eigenvalues and normalized eigenfunctions of the corresponding Sturm-Liouville problem of Eq. (10) with zero flux boundary conditions; the r.h.s. of Eq. (13) is set to be zero. Zeroth eigenvalue and eigenfunction correspond to the trivial solution. Other eigenvalues $\lambda_{n,i}$, $n=1, 2, \dots$, are all positive definite and form a discretely increasing sequence. Then the Laplace transformation of $G_i(z)$ is expressed as

$$\hat{G}_i(s) = G_i(\infty)/s + \hat{g}_i(s) \quad (20)$$

where

$$G_i(\infty) = \frac{Sh_i}{a_i Pe_i} \quad (21)$$

$$\hat{g}_i(s) = Sh_i \sum_{n=1}^{\infty} \frac{\{\phi_{n,i}(a_i)\}^2}{s + \lambda_{n,i}} \quad (22)$$

Equation (20) is a special case of Eq. (48) derived in Section 4. The inverse transformation of Eq. (20) gives:

$$G_i(z) = G_i(\infty) + g_i(z) \quad (23)$$

where $g_i(z)$ has singularity at $z=0$ and converges to zero when z tends to infinity, and $G_i(\infty)$ is the secular term. Because of singularity in $g_i(z)$, contribution of the neighbour region of $z=0$ is dominant in the integrals involving $g_i(z)$ in Eq. (18). An order estimation is possible for the neighbour region by the mean deviation of $g_i(z)$:

$$\langle z \rangle_i \equiv - \frac{d\hat{g}_i(0)}{ds} / \hat{g}_i(0) \quad (24)$$

Therefore if the condition, $\langle z \rangle_i \ll l$, $i=1, 2$, is satisfied, Eq. (18) is asymptotically equivalent to:

$$\begin{aligned} j(z) + \int_0^{\infty} g_1(z_0) j(z - z_0) dz_0 + \int_0^{\infty} g_2(z_0) j(z + z_0) dz_0 \\ = 1 - G_1(\infty) \int_0^z j(z_0) dz_0 - G_2(\infty) \int_z^l j(z_0) dz_0 \end{aligned} \quad (25)$$

almost everywhere except in the inlet and outlet regions. The l.h.s. of Eq. (25) contains excess terms compared with Eq. (18):

$$\text{excess terms} = \int_z^{\infty} g_1(z_0) j(z - z_0) dz_0 + \int_{l-z}^{\infty} g_2(z_0) j(z + z_0) dz_0 \quad (26)$$

It is shown in the following derivation of the solution of Eq. (25) that the excess terms vanish when z and $l-z$ tend to infinity: in the asymptotic state.

The solution of Eq. (25) is assumed to be the following form:

$$j(z) = j(0) e^{pz} \quad (27)$$

where p is a constant parameter. Equation (25) is simplified by substitution of Eq. (27) as

$$R(p) j(z) = 1 - G_1(\infty) \int_0^z j(z_0) dz_0 - G_2(\infty) \int_z^l j(z_0) dz_0 \quad (28)$$

where

$$R(p) = 1 + \hat{g}_1(p) + \hat{g}_2(-p) \quad (29)$$

The parameter p is determined by the following self-consistent equation derived by differentiating Eq. (28).

$$R(p) = -\Delta G(\infty)/p \quad (30)$$

where

$$\Delta G(\infty) = G_1(\infty) - G_2(\infty) \quad (31)$$

In the following discussions, $\Delta G(\infty)$ is assumed to be non-zero. However, general results remain valid in the limiting case: $\Delta G(\infty) \rightarrow 0$.

Existence of the solution of Eq. (30) can be proved. The orthonormal set of Sturm-Liouville problem is shown to be uniformly bounded, and n -th eigenvalue is asymptotically proportional to n^2 ⁵⁾. These general results of the Sturm-Liouville problem verify convergence of $\hat{g}_i(0)$ in Eq. (22), therefore $R(0)$ exists as a finite number. For the first time, $\Delta G(\infty)$ is assumed to be a negative value. When p approaches the 1st eigenvalue for the dilute flow side, $\lambda_{1,2}$, $R(p)$ diverges to infinity. Because the r.h.s. of Eq. (30) diverges at $p=0$ and is monotonically decreasing, Eq. (30) must have at least one solution: $0 < p < \lambda_{1,2}$. A similar argument in the case, $\Delta G(\infty) > 0$, shows the existence of the solution: $0 > p > -\lambda_{1,1}$. The asymptotic trans-wall flux $j(z)$ derived in such a manner varies therefore more slowly than $g_2(z)$ when $p > 0$, or than $g_1(z)$ when $p < 0$. An additional result for the initial value determined by Eq. (28):

$$j(0) = p / \{e^{p l} G_2(\infty) - G_1(\infty)\} \quad (32)$$

shows that Eq. (26) is asymptotically vanishing.

Equation (28) is equivalent to well-known equations of mass balance for the concentration plug flow system. Therefore $R(p)$ is considered to be the overall mass transfer resistance for the present system. Wall resistance is renormalized to R with additional terms of boundary layer resistances in Eq. (29). R is the asymptotic overall mass transfer resistance normalized by the wall resistance. The effects of boundary layers are not simple summation, but they are interrelated by way of Eq. (30). The equation determining R is obtained by rearranging Eq. (29) as

$$R = 1 + \hat{g}_1(-\Delta G(\infty)/R) + \hat{g}_2(\Delta G(\infty)/R) \quad (33)$$

The asymptotic trans-wall flux $j(z)$ is rewritten explicitly as

$$j(z) = \frac{\Delta G(\infty)}{R \{G_1(\infty) - G_2(\infty) e^{-(\Delta G(\infty)/R) l}\}} e^{-(\Delta G(\infty)/R) z} \quad (34)$$

The total flux is therefore obtained as follows.

$$\int_0^l j(z) dz = (1 - e^{-(\Delta G(\infty)/R) l}) / \{G_1(\infty) - G_2(\infty) e^{-(\Delta G(\infty)/R) l}\} \quad (35)$$

An equivalent expression of Eq. (33) for the co-current system is similarly derived as follows.

$$R = 1 + \hat{g}_1(-\sum G(\infty)/R) + \hat{g}_2(-\sum G(\infty)/R) \quad (36)$$

where

$$\sum G(\infty) = G_1(\infty) + G_2(\infty) \quad (37)$$

4. Equivalence with the Nunge-Gill-Blanco Method

The generalized eigenvalue problem proposed by Nunge, Gill and Blanco^{2,7)} for the present system is formulated as:

$$\frac{d^2 X}{dx^2} + \lambda p(x) X = 0; \quad 0 < x < 1 \quad (38)$$

$$\frac{dX}{dx}(0) = \frac{dX}{dx}(1) = 0 \quad (39)$$

$$X(a-) - X(a+) = -Sh_1^{-1} \frac{dX}{dx}(a-) = -Sh_2^{-1} \frac{dX}{dx}(a+) \quad (40)$$

where

$$p(x) = p_1(x); \quad 0 < x < a \quad (41)$$

$$p(x) = \pm p_2(x); \quad a < x < 1 \quad (42)$$

Hereafter the upper and lower signs are to correspond to co-current and counter-current flows, respectively. Equations (38)–(40) are derived by separating variables in the basic equations (1)–(7). The generalized boundary value problem defined by the above equations differs from usual Sturm-Liouville problems in that $p(x)$ changes signs in the interval and the solution $X(x)$ contains discontinuity at the wall surface $x=a$, therefore rigorous mathematical properties of the orthogonal expansion have not been clarified as pointed out by Tien and Srinivasan¹⁰⁾. However, the formal solution of Eqs. (38)–(40) with the determining equation of the eigenvalue spectrum can be derived. Now we assume eigenvalue λ such that $\lambda \neq \lambda_{n,1}$, $\lambda \neq \pm \lambda_{m,2}$. Then the value defined by Eq. (40) must not be eliminated, therefore the uniqueness of the solution of the boundary value problem defined in the interval, $(0, a)$ or $(a, 1)$, determines the corresponding eigenfunction $X_\lambda(x)$ with the arbitrary proportional coefficient j .

$$X_\lambda(x) = -j \hat{G}_1(x, -\lambda); \quad 0 < x < a \quad (43)$$

$$X_\lambda(x) = j \hat{G}_2(1-x, \mp \lambda); \quad a < x < 1 \quad (44)$$

where $\hat{G}_i(x, s)$ is defined by the following boundary value problem.

$$\frac{d^2}{dx^2} \hat{G}_i(x, s) = s q_i(x) \hat{G}_i(x, s) \quad (45)$$

$$\frac{d \hat{G}_i}{dx}(0, s) = 0 \quad (46)$$

$$\frac{d \hat{G}_i}{dx}(a_i, s) = Sh_i \quad (47)$$

The explicit form is immediately given as

$$\hat{G}_i(x, s) = Sh_i \sum_{n=0}^{\infty} \frac{\phi_{n,i}(a_i) \phi_{n,i}(x)}{s + \lambda_{n,i}} \quad (48)$$

which coincides with the Laplace transform of $G_i(x, z)$ defined by Eqs. (10)–(13) in the convergence region. Equation (40) shows that the parameter j is the discontinuous jump at the semi-permeable wall:

$$j = X_\lambda(a-) - X_\lambda(a+) \quad (49)$$

Equations (43), (44) and (49) derive the following equation which determines the eigenvalue.

$$\hat{G}_1(-\lambda) + \hat{G}_2(\mp \lambda) + 1 = 0 \quad (50)$$

where $\hat{G}_i(s)$ is the same as the Laplace transform of $G_i(z)$ expressed in Eqs. (20)–(22), but defined for all

real numbers beyond the convergence region. Equation (50) is rewritten as

$$1 + \hat{g}_1(-\lambda) + \hat{g}_2(\mp\lambda) = \frac{G_1(\infty) \pm G_2(\infty)}{\lambda} \quad (51)$$

which coincides with Eq. (29) in the counter-current case. Every eigenvalue except $\lambda_{n,1} \pm \lambda_{m,2}$ must satisfy Eq. (50) or Eq. (51).

Equation (50) cannot cover all the eigenvalues completely; e.g. the trivial eigenvalue $\lambda=0$ does not satisfy Eq. (50). When coincidence happens in the two systems of eigenvalues: $\lambda_{n,1} = \pm \lambda_{m,2} = \lambda$ for some n and m , the value λ is shown to be an eigenvalue of the composite system, Eqs. (38)–(40). The trivial eigenvalue corresponds to that special case. For the counter-current system, no such particular eigenvalue exists except the trivial one. The co-current system may allow those particular eigenvalues with corresponding eigenfunctions which are expressed by the eigenfunctions of each component flow.

In the Nunge-Gill-Blanco method²⁾, the asymptotic value of the dimensionless overall mass transfer resistance R is determined by the first non-zero eigenvalue $\lambda_1 = \lambda_1^+$ or λ_1^- as follows.

$$R = \frac{G_1(\infty) \pm G_2(\infty)}{\lambda_1} \quad (52)$$

where the first positive or negative eigenvalue, λ_1^+ or λ_1^- , is degenerated to zero in the counter-current system, and there exists no negative eigenvalue in the co-current system. The resulting expression of Eq. (50) by substituting Eq. (52) for λ is coincident with Eqs. (33) and (36). It is shown in the asymptotic analysis of the integral equation that the asymptotic solution corresponds to the first eigenvalue: $-\lambda_{1,1} < p < \lambda_{1,2}$. Therefore the asymptotic solution determined by Eqs. (33) and (36) is equivalent to the asymptotic form by the Nunge-Gill-Blanco method.

5. Example of the Plug Flow System

As an example of the asymptotic solution derived in this paper, a system composed of plug flows is examined. This is a particular case when the integral kernel can be solved as an elementary function. Although the plug flow model is over-simplified, characteristic behavior of diffusional interaction of two flows via a semi-permeable wall is shown in this model. The plug flow solution of $\hat{g}_i(s)$ defined by Eq. (22) is expressed as

$$\hat{g}_i(s) = \frac{Sh_i}{\sqrt{Pe_i s}} \coth \sqrt{Pe_i s} a_i - \frac{G_i(\infty)}{s} \quad (53)$$

The mean deviation of $g_i(z)$ defined by Eq. (24) for this case is calculated as

$$\langle z \rangle_i = (1/15) a_i^2 Pe_i \quad (54)$$

Therefore the asymptotic state corresponds to the case

where Graetz numbers of both flows tend to zero. Equations (33) and (36) determining the asymptotic overall mass transfer resistance is expressed, for the counter-current system, as

$$\begin{aligned} & \frac{\sqrt{a_2 Sh_2 R}}{\sqrt{1 - G_1(\infty)/G_2(\infty)}} \cot \sqrt{1 - \frac{G_1(\infty)}{G_2(\infty)}} \sqrt{\frac{a_2 Sh_2}{R}} \\ & - \frac{\sqrt{a_1 Sh_1 R}}{\sqrt{G_2(\infty)/G_1(\infty) - 1}} \coth \sqrt{\frac{G_2(\infty)}{G_1(\infty)} - 1} \sqrt{\frac{a_1 Sh_1}{R}} \\ & = 1; \quad \Delta G(\infty) < 0 \end{aligned} \quad (55)$$

$$\begin{aligned} & \frac{\sqrt{a_1 Sh_1 R}}{\sqrt{1 - G_2(\infty)/G_1(\infty)}} \cot \sqrt{1 - \frac{G_2(\infty)}{G_1(\infty)}} \sqrt{\frac{a_1 Sh_1}{R}} \\ & - \frac{\sqrt{a_2 Sh_2 R}}{\sqrt{G_1(\infty)/G_2(\infty) - 1}} \coth \sqrt{\frac{G_1(\infty)}{G_2(\infty)} - 1} \sqrt{\frac{a_2 Sh_2}{R}} \\ & = 1; \quad \Delta G(\infty) > 0 \end{aligned} \quad (56)$$

and for the co-current system

$$\begin{aligned} & \frac{\sqrt{a_1 Sh_1 R}}{\sqrt{1 + G_2(\infty)/G_1(\infty)}} \cot \sqrt{1 + \frac{G_2(\infty)}{G_1(\infty)}} \sqrt{\frac{a_1 Sh_1}{R}} \\ & + \frac{\sqrt{a_2 Sh_2 R}}{\sqrt{1 + G_1(\infty)/G_2(\infty)}} \cot \sqrt{1 + \frac{G_1(\infty)}{G_2(\infty)}} \sqrt{\frac{a_2 Sh_2}{R}} = 1 \end{aligned} \quad (57)$$

Equations (55)–(57) are reduced to the following simple form when the arguments of cotangent and hyperbolic cotangent functions are sufficiently small.

$$R = 1 + \frac{1}{3}(a_1 Sh_1 + a_2 Sh_2) \quad (58)$$

Equation (58), which is exact when $\Delta G(\infty) = 0$, shows simple additivity of the wall resistance and two boundary layer resistances. Beyond those regions, the effects of convective diffusion of both flows become intercorrelated. When Pe_2 tends to infinity, Eq. (56) is reduced to the solution of single flow with a homogeneous boundary condition, $C_2(x, z) = 0$, as follows:

$$\sqrt{a_1 Sh_1 R} \cot \sqrt{a_1 Sh_1 / R} = 1 \quad (59)$$

Equation (59) is qualitatively similar to the asymptotic solution in the parabolic flow system⁶⁾. Similar arguments are possible in the case of arbitrary velocity profile.

An example of the asymptotic solution is shown in Figs. 2 and 3, compared with a numerical solution for the counter-current plug flow system; $a_1 = a_2 = 0.5$, $Sh_1 = Sh_2 = 5$, $Pe_1 = 500$, $Pe_2 = 1000$. The numerical solution was calculated by successive iterations for the following modified integral equation equivalent to Eq. (18).

$$j(z) = \left[1 + \int_0^L G(z, z_0) \{j(z) - j(z_0)\} dz_0 \right] / \left\{ 1 + \int_0^L G(z, z_0) dz_0 \right\}$$

where

$$G(z, z_0) = G_1(z - z_0); \quad z > z_0$$

$$G(z, z_0) = G_2(z_0 - z); \quad z < z_0$$

In Fig. 2, the asymptotic trans-wall flux given by Eq.

(34) agrees well with the numerical solution except in the inlet and outlet regions. If the condition, $\langle z \rangle_i \ll l$, $i=1, 2$, is satisfied, the effect of the inlet and outlet regions becomes negligible. In the asymptotic state described above, the asymptotic total flux given by Eq. (35) gives a good approximation as shown in Fig. 3.

6. Series Solution for Arbitrary Velocity Profile

Simple analytical expressions cannot easily be obtained for the propagators for various velocity profiles other than plug flow. In this section, a series expansion method is described to derive the function $\hat{G}(x, s)$ defined by Eqs. (45)–(47) for the arbitrary velocity profile $q(x)$. Hereafter the subscript $i=1, 2$ is omitted for simplicity of description.

Let $\hat{G}_0(x, s)$ be the plug flow solution of Eqs. (45)–(47); $q(x)$ is replaced by Pe in Eq. (45). The explicit form of $\hat{G}_0$ is

$$\hat{G}_0(x, s) = \frac{Sh}{\sqrt{sPe}} \frac{\cosh \sqrt{sPe} x}{\sinh \sqrt{sPe} a} \quad (60)$$

Let L be a linear operator acting on any function $f(x, s)$ defined in the region, $0 < x < a$. L is defined by the following boundary value problem.

$$\frac{\partial^2}{\partial x^2} Lf = sPeLf + s\delta q(x)f \quad (61)$$

$$\frac{\partial}{\partial x} Lf(0, s) = \frac{\partial}{\partial x} Lf(a, s) = 0 \quad (62)$$

where

$$\delta q(x) = q(x) - Pe \quad (63)$$

Then the following series expression of $\hat{G}$ is easily shown.

$$\hat{G}(x, s) = \sum_{n=0}^{\infty} L^n \hat{G}_0(x, s) \quad (64)$$

The explicit form of operator L is obtained as

$$Lf(x, s) = \int_0^a s\delta q(t)f(t, s) \frac{1}{\sqrt{sPe}} \sinh \sqrt{sPe}(x-t) dt - \frac{\cosh \sqrt{sPe} x}{\sinh \sqrt{sPe} a} \int_0^a s\delta q(t)f(t, s) \frac{1}{\sqrt{sPe}} \times \cosh \sqrt{sPe}(a-t) dt \quad (65)$$

Application of Eq. (64) to any velocity profile gives the correction terms for the plug flow solution derived in the preceding section. In the integral equation approach, the wall surface value $\hat{G}(s)$ is important. The wall surface value of Eq. (65) is simplified as

$$Lf(a, s) = - \int_0^a s\delta q(t)f(t, s) \frac{1}{\sqrt{sPe}} \frac{\cosh \sqrt{sPe} t}{\sinh \sqrt{sPe} a} dt \quad (66)$$

In particular, the first correction term for $\hat{G}(s)$ is given as follows.

$$L\hat{G}_0(s) = \frac{Sh}{Pe} \frac{1}{1 - \cosh 2\sqrt{sPe} a} \int_0^a \delta q(t) \cosh 2\sqrt{sPe} t dt \quad (67)$$

$$(a_1 = a_2 = 0.5, Sh_1 = Sh_2 = 5, Pe_1 = 500, Pe_2 = 1000, l = 80; \langle z \rangle_1 = 8.33, \langle z \rangle_2 = 16.66)$$

Fig. 2 Trans-wall flux for counter-current plug flow system

Fig. 3 Total flux for counter-current plug flow system (identical conditions with Fig. 2 except l)

To estimate convergency of the series solution, the n -th power flow case is examined in the limiting state; $s \rightarrow 0$. The velocity profile of the n -th power flow is expressed as

$$q(x) = \frac{n+1}{n} Pe \left\{ 1 - \left(\frac{x}{a} \right)^n \right\} \quad (68)$$

The first correction term for $\hat{G}(0)$ is obtained by substitution of Eq. (68) in Eq. (67) as follows.

$$\lim_{s \rightarrow 0} L\hat{G}_0(s) = \frac{2}{3(n+3)} aSh \quad (69)$$

The second term is

$$\lim_{s \rightarrow 0} L^2 \hat{G}_0(s) = \frac{2}{3(n+3)(2n+3)} aSh \quad (70)$$

In this particular case, all the higher-order terms vanish. Therefore the exact solution for $\hat{G}(0)$ is obtained as follows.

$$\lim_{s \rightarrow 0} \hat{G}(s) = \frac{G(\infty)}{s} + \left\{ \frac{1}{3} + \frac{2}{3(n+3)} + \frac{2}{3(n+3)(2n+3)} \right\} aSh - \frac{G(\infty)}{s} + \frac{2n^2 + 13n + 17}{3(n+3)(2n+3)} aSh \quad (71)$$

Equation (71) coincides with the numerical solution in the parabolic flow case; that is, $n=2^{(1)}$. Up to the first-order term, the series solution is shown to give a good approximation.

Table 1 Summary of analytical methods described in this paper

	Integral equation approach	Asymptotic analysis	Nunge, Gill, Blanco method
Initial requirement	Single flow propagator for arbitrary velocity profile; $G_1(x, z)$ or $\hat{G}_1(x, s)$ (e.g. Eq. (48) or Eq. (64))		
Basic equations	Integral equation for trans-wall flux (e.g. Eq. (18))	Determining equation for asymptotic overall mass transfer resistance (e.g. Eq. (33))	Generalized eigenvalue problem (Eqs. (38)–(40)) Eigenvalues and eigenfunctions determined by Eqs. (43), (44) and (50)
Solutions	Trans-wall flux and concentration distributions derived by numerical methods	Asymptotic trans-wall flux (e.g. Eq. (34))	Trans-wall flux and concentration distributions derived by N.G.B. method

Conclusions

The integral equation approach is useful for analysis of mutual boundary value problems in transport phenomena. The asymptotic analysis gives the foundation to treat the integral equation and to derive the transport characteristics of the system. The eigenvalue spectrum and corresponding eigenfunctions of mutual boundary value problems are determined by those of each single flow problem by way of the transform equation derived in the asymptotic analysis. Thus solutions of various single flow problems can be applied straightforwardly to the mutual boundary value problems.

Nomenclature

a	= dimensionless width of the concentrated flow
a_1	= a
a_2	= $1 - a$ = dimensionless width of the dilute flow
$C(x, z)$	= dimensionless concentration
$f(x, s)$	= arbitrary continuous function
$G(z)$	= wall surface value of $G(x, z)$
$G(x, z)$	= propagator function defined by Eqs. (10)–(13)
$g(z)$	= singular part of $G(z)$
j	= parameter defined by Eqs. (43) and (44)
$j(z)$	= dimensionless trans-wall flux
L	= linear operator defined by Eqs. (61)–(63)
l	= dimensionless axial length
Pe	= Peclet number (average value of $p(x)$)
p	= parameter defined by Eq. (27)
$p(x)$	= dimensionless velocity profile
$q(x)$	= dimensionless velocity profile defined by Eq. (15)
R	= dimensionless asymptotic overall mass transfer resistance
Sh	= wall Sherwood number
s	= Laplace parameter
$X_2(x)$	= eigenfunction of the generalized eigenvalue problem belonging to eigenvalue λ
x	= dimensionless transverse coordinate

z	= dimensionless axial coordinate
$\langle z \rangle$	= mean deviation of $g(z)$ defined by Eq.(24)
$\delta q(x)$	= $q(x) - Pe$
$\Delta G(\infty)$	= $G_1(\infty) - G_2(\infty)$
λ	= eigenvalue of the generalized eigenvalue problem; Eqs. (38)–(42)
$\lambda_{n,i}$	= n -th eigenvalue of the Sturm-Liouville problem of i -th flow
$\phi_{n,i}$	= n -th eigenfunction of the Sturm-Liouville problem of i -th flow
$\Sigma G(\infty)$	= $G_1(\infty) + G_2(\infty)$
$\langle \text{Subscripts} \rangle$	
i	= 1, 2
1	= for concentrated flow
2	= for dilute flow
m	= non-negative integer
n	= non-negative integer
0	= for plug flow (in Section 6)
$\langle \text{Superscript} \rangle$	
$\wedge$	= Laplace transform

Literature Cited

- 1) Bentwich, M.: *J. Heat Transfer, Trans. ASME, Ser. C*, **95**, 458 (1973).
- 2) Blanco, J. A. and W. N. Gill: *Chem. Eng. Prog. Symp. Ser.*, No. 77, **63**, 66 (1967).
- 3) Colton, C. K., K. A. Smith, P. Stroeve and E. W. Merrill: *AIChE J.*, **17**, 773 (1971).
- 4) Cooney, D. O., S. S. Kim and E. J. Davis: *Chem. Eng. Sci.*, **29**, 1731 (1974).
- 5) Courant, R. and D. Hilbert: "Methods of Mathematical Physics", Vol. 1, Interscience Publishers (1953).
- 6) Grimsrud, L. and A. L. Babb: *Chem. Eng. Prog., Symp. Ser.*, No. 66, **62**, 20 (1966).
- 7) Nunge, R. J. and W. N. Gill: *Int. J. Heat Mass Transfer*, **8**, 873 (1965).
- 8) Nunge, R. J. and W. N. Gill: *AIChE J.*, **12**, 279 (1966).
- 9) Seban, R. A., T. C. Hsieh, and R. Greif: *J. Heat Transfer, Trans. ASME, Ser. C*, **94**, 391 (1972).
- 10) Tien, C. and S. Srinivasan: *AIChE J.*, **15**, 39 (1969).
- 11) Walker, G. and T. Davies: *ibid.*, **20**, 881 (1974).