CONVOLUTION MASS TRANSFER FROM A CYLINDER TO THE GAS FLOWING LAMINARLY IN A PARALLEL-PLATE DUCT

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Convective sublimation from a cylinder fixed in a parallel-plate duct has been studied, both theoretically and experimentally. In the theory, a simple and efficient finite difference technique has been developed to solve the Navier-Stokes and diffusion equations. The procedure contains a transformation of the coordinate system to facilitate the treatment for curved boundaries. The calculated results concerning the average Sherwood number are in very close agreement with those of the experiment.

Through a comparison of the results for bounded flow in a duct with those for unbounded flow in an infinite region, it is found that the duct wall tends to depress the development of the wake behind a cylinder, and to promote mass transfer from it.

Introduction

Heat and/or mass transfer between a rigid body and its surrounding stream is of great importance in many chemical processes. A large number of experimental and theoretical articles on this subject are available. It is well known that the numerical solution for a forced-convection fluid flow problem gives reliable information when a simple body is immersed in an infinite region of a low Reynolds number flow field.1,2,4,6 If a duct wall is sufficiently close to the body immersed, the interaction between the wall and the body has a great influence on the flow pattern around the body9 and on the mass transfer processes between the fluid and the body. Quantitative evaluation of this interaction has been rarely conducted.

This paper deals with the influence of the Reynolds number on the steady-state laminar mass transfer from a cylinder fixed in a parallel-plate duct. Numerical solutions of the Navier-Stokes and diffusion equations are obtained by using a finite difference method. In the analysis, the usual difficulties encountered in the development of a finite difference scheme in an irregularly-shaped domain have been overcome by application of a conformal transformation of the flow region. The numerical results for the mass transfer rate were compared with experimental ones obtained from sublimation experiments using a naphthalene cylinder in a nitrogen stream, and further with theoretical results for an unbounded region.8

1. Theory

1.1 Mathematical formulation

Consider the convective mass transfer from a centrally placed, fixed cylinder in a parallel-plate duct to an incompressible Newtonian fluid flowing inside the duct as shown in Fig. 1(a). The assumptions are as follows:

i) The system is isothermal and the cylinder surface maintains a saturated concentration of the subliming species.

ii) The geometric change of the cylinder due to sublimation is so slow that the transfer processes can be considered to be steady in nature.

iii) Both at the inlet and at the outlet, the fluid has uniform velocity profiles of equal magnitude.

iv) The fluid has no mass fraction of the vapor of the sublimable material at the inlet and has no axial concentration gradient at the outlet.7

Under these assumptions, using the characteristic velocity: \( \mu/p \cdot y_1 \); characteristic length: \( y_1 \); and characteristic concentration: \( C_0 \), the governing non-dimensional equations can be written in rectangular coordinates as:

\[
\begin{align*}
\frac{\partial \phi}{\partial Y} + \frac{\partial \phi}{\partial X} + \frac{\partial \phi}{\partial Y} \cdot \frac{\partial \zeta}{\partial Y} - \frac{\partial^2 \zeta}{\partial X^2} + \frac{\partial^2 \zeta}{\partial Y^2} & = 0 \\
\zeta &= \frac{\partial^2 \phi}{\partial X^2} + \frac{\partial^2 \phi}{\partial Y^2} \\
- \frac{\partial \phi}{\partial Y} \cdot \frac{\partial Z}{\partial X} + \frac{\partial \phi}{\partial X} \cdot \frac{\partial Z}{\partial Y} + \frac{1}{Sc} \left[ \frac{\partial^2 Z}{\partial X^2} + \frac{\partial^2 Z}{\partial Y^2} \right] & = 0
\end{align*}
\]
Where the stream function and vorticity are related to the velocity components by the following equations:

\[ U = -\frac{\partial \phi}{\partial Y}, \quad V = \frac{\partial \phi}{\partial X} \]
\[ \zeta = \frac{\partial V}{\partial X} - \frac{\partial U}{\partial Y} \]

The boundary conditions are formulated as follows:

\[ a^0) \quad R = R_0; \quad \phi = 0, \quad \zeta = \partial^2 \phi / \partial R^2, \quad Z = 1 \]
\[ b^0) \quad Y = 0, \quad (0 < X < X_1 \text{ or } X_3 < X < X_4); \quad \phi = 0, \quad \zeta = 0, \quad \partial Z / \partial Y = 0 \]
\[ c^0) \quad X = 0, \quad 0 \leq Y \leq 0.5; \quad \phi = -Re \cdot Y, \quad \zeta = \partial^2 \phi / \partial X^2, \quad Z = 0 \]
\[ d^0) \quad X = X_4, \quad Y = 0, \quad 0 \leq Y \leq 0.5; \quad \phi = -Re \cdot \partial Y / \partial X, \quad \zeta = \partial^2 \phi / \partial X^2, \quad \partial Z / \partial X = 0 \]
\[ e^0) \quad Y = 0.5, \quad 0 < X < X_4; \quad \phi = -0.5 \cdot Re, \quad \zeta = \partial^2 \phi / \partial Y^2, \quad \partial Z / \partial Y = 0 \]

\[ 1.2 \ Coordinate \ transformation \]

As shown in Fig. 1(a), the physical plane is irregularly-shaped and unsuitable for being analyzed by the simple finite difference method. So the physical \((X, Y)\) plane is mapped into a uniform rectangular region in a transformed \((\xi, \eta)\) plane (Fig. 1(b)). For simplicity, the transformed coordinates \(\xi\) and \(\eta\), where \(1 \leq \xi \leq \xi_4\), and \(1 \leq \eta \leq \eta_1\), are chosen to be mutually orthogonal. For this transformation to be valid, it can be shown that the following conditions apply.\(^{10,11}\)

\[ \partial^2 \xi / \partial X^2 + \partial^2 \xi / \partial Y^2 = 0 \]
\[ \partial^2 \eta / \partial X^2 + \partial^2 \eta / \partial Y^2 = 0 \]

The above relations between physical and mapped domain are sufficient to give the following equations:

\[ \partial \xi / \partial X = (1/J) \cdot (\partial Y / \partial \eta) \]
\[ \partial \xi / \partial Y = -(1/J) \cdot (\partial X / \partial \eta) \]
\[ \partial \eta / \partial X = -(1/J) \cdot (\partial Y / \partial \xi) \]
\[ \partial \eta / \partial Y = (1/J) \cdot (\partial X / \partial \xi) \]
\[ J = (\partial X / \partial \xi) \cdot (\partial X / \partial \eta) - (\partial X / \partial \xi) \cdot (\partial Y / \partial \xi) \]
\[ \alpha = (\partial X / \partial \xi)^2 + (\partial \eta / \partial \eta)^2 \]
\[ \beta = (\partial X / \partial \xi) \cdot (\partial X / \partial \eta) + (\partial Y / \partial \xi) \cdot (\partial Y / \partial \eta) \]
\[ \gamma = (\partial X / \partial \xi)^2 + (\partial \eta / \partial \eta)^2 \]

\[ DX = \alpha (\partial^2 X / \partial \xi^2) - 2\beta (\partial^2 X / \partial \xi \partial \eta) + \gamma (\partial^2 X / \partial \eta^2) \]
\[ DY = \alpha (\partial^2 Y / \partial \xi^2) - 2\beta (\partial^2 Y / \partial \xi \partial \eta) + \gamma (\partial^2 Y / \partial \eta^2) \]

\[ (1/J) \cdot \{(\partial X / \partial \xi) \cdot DX - (\partial X / \partial \xi) \cdot DY\} = \tau = \{(\partial X / \partial \xi) \cdot DY - (\partial X / \partial \eta) \cdot DX\} \]

Since all numerical computations must be done in the rectangular transformed plane, it is necessary to interchange the dependent variables in Eqs. (11) and (12), thus,

\[ \frac{\partial^2 \xi}{\partial \xi^2} - 2\beta \frac{\partial^2 \xi}{\partial \xi \partial \eta} + \gamma \frac{\partial^2 \xi}{\partial \eta^2} + \sigma \frac{\partial \xi}{\partial \eta} + \tau \frac{\partial \xi}{\partial \xi} = 0 \]

Using Eqs. (13) to (26), the governing Eqs. (1) to (3) can be written as

\[ \frac{\partial \phi}{\partial \eta} \cdot \frac{\partial^2 Z}{\partial \xi^2} + \frac{\partial Z}{\partial \eta} \cdot \frac{\partial^2 \phi}{\partial \xi^2} = \frac{1}{J} \]

In terms of the transformed coordinates, the boundary conditions are as follows:

\[ a^1) \quad \eta = 1, \quad \xi_1 \leq \xi \leq \xi_3; \quad \phi = 0, \quad \zeta = \gamma (\partial^2 \phi / \partial \xi^2), \quad Z = 1 \]
\[ b^1) \quad \eta = 1, \quad (1 \leq \xi < \xi_1 \text{ or } \xi_3 < \xi \leq \xi_4); \quad \phi = 0, \quad \zeta = 0, \quad \partial Z / \partial \eta = 0 \]
\[ c^1) \quad \xi = 1, \quad 1 \leq \eta \leq \eta_1; \quad \phi = -Re \int_0^\eta (\partial Y / \partial \eta) d\eta, \quad \zeta = 0 \]
\[ \zeta = (1/J^2) \cdot [\alpha(\partial^2 \phi/\partial \xi^2) + \gamma(\partial^2 \phi/\partial \eta^2) + \sigma(\partial \phi/\partial \eta)] , \]
\[ Z = 0 \]  
\[ d^1 \quad \xi = \xi_4 , \quad 1 \leq \eta \leq \eta_1 ; \]
\[ \phi = -Re \int_0^\eta (\partial Y/\partial \eta) d\eta ; \]
\[ \zeta = (1/J^2) \cdot [\alpha(\partial^2 \phi/\partial \xi^2) + \gamma(\partial^2 \phi/\partial \eta^2) + \sigma(\partial \phi/\partial \eta)] , \]
\[ \partial Z/\partial \xi = 0 \]  
\[ e^1 \quad \eta = \eta_1 , \quad 1 < \xi < \xi_4 ; \]
\[ \phi = -0.5Re , \quad \zeta = (\gamma/J^2) \cdot (\partial^2 \phi/\partial \eta^2) , \]
\[ \partial Z/\partial \eta = 0 \]  

1.3 Numerical technique

First, all the derivatives in Eqs. (25) and (26) are approximated by a second-order central difference expression.\(^\text{a2}\) When the solid boundary mesh points for \((X, Y)\) and \((\xi, \eta)\) are given, the set of nonlinear simultaneous difference equations can be solved by the point Gauss-Seidel iteration.\(^\text{a3}\) Then the coordinate values for \(X\) and \(Y\) can be specified at all the equi-spaced \(\xi, \eta\)-points of the transformed plane. Using the obtained \(X\) and \(Y\), all the coordinate transformation-parameters, such as \(J, \alpha, \beta, \gamma, DX, DY, \sigma\) and \(\tau\), are evaluated numerically and stored in a magnetic disc memory area of an electronic computer as mesh-points information. Then, under the given transport parameters, such as Reynolds number and Schmidt number, a numerical computation of the transformed transport equations can be carried out. To retain computational stability, unsteady terms, \(J(\partial \xi/\partial T)\) and \(J(\partial Z/\partial T)\), are added to Eqs. (27) and (29), respectively. When the procedure using backward difference schemes for the time derivatives, i.e. the implicit method, is adopted in the finite difference discretization, computation can be carried out easily and with stability, even though central difference schemes are adopted in convective and diffusive terms. The solution for steady state is obtained as an asymptotic solution for large time of an unsteady state. Figure 2 illustrates the salient features of the

\(^a^2\) For \(\alpha, \beta\) and \(\gamma\), Eqs. (18) to (20) are usable.

\(^a^3\) The mesh widths in the transformed plane, \(d\xi\) and \(d\eta\), are both taken as unity for convenience; then there are \(\xi_4 \times \eta_1\) mesh points in the plane.
computational procedure adopted here.

2. Experimental

2.1 Apparatus and method

The apparatus used in the experiments is the rectangular duct as shown in Fig. 3. It is made of 0.006 m-thick brass and consists of entry section: 1, test section: 2 and exist section: 3. A naphthalene cylinder*4 of 0.01 m diameter denoted by D is centered in the test section. The glass-wool filters (Toyo Roshi Co., Ltd. GC-90) denoted by C and E help to assure uniform fluid velocity at the inlet and outlet of the test section, respectively.

The experiment was conducted in an air bath maintained at 313±0.1 K. At low flow rates, an evolution of the density-driven convection caused by concentration variation may be anticipated. To make clear the effect of its convection, two different flow types, namely upward and downward, were selected. Sublimation rates were determined by measurement of naphthalene concentration in the effluent gas using a gas chromatograph.

2.2 Physical properties

Since no experimental value for the mutual diffusion coefficient of the naphthalene-nitrogen system is available in the literature, it was estimated through the correction12) of the value predicted by the Fuller equation.3) Other properties were quoted from the "Chemical Engineers' Handbook of Japan." These values are shown in Table 1.

3. Results and Discussion

Figure 4 shows the contours of \( \zeta \) and \( \eta \), thereby giving the actual 51 \( \times \) 21 mesh pattern in the physical plane. It is clear that any mesh in the field is nearly equal in area and therefore numerical analysis of the governing equations is expected to give an accurate result.

Figure 5 shows the stream function and concentration contours obtained*5 for \( Re_{cy} = 5, 10, 20 \) and 40. The flow pattern of \( Re_{cy} = 5 \) (Fig. 5a) is almost symmetrical about the cylinder and is similar to a creeping fluid flow pattern around a cylinder. In the case of \( Re_{cy} = 10 \) (Fig. 5b), a small wake appears behind the cylinder. The scale of the wake increases with \( Re_{cy} \).

The non-dimensional length of the wake and the separation angle from the rear stagnation point are tabulated in Table 2, compared with the following theoretical correlations for an unbounded flow around a cylinder.7)

\[
L_w = \frac{l_w}{2r_0} = 0.12Re_{cy} - 0.748 \quad (35)
\]

\[
\theta_s = 40.4 \cdot (ln Re_{cy} - 1.83)^{0.456} \quad (36)
\]

It can be seen from the table that the evolution or development of the wake is depressed*6 by the existence of a wall.

Average Sherwood numbers,*7 corresponding to a

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*4 Prepared by coating naphthalene on a wooden cylinder.

*5 The computer used is the ACOS 77/900, and the CPU time required was about 200 seconds for convergence in each condition.

*6 In our analysis, it can be presumed that the outlet boundary condition, the velocity being flat, contributes to the depression of the wake development at high Reynolds number flow.

*7 When saturation concentration is adopted as a driving force, the following relation is obtained:

\[
Sh = Sc \cdot Re \cdot Z_{sat}/\pi
\]
non-dimensional mass transfer rate, obtained both from the numerical analysis and from the experiments are shown in Fig. 6. A theoretical correlation line calculated from Eq. (37) for an unbounded flow is also shown in the figure and compared with this work.

\[
Sh = 0.462 \cdot (R_{\text{cyl}} \cdot Sc)^{0.1} + \frac{2.5}{\left[ 1 + (1.25Sc^{1.667} + 2.5)^{0.4} \right]} \cdot \frac{(R_{\text{cyl}} \cdot Sc)^{0.7}}{1 + 2.79(R_{\text{cyl}} \cdot Sc)^{0.2}}
\]

(37)

Figure 6 shows that a bounded flow gives a higher mass transfer rate than an unbounded one. This is because, near the cylinder, the fluid velocity in the former is larger than that in the latter. Even at very low Reynolds number, the difference in experimental Sherwood number between two flow types, upward and downward, is so small that density-driven convection can be neglected. Regardless of its convection, the theoretical line is in fair agreement with the experimental values.\(^{8\text{a}}\) In a flow range with a high Reynolds number, there appears some difference between the theoretical and the experimental results. This may be attributed to our experimental incompleteness with respect to the isothermal condition. At high flow rate, the large consumption of heat of sublimation results in lowering of the surface temperature. Under the condition mentioned above, the surface concentration of the subliming material is lower than that under the exact isothermal condition. Consequently, sublimation rate decreases.

**Conclusion**

Convective mass transfer from a cylinder fixed in a parallel-plate duct has been studied, both theoretically and experimentally. In the theory, a simple and efficient finite difference technique has been developed. The procedure contains a transformation of the coordinate system to facilitate the treatment for curved boundaries. The numerical results concerning average Sherwood number are very close to the experimental ones. This shows that the numerical procedure postulated here is appropriate.

Through a detailed comparison of the bounded flow with the unbounded one, it is found that the duct wall tends to depress the evolution or the development of the wake behind the cylinder and to accelerate mass transfer from the cylinder.

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\(^{8\text{a}}\) To support the validity of neglecting the density convection, some computations were carried out by considering a gravitational term (in this system, \(Gr = 1500\)). Scarcely any difference could be found in the results.
Acknowledgment

The authors wish to thank Mr. T. Okumura for carrying out a part of the experimental work.

Nomenclature

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C$</td>
<td>concentration [mol/m$^3$]</td>
</tr>
<tr>
<td>$C_0$</td>
<td>saturated concentration [mol/m$^3$]</td>
</tr>
<tr>
<td>$D$</td>
<td>binary diffusion coefficient [m$^2$/s]</td>
</tr>
<tr>
<td>$DX$</td>
<td>coordinate transformation parameter defined by Eq. (21) [-]</td>
</tr>
<tr>
<td>$DY$</td>
<td>coordinate transformation parameter defined by Eq. (22) [-]</td>
</tr>
<tr>
<td>$J$</td>
<td>coordinate transformation parameter defined by Eq. (17) [-]</td>
</tr>
<tr>
<td>$L_w$</td>
<td>dimensionless wake length, $l_w/(2r_0)$ [m]</td>
</tr>
<tr>
<td>$l_w$</td>
<td>wake length [m]</td>
</tr>
<tr>
<td>$r_0$</td>
<td>saturated partial pressure of sublimable material [Pa]</td>
</tr>
<tr>
<td>$R$</td>
<td>dimensionless radial distance, $r/r_0$ [-]</td>
</tr>
<tr>
<td>$r$</td>
<td>radial coordinate [m]</td>
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<td>$Re$</td>
<td>Reynolds number based on duct height, $\rho \cdot u_0 \cdot u_0/\mu$ [-]</td>
</tr>
<tr>
<td>$Re_{st}$</td>
<td>Reynolds number based on cylinder diameter, $2 \rho \cdot u_0 \cdot u_0/\mu$ [-]</td>
</tr>
<tr>
<td>$R_0$</td>
<td>dimensionless radius of cylinder, $r_0/z_0$ [-]</td>
</tr>
<tr>
<td>$r_0$</td>
<td>radius of cylinder [m]</td>
</tr>
<tr>
<td>$Sc$</td>
<td>Schmidt number, $\mu/\rho \cdot D$ [-]</td>
</tr>
<tr>
<td>$Sh$</td>
<td>Sherwood number, $Sc \cdot Re \cdot Z_{out}/\pi$ [-]</td>
</tr>
<tr>
<td>$U$</td>
<td>$x$-component of dimensionless fluid velocity, $\rho \cdot u_0 \cdot u_0/\mu$ [-]</td>
</tr>
<tr>
<td>$u$</td>
<td>$x$-component of fluid velocity [m/s]</td>
</tr>
<tr>
<td>$u_0$</td>
<td>$x$-component of fluid velocity at inlet or outlet [m/s]</td>
</tr>
<tr>
<td>$V$</td>
<td>$y$-component of dimensionless fluid velocity, $\rho \cdot u_0 \cdot u_0/\mu$ [-]</td>
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<tr>
<td>$v$</td>
<td>$y$-component of fluid velocity [m/s]</td>
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<td>$X$</td>
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<td>duct height [m]</td>
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<tr>
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<tr>
<td>$Z_{out}$</td>
<td>dimensionless outlet concentration [-]</td>
</tr>
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</table>

\[\beta = \text{coordinate transformation parameter defined by Eq. (19)}\]
\[\gamma = \text{coordinate transformation parameter defined by Eq. (20)}\]
\[\Delta \eta = \text{finite grid-size in } \eta \text{ direction} \]
\[\Delta \zeta = \text{finite grid-size in } \zeta \text{ direction} \]
\[\xi = \text{vorticity defined by Eq. (5)} \]
\[\eta = \text{transformed dimensionless coordinate (} \eta; \text{ see Fig. 1) } \]
\[\theta_A = \text{separation angle \[^{\circ}\]} \]
\[\mu = \text{viscosity [kg/(m \cdot s)]} \]
\[\rho = \text{density [kg/m}^3\text{]} \]
\[\sigma = \text{coordinate transformation parameter defined by Eq. (23)} [-] \]
\[\tau = \text{coordinate transformation parameter defined by Eq. (24)} [-] \]
\[\phi = \text{stream function defined by Eq. (4)} [-] \]

Literature Cited


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