The necessary radius and length of material to avoid end effect in determining thermal conductivity by the transient method is discussed mathematically and experimentally. It is found that the dimensions of the material can be determined by the conditions $b \geq \sqrt[4]{a^2 \alpha t}$ (Eq.(24)), and $l \geq (0.4)\sqrt[4]{a^2 \alpha t}$ (Eq.(30)), by using the data after the elapsed time given by $t \geq 20a^2/\alpha$ (Eq.(16)). These critical conditions are demonstrated to be valid experimentally and may be applied widely to the transient method.

Introduction

The usual methods of measuring the thermal conductivity of a material are classified as steady state method and transient method. In each method, various kinds of procedure have been used.

From a given time, an electric current is passed through a thin straight wire, placed in the homogeneous material of which the thermal conductivity is to be measured. The constant heat production in the wire causes a cylindrical temperature field in the material. The rise of the surface temperature of the wire is dependent on the thermal properties of this material. This is the principle of measuring the thermal conductivity by the transient method. The transient method is useful for materials of lower values of thermal conductivity and has the advantage that thermal conductivity can be measured during short elapsed time$^{5,6}$. However, the relation between material dimensions and end effect has not been discussed sufficiently$^4$. It is the purpose of this paper to deduce the relation theoretically and demonstrate it experimentally.

Mathematical Representation of the End Effect

A thin wire, used as a heater, is immersed along the axis of the cylindrical material, as shown in Fig. 1. The outer temperature of the material is maintained at $T_o$. As the wire is usually a good conductor, the radial temperature distribution in the wire is assumed to be negligibly small. A constant heat ($A_o$) is produced in the wire. A thermal contact-resistance exists at the boundary between wire and material. Then the temperature profile in the material is determined by the following equations and boundary conditions;

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} - \frac{1}{\alpha} \frac{\partial T}{\partial t} = 0$$  

$T=T_o, \quad t=0$  

$T=T_o, \quad z=\pm l$  

![Fig. 1 The coordinate system](image-url)
\[ T = T_0, \quad r = b \]  
\[ -\lambda \frac{\partial T}{\partial r} = H(T_h - T), \quad r = a \]  
\[ (4) \]

The temperature in the wire is determined by the following equation and initial condition:

\[ \frac{\partial^2 T_h}{\partial z^2} - \frac{2H}{a_1} (T_h - T_a) - \frac{1}{\alpha_1} \frac{\partial T_h}{\partial t} = -A_0 \frac{a^2}{T_0} \]  
\[ T_h = T_0, \quad t = 0 \]  
\[ (6) \]

Using the non-dimensional variables, the above equations and boundary conditions become as follows:

In material

\[ \frac{\partial^2 \theta}{\partial z^2} + \frac{1}{r_s} \frac{\partial \theta}{\partial r_s} + \frac{\partial^2 \theta}{\partial z_+^2} - \frac{\partial \theta}{\partial \tau} = 0 \]  
\[ \theta = 0, \quad \tau = 0 \]  
\[ \theta = 0, \quad z_+ = \pm l/a \]  
\[ \theta = 0, \quad r_+ = b/a \]  
\[ \frac{\partial \theta}{\partial r_s} = - \frac{a H}{\lambda} (\theta_h - \theta), \quad r_s = 1 \]  
\[ (5) \]

In wire

\[ \frac{\partial^2 \theta_h}{\partial z_+^2} + \frac{2H a}{\lambda_1} (\theta_h - \theta) - \frac{\alpha}{\alpha_1} \frac{\partial \theta_h}{\partial \tau} = -A_0 \frac{a^2}{T_0} \]  
\[ \theta_h = 0, \quad \tau = 0 \]  
\[ (7) \]

From the above equations and boundary conditions, the Laplace transformation \( \Theta(1,0,\rho) \) of the surface temperature \( \theta(1,0,\tau) \) at the center \((r_s = 1, z_+ = 0)\) of the wire becomes

\[ \Theta(1,0,\rho) = \frac{A_0 a^2}{4 \lambda_1 T_0} \int_{\rho = 0}^{\rho} \frac{4(-1)^n ((2n+1))}{(2n+1)!} F_n \]  
\[ \times \left[ 1 + \frac{\alpha}{\alpha_1} \right] - F_n \theta_h \left[ 1 + \frac{\alpha}{\alpha_1} \right] \]  
\[ \times \frac{\lambda_1}{2aH} \left[ \frac{(2n+1)(\alpha/a)^2}{2} + \alpha/a_1 \right] \]  
\[ (8) \]

1) Infinite material

In an infinite material, the transform \( \Theta(b \to \infty, l \to \infty) \) may be obtained from Eq.(8), taking account of \( b \to \infty \) and \( l \to \infty \)

\[ \Theta(b \to \infty, l \to \infty) = \frac{A_0 a^2}{2L T_0} \frac{K_0(\sqrt{\rho})}{\sqrt{\rho} + (1 + B) \sqrt{\rho} K_1(\sqrt{\rho})} \]  
\[ (9) \]

Now, in the case that the elapsed time is so large that \( \tau > 1 \) is valid enough, \( \Theta(\infty, \infty, \tau) \) may be approximately expanded in ascending series of a parameter \( \rho \), taking account of \( \rho \ll 1 \);

\[ \Theta(\infty, \infty, \tau) = \frac{A_0 a^2}{2L T_0} \frac{1}{\rho^2} \ln \frac{C_0 \sqrt{\rho}}{4} + \frac{1}{2} \ln \left( \frac{1}{2} (1 - 2B) \ln \frac{C_0 \sqrt{\rho}}{4} \right) \]  
\[ - \frac{1}{2} (1 - 2A) \ln \left( \frac{C_0 \sqrt{\rho}}{4} \right)^2 + O(\rho^2) \]  
\[ (10) \]

From the inversion theorem of the Laplace transformation, the surface temperature becomes

\[ \theta(\infty, \infty, \tau) = \frac{A_0 a^2}{2L T_0} \left[ \ln \frac{4\pi}{C} + \frac{1}{4\pi} (1 - 2B) \right] \]  
\[ + (1 - 2A) \ln \frac{4\pi}{C} + O(1/\tau^2) \]  
\[ (11) \]

If the elapsed time \( \tau \) is so large that the second term of the right side in Eq.(11) is insignificant in magnitude, the thermal conductivity may be determined graphically from the slope of the line, by fitting a straight line to the plot of \( \theta(\infty, \infty, \tau) \) against \( \ln(\tau) \). To estimate such a critical value of \( \tau \), the magnitude of \( A \) and \( B \) must be given. \( A \) is expressed as \( c_1 \rho_1 / 2c_0 \rho_0 \), using the physical properties of material and wire. Generally, the relations of \( c_1 < c \) and \( \rho_1 > \rho_0 \) are valid, and values of \( A \) seem to be 1. In the experiment, constantan wire and polyethylene were used as heater and material, respectively. In this case \( A \) becomes 0.84. Then \( (1 - 2A) \approx -1 \). On the other hand, \( B \) is expressed as \( \lambda_1 a H \).

If the thermal contact-resistance \( (1/H) \) becomes larger and larger, \( B \)-value becomes significant. When the thermal contact-resistance is small and insignificant, \( B \approx 0 \), the critical value of \( \tau \) becomes, within an error of 1%,

\[ \tau \approx 40 \]  
\[ (12) \]

In the significant value of \( 1/H \), the critical value of \( \tau \) may be estimated by taking account of the time lag by the thermal contact-resistance. This can be shown by putting \( \tau + \tau_0 \) instead of \( \tau \) in Eq.(11);

\[ \theta(\infty, \infty, \tau) = \frac{A_0 a^2}{4 \lambda_1 T_0} \left[ \ln \frac{4\pi}{C} + \ln \left( 1 + \frac{\tau_0}{\tau} \right) + \frac{1}{2} \frac{1}{(\tau + \tau_0)^2} \right] \]  
\[ \times \left( 1 - (2B) + (1 - 2A) \ln \frac{4\pi}{C} \left( 1 + \frac{\tau_0}{\tau} \right) \right) \]  
\[ + O \left( \frac{1}{(\tau + \tau_0)^2} \right) \]  
\[ + (1 - 2A) \ln \frac{4\pi}{C} + O \left( \frac{1}{\tau^2} \right) \]  
\[ \times \frac{1}{2} \]  
\[ \left( \frac{1 - 2B}{2(\lambda_1 a H)} \right) \]  
\[ (13) \]

If \( \tau_0 \) is chosen so as to satisfy the equation

\[ 2\tau_0 + (1 - 2B) + (1 - 2A) \ln (4\pi \tau/C) = 0 \]  
\[ (14) \]

Eq.(13) becomes

\[ \theta(\infty, \infty, \tau) = \frac{A_0 a^2}{4 \lambda_1 T_0} \left[ \ln \frac{4\pi}{C} \right] \]  
\[ + (1/2\pi) (1 - 2A) \ln (\tau \tau^* + O(1/\tau^2)) \]  
\[ \times \frac{1}{2} \]  
\[ \left( \frac{1 - 2B}{2(\lambda_1 a H)} \right) \]  
\[ (15) \]

\( \tau^* \) may be preferably determined by using the relation \( \tau^* = (\tau_1 \tau_2)^{1/2} \) deduced from \( \ln (\tau^*) = (\ln \tau_1 + \ln \tau_2)/2 \), where \( \tau_1 \) and \( \tau_2 \) are the elapsed time at the beginning and the end of the experimental data needed to determine the thermal conductivity. From Eq.(15), the critical value of \( \tau \) becomes, within an error of 1%,

\[ \tau \approx 20 \]  
\[ \times \frac{1}{20a^2/\alpha} \]  
\[ (16) \]

From the above result, the surface temperature may be expressed as
\[ \theta(\infty, \infty, \tau) = (A_0 \alpha^2 / 4 \Delta T_0) \ln(4/C)(\tau + \tau_0) \quad (17) \]

The time lag \( \tau_0 \) may be determined by Eq.(14), but is best determined experimentally. The reciprocal of the derivative with respect to \( t \) in Eq.(17) is

\[ \left( \frac{\partial \theta}{\partial t} \right)^{-1} = (4 \Delta T_0 / A_0 \alpha^2) (t + t_0) \quad (18) \]

If we plot \( (\partial \theta / \partial t) \) against \( t \), again we find a straight line with a slope \( (4 \Delta T_0 / A_0 \alpha^2) \). The axis \( (\partial \theta / \partial t) = 0 \) cuts this straight line at \(-t_0\). We do not use this line to determine \( \lambda \), since by graphical differentiation we get an inadmissible spread of the points. We use it only to determine \( t_0 \). Then \( \tau_0 \) is given as the non-dimensional representation of \( t_0 \).

2) Semi-infinite material \((a < r < b, l \to \infty)\)

Let us consider how large the radius of the material must be to obtain \( \lambda \) by the same procedure as an infinite material. In this case, Eq.(8) becomes, taking account of \( l \to \infty \),

\[ \Theta(b, \infty) = \left( A_0 \alpha^2 / 2 \Delta T_0 \right) \frac{1}{l} \ln b/a \]

To obtain a large time solution, the transform \( \Theta(b, \infty) \) is expanded in ascending powers of \( \sqrt{b} \) and \( \sqrt{b} \) as follows, taking account of \( \sqrt{b} \ll 1 \) and \( \sqrt{b} b/a \ll 1 \);

\[ \Theta(b, \infty) = \left( A_0 \alpha^2 / 2 \Delta T_0 \right) \frac{1}{l} \ln b/a + \left( C \right) \frac{(1 - 2A)}{2} \]

From the inversion theorem, the surface temperature becomes

\[ \Theta(b, \infty, \tau) = \left( A_0 \alpha^2 / 2 \Delta T_0 \right) \ln b/a \]

\[ \lambda \text{ can be determined from the constant value given by Eq.}(21) \text{. It is, however, an advantage of the transient method to determine} \lambda \text{ before the surface temperature has a constant value. So, Eq.}(21) \text{ is not used to determine} \lambda \text{ in this paper. To apply the transient method effectively, the experiment must be finished before the surface temperature reaches the steady value. From this point of view, it seems that there is a region of time needed effectively in the experiment. This region of time may be determined from the condition that the initial temperature wave caused by the wire exists far from the wire and does not reach the outer wall of the material. In the region of time as above,} \sqrt{b} \ll 1 \text{ and} \sqrt{b} b/a \gg 1 \text{ are valid. Therefore, Eq.(19) may be expanded in ascending powers of} \sqrt{b} \text{ and in descending powers of} \sqrt{b} b/a \text{ as follows;}

\[ \Theta(b, \infty) = \left( A_0 \alpha^2 / 2 \Delta T_0 \right) \ln b/a \]

\[ \text{Using the inversion theorem, the surface temperature becomes}

\[ \theta(b, \infty, \tau) = \frac{A_0 \alpha^2}{2 \Delta T_0} \ln \left( \frac{C \mu}{4} \right) + \frac{1}{4} \left( \frac{a^2}{b^2} \right) \left[ 1 + \frac{b}{2} \left( \frac{1 - 2A}{2} \right) \ln \left( \frac{C \mu}{4} \right) \right] + O(e^{-\sigma b / \sqrt{C}}) \]

3) Semi-infinite material \((b \to \infty, -\infty < z < \infty)\)

Next, let us consider how long the material must be to obtain \( \lambda \) by the same procedure as an infinite material. In this case, Eq.(8) becomes, taking account of \( b \to \infty \),

\[ \Theta(b, \infty) = \left( A_0 \alpha^2 / 2 \Delta T_0 \right) \ln b/a \]

\[ \text{The condition for satisfying within an error of 1% if} \tau \text{ is determined from both Eq.(16) and the relation}

\[ \eta = b/a \sqrt{4 \tau / C} < 5 \text{ or} \ b > 5 \sqrt{4 \tau / C} \]

Using the inversion theorem, the surface temperature becomes

\[ \theta(b, \infty, \tau) = \frac{A_0 \alpha^2}{2 \Delta T_0} \ln \left( \frac{C \mu}{4} \right) + \frac{1}{4} \left( \frac{a^2}{b^2} \right) \left[ 1 + \frac{b}{2} \left( \frac{1 - 2A}{2} \right) \ln \left( \frac{C \mu}{4} \right) \right] + O(e^{-\sigma b / \sqrt{C}}) \]

\[ \text{Using the inversion theorem, the surface temperature becomes}

\[ \theta(b, \infty, \tau) = \frac{A_0 \alpha^2}{2 \Delta T_0} \ln \left( \frac{C \mu}{4} \right) + \frac{1}{4} \left( \frac{a^2}{b^2} \right) \left[ 1 + \frac{b}{2} \left( \frac{1 - 2A}{2} \right) \ln \left( \frac{C \mu}{4} \right) \right] + O(e^{-\sigma b / \sqrt{C}}) \]
In Eq. (27), the term including $A_0$ is the greatest in magnitude. The terms including $A_1, A_2, \ldots$ become smaller and smaller exponentially in magnitude. So, only the value at $n=0$ of Eq. (27) may be discussed to estimate the end effect. Moreover, $A_0\tau$ is small even though $\tau$ is large, because $a/l$ is small. If $A_0\tau$ is small, the following relation may be used:

$$\int_0^\infty \left( e^{-x\tau/\xi} \right) d\xi = -C - \ln A_0\tau + A_0\tau + O\left( (A_0\tau)^n \right)$$

Then Eq. (28) becomes

$$\theta(\infty, l, \tau) = \frac{A_0a^2}{2xT_0} \left[ C + \ln A_0\tau - A_0\tau + O\left( R_p^2\tau^3 \right) \right]$$

$$+ \frac{e^{-x\tau}}{2\tau} \left( \left( 1 + 2B - \frac{2\lambda}{aH} \right) + (1 - 2A) \ln 4\tau \right)$$

$$+ O\left( A_0\ln A_0\tau \right)$$

In Eq. (29), if the terms in the right side $C + \ln (A_0\tau)$ are remarkably larger than the other terms, $\lambda$ may be determined by the slope of the plot $\theta(\infty, l, \tau)$ against $\ln(t)$. The fifth term including the thermal contact-resistance can be removed by taking account of the time lag as in an infinite material. Then, the right side in Eq. (29) can be described by the terms $C + \ln(A_0\tau)$, within an error of 1%, if $\tau$ is determined by both Eq. (16) and the relation

$$A_0\tau \lesssim 0.04 \quad \text{or} \quad l \gtrsim (\pi/0.4)\sqrt{a\bar{t}} \quad \text{(30)}$$

**4) Finite material**

Let us discuss the radius and length of the material and the time for measurement required to obtain $\lambda$ by the same procedure as in an infinite material. The same procedure as in an infinite material is valid if various factors are determined by the following conditions;

a) the wire must be so small in radius and good in thermal conductivity that the radial temperature profile in the wire becomes negligibly small compared to that in the material.

b) the radius of the material to be measured is determined by Eq. (24).

c) the length of the material is determined by Eq. (30).

d) the effective data needed to obtain $\lambda$ must be chosen from the region of time which satisfies Eq. (16).

When we want to measure the thermal conductivity of a material, we often can estimate its rough value as a first step. Then the thermal diffusivity ($\alpha$) can be estimated roughly. If the radius of the wire is given, the effective data needed to obtain $\lambda$ must be chosen after the elapsed time satisfied by Eq. (16). Then, if the effective data were chosen in the region of time $t = t_t - t_0 \left( t_0 > t_t \right)$, $t_t$, of course, must be larger than $20a^3/\alpha$, $b$ must be larger than $5\sqrt{\alpha t_t}$, and $l$ larger than $(\pi/0.4)\sqrt{a\bar{t}}$.

**Apparatus and Procedure of Experiments**

Polyethylene sold at a shop was used as a sample which was formed cylindrically. The cylindrical sample was divided into two pieces along the axis. Along the axis, $C$–$A$ thermocouples ($0.3 \text{ mm}$) enamelled for insulation and constantan wire ($0.3 \text{ mm}$) used as line heater were attached.

**Fig. 2** is a schematic diagram of the experimental apparatus. Sample (S) is maintained at a specified temperature level in a vessel ($T_s$). An electric current is passed through a thin constantan wire, using a storage battery (12 volts) (B). Magnitude of the electric current used depends on the thermal conductivity of the material, to make the temperature rise a few degrees. In the present experiment, electric currents were passed through for two minutes in the range of 0.4 to 0.8 amp., and temperature rise in the wire was measured against elapsed time for two minutes. Using the data, thermal conductivity $\lambda$ was obtained by the following equation, deduced from Eq. (17):

$$\lambda = \frac{4a^2}{\pi} \left[ (T_2 - T_1) \cdot \ln \left( \frac{t_2 + t_0}{t_1 + t_0} \right) \cdot (t_1 + t_0)^{-1} \right]$$

**Experimental Results and Discussion**

Experimental results are indicated in **Figs. 3, 4** and 5. All of the data are chosen in the region of non-dimensional time $\tau$ 100 to 1000. Fig. 3 shows that the radius of polyethylene must be larger than 2.5 cm to
Fig. 3 Influence of the radius of the material on the thermal conductivity at 20°C

Fig. 4 Influence of the length of the material on the thermal conductivity at 17°C

Fig. 5 Temperature dependence of the thermal conductivity of polyethylene

avoid the end effect caused by the radius. The length of polyethylene is fixed at 4 cm. Fig. 4 shows that the length of polyethylene must be greater than 4 cm to avoid the end effect caused by the length. The radius is fixed at 2.5 cm. The critical values of the radius and the length may be determined theoretically by Eqs.(24) and (30) as follows;

\[ b > 5a^2 \tau = 5 \times 1.5 \times 10^{-2} \times \sqrt{1000} = 2.4 \text{ cm} \]

\[ l > (\pi/0.4)a\sqrt{\tau} = (\pi/0.4) \times 1.5 \times 10^{-2} \times \sqrt{1000} = 3.7 \text{ cm} \]

The theoretical estimation is in good agreement with experimental results. For illustration, Fig. 5 shows the temperature dependence of the thermal conductivity of polyethylene.

From the above discussion, it was found that Eqs.(16), (24) and (30) might be applied to determine the critical dimensions of a material in measuring its thermal conductivity by the transient method.

Nomenclature

- \( A_0 \) = heat transmitted from the wire per unit volume and unit time \([\text{kcal/m}^2\text{hr}]\)
- \( A = c_1\rho_1/2C \) \([-]\)
- \( a = \) the radius of the wire \([\text{cm}]\)
- \( B = (\lambda/2aH)(c_1\rho_1/\rho) \) \([-]\)
- \( b = \) the radius of the material \([\text{cm}]\)
- \( c_1, c_2 = \) the specific heat of the material and the wire \([\text{kcal/kg°C}]\)
- \( C = \) Euler’s constant \(= 0.5722 \) \([-]\)
- \( F_0 = I_0(\sigma_0)K_0(\sigma_0\beta) = I_0(\sigma_0\beta)K_0(\sigma_0) \) \([-]\)
- \( F_1 = I_1(\sigma_0)K_1(\sigma_0\beta) + I_0(\sigma_0\beta)K_1(\sigma_0) \) \([-]\)
- \( H = \) the “outer conductivity” at \( r = a \) (a measure of the contact-resistance between wire and material) \([\text{kcal/m}^2\text{-hr°C}]\)
- \( l = \) the length of the material \([\text{cm}]\)
- \( r, z, t = \) the coordinates and the time \([\text{cm}]\)
- \( \tau = \) the thermal diffusivity of the material and the wire \([\text{m}^2/\text{hr}]\)
- \( \theta = \) \(T - T_0\)/\(T_0\) \([-]\)
- \( \theta_b = 1/3 \left( (T_b - T_0)/T_0 \right) \) \([-]\)
- \( \lambda, \lambda_1 = \) the thermal conductivity of the material and the wire \([\text{kcal/m·hr°C}]\)
- \( \rho, \rho_1 = \) the density of the material and the wire \([\text{kg/m}^3]\)
- \( \zeta = b/a\sqrt{\tau} \) \([-]\)

Literature Cited

3) Van der Held, E. F. M. and F. G. Van Drunen: Physica, 15, 865 (1949)