THE THICKNESS OF A VISCOUS LIQUID FILM ON A ROTATING DISK*

SHIRO MATSUMOTO, KUNIKI SAITO
AND YOICHI TAKASHIMA

Research Laboratory of Nuclear Reactor, Tokyo Institute of
Technology, Tokyo, Japan

Various theoretical solutions including the author's for the thickness of a viscous liquid film on a rotating disk are presented in terms of two dimensionless parameters and are compared with experimental results.

The film thickness on the disk was measured by a micrometer which is movable with a needle used as a surface detector.

The numerical solution of the equations as simplified by von Karman's method and Bruin's solution agree well with the experimental results in a wide range of operating conditions. Polynomial approximation with orders up to fourth and fifth degree is usable to predict the film thickness.

Introduction

The atomization by a rotating disk is widely used as a tool for spray drying and a promotor of chemical reaction or absorption between gas and liquid. How to control the drop size is one of the most important techniques for the design of equipment and for maintaining good product quality. The velocity profile and the thickness of liquid film on a rotating disk are closely related to this problem and have been studied by several investigators^{1,2,4,6,11,12}.

Recently, the authors analyzed film thickness on a disk by the numerical integration of non-linear ordinary differential equations simplified by von Karman's method from Navier-Stokes equations^{7,8}. Similar studies were made by Nikolaev *et al.*¹⁰, Bruin¹ and Miyasaka⁹. However, those theoretical studies of film thickness are not yet sufficiently checked by the experimental results, because of the lack of reliable data³.

Here, the authors have summarized these theoretical studies and show some experimental results on the film thickness on a rotating disk in which the height of the film surface was measured by moving a needle and touching it to the surface. Finally, theoretical and experimental results are compared.

Theoretical Results

The flow rate, Q , flowing on a rotating disk is ob-

tained by integrating the radial velocity distribution on the disk, as follows:

$$Q = 2\pi r \int_0^{z_0} u dz \quad (1)$$

where r is the radial position, u the radial velocity and z_0 the surface position of liquid film.

Introducing the following dimensionless parameters

$$\xi = z \sqrt{\frac{\omega}{\nu}} \quad (2)$$

$$F = \frac{u}{r\omega} \quad (3)$$

where ξ is the dimensionless coordinate, F the dimensionless radial velocity and ω the angular velocity of disk, Eq.(1) is transformed in terms of dimensionless quantities, as follows:

$$Q^+ = \frac{Q}{2\pi r^2 \sqrt{\nu \omega}} = \int_0^{\xi_0} F d\xi \quad (4)$$

where Q^+ is the dimensionless flow rate.

Oyama and Endou¹¹ presented the following dimensionless ordinary differential equations simplified by von Karman's method⁵ from Navier-Stokes equations.

$$F^2 + F'H - F'' = 0 \quad (5)$$

$$2FG + HG' - G'' = 0 \quad (6)$$

$$2F + H' = 0 \quad (7)$$

The boundary conditions were

$$\begin{aligned} \xi=0: F=0, G=1, H=0 \\ \xi=\xi_0: F'=0, G'=0 \end{aligned} \quad (8)$$

where G and H are the dimensionless tangential and

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〒152 東京都目黒区大岡山2-12-1

東京工業大学原子炉工学研究所 松本史朗

Table 1 Values of a_1 , b_1 and a

ξ_o	4th degree		5th degree		Oyama & Endou a
	a_1	b_1	a_1	b_1	
0.3	0.2984	-0.01781	0.2986	-0.01787	-0.4949
0.4	0.3934	-0.04127	0.3942	-0.04174	-0.4844
0.5	0.4810	-0.07704	0.4830	-0.07913	-0.4646
0.6	0.5567	-0.1234	0.5604	-0.1301	-0.4345
0.7	0.6174	-0.1758	0.6224	-0.1924	-0.3965
0.8	0.6630	-0.2279	0.6668	-0.2620	-0.3545
0.9	0.6960	-0.2744	0.6935	-0.3356	-0.3126
1.0	0.7204	-0.3121	0.7039	-0.4111	-0.2733

axial velocities, respectively, and prime denotes the differential with respect to ξ .

Oyama and Endou applied a polynomial approximation of the second degree for the radial and the tangential velocity distribution, and by applying Pohlhausen's method to two-dimensional laminar boundary layer flow, obtained the following equations.

$$Q^+ = -\frac{2}{3}a\xi_o^3 \quad (9)$$

$$\frac{8192}{1125}\xi_o^{12}a^4 - \frac{1024}{45}\xi_o^8a^3 + \left(\frac{352}{15}\xi_o^4 - \frac{512}{675}\xi_o^8\right)a^2 + \left(\frac{64}{45}\xi_o^4 - 8\right)a - 4 = 0 \quad (10)$$

where a is the coefficient of polynomial approximation. If the value of ξ_o is much less than unity, the value of a becomes $-1/2$ and Eq.(9) becomes

$$Q^+ = \frac{1}{3}\xi_o^3 \quad (11)$$

This result was same as that analyzed by Emslie *et al.*²⁾, Tanasawa *et al.*¹²⁾ and Hinze and Milborn⁴⁾.

The authors applied polynomial approximation of higher degrees for the radial, the tangential and the axial velocity distributions. The coefficients of the polynomials were determined from the boundary conditions as shown in Eq.(8) and from the further boundary conditions obtained by putting $\xi=0$ in Eqs.(5), (6) and (7), and in differentials of those equations. The resultant polynomial approximation of higher degrees were given by

$$F = a_1\xi - \frac{1}{2}\xi^2 - \frac{1}{3}b_1\xi^3 - \frac{1}{12}b_1^2\xi^4 - \frac{1}{60}a_1\xi^5 \quad (12)$$

$$G = 1 + b_1\xi + \frac{1}{3}a_1\xi^3 + \frac{1}{12}(a_1b_1 - 1)\xi^4 - \frac{1}{15}b_1\xi^5 \quad (13)$$

$$H = -a_1\xi^2 + \frac{1}{3}\xi^3 + \frac{1}{6}b_1\xi^4 + \frac{1}{30}b_1\xi^5 + \frac{1}{180}a_1\xi^6 \quad (14)$$

a_1 and b_1 must also be determined from the free boundary, $F'(\xi_o)=0$ and $G'(\xi_o)=0$.

Consequently, the thickness of the film on the disk was given by the following equations.

3rd degree:

$$Q^+ = \frac{\xi_o^3}{2(1+\xi_o^4)} - \frac{1}{6}\xi_o^3 + \frac{\xi_o^7}{12(1+\xi_o^4)} \quad (15)$$

4th degree:

$$Q^+ = \frac{1}{2}a_1\xi_o^2 - \frac{1}{6}\xi_o^3 - \frac{1}{12}b_1\xi_o^4 - \frac{1}{60}b_1\xi_o^5 \quad (16)$$

where a_1 and b_1 are the roots of the equations

$$\begin{aligned} a_1 - \xi_o - b_1\xi_o^2 - \frac{1}{3}b_1\xi_o^3 &= 0 \\ b_1 + a_1\xi_o^2 + \frac{1}{3}(a_1b_1 - 1)\xi_o^3 &= 0 \end{aligned} \quad (17)$$

5th degree:

$$Q^+ = \frac{1}{2}a_1\xi_o^2 - \frac{1}{6}\xi_o^3 - \frac{1}{12}b_1\xi_o^4 - \frac{1}{60}b_1\xi_o^5 - \frac{1}{360}a_1\xi_o^6 \quad (18)$$

where the algebraic equations with respect to a_1 and b_1 are given by

$$\begin{aligned} a_1 - \xi_o - b_1\xi_o^2 - \frac{1}{3}b_1\xi_o^3 - \frac{1}{12}a_1\xi_o^4 &= 0 \\ b_1 + a_1\xi_o^2 + \frac{1}{3}(a_1b_1 - 1) - \frac{1}{3}b_1\xi_o^4 &= 0 \end{aligned} \quad (19)$$

The undetermined coefficients of the polynomials were calculated from Eqs.(10), (17) and (19). They are presented in **Table 1**.

Further, the authors obtained a numerical result from the numerical integration of Eq.(4) by the complete numerical analysis of Eqs.(5), (6) and (7) with the boundary conditions, Eq.(8). By putting

$$X_1 = F, X_2 = G, X_3 = H, X_4 = F', X_5 = G' \quad (20)$$

Eqs.(5), (6) and (7) are transformed into a set of ordinary differential equations as follows:

$$\begin{aligned} \dot{X}_1 &= X_4 \\ \dot{X}_2 &= X_5 \\ \dot{X}_3 &= -2X_1 \\ \dot{X}_4 &= X_1^2 + X_3X_4 - X_2^2 \\ \dot{X}_5 &= 2X_1X_2 + X_3X_5 \end{aligned} \quad (21)$$

where the dots denotes the differential.

Therefore, the boundary conditions become

$$\begin{aligned} X_1(0) &= 0, X_2(0) = 1, X_3(0) = 0 \\ X_4(\xi_o) &= 0, X_5(\xi_o) = 0 \end{aligned} \quad (22)$$

A quasilinearization method has been used for obtaining the numerical solution of these nonlinear boundary value problems. The details of these numerical calculations are shown elsewhere⁷⁾.

Bruin¹⁾ and Nikolaev *et al.*¹⁰⁾ independently analyzed the velocity profiles in a liquid film on a rotating conical disk. From the velocity profile, the more general relation between the dimensionless flow rate and the film thickness was given by the equation*

$$Q^+ = \frac{1}{4} \left(1 - \frac{\cot\beta}{Fr} \right) \frac{\sinh 2\xi_o - \sin 2\xi_o}{\cosh 2\xi_o + \cos 2\xi_o} \quad (23)$$

where β is the half angle at apex of cone and Fr Froude

* The solution by Nikolaev *et al.* was presented in the form of more complex formula.

Fig. 1 Comparison between various theoretical solutions of film thickness

Fig. 2 Schematic diagram of experimental apparatus

number defined by

$$Fr = \frac{r_o \omega^2 \sin \beta}{g} \quad (24)$$

Miyasaka⁹⁾ presented another numerical solution. It is given by using the dimensionless flow rate and the film thickness, as follows:

$$\xi_o = \left(\frac{z_o}{r_o} \right) \left(\frac{m Re_o}{2} \right)^{1/2} \quad (25)$$

$$Q^+ = \frac{1}{2\sqrt{2}} \left(\frac{r_o}{r} \right)^2 \left(\frac{Re_o}{m} \right)^{1/2} \quad (26)$$

where r_o is the radius of feed nozzle, m and Re_o the dimensionless rotational parameter and Reynolds number defined by

$$m = \frac{r_o \omega}{w_o}, \quad Re_o = \frac{2r_o w_o}{\nu} \quad (27)$$

w_o is the feed velocity by the nozzle.

As a result, the relation between the dimensionless thickness of a liquid film on the disk and the dimensionless flow rate is shown in Fig. 1. In this figure, one-point dash line, two-point dash line and three-point dash line represent polynomial approximations with orders up to third, fourth and fifth degree, respectively. The dash line corresponds to Eq.(11), the dot line to the solution by Oyama and Endou. The solid line corresponds to the numerical solution. The solutions by Bruin and Nikolaev *et al.* were same as the numerical solution in the range of this representation.

Fig. 3 Detail of disk

Fig. 4 Schematic measuring apparatus of film thickness

These theoretical solutions will be compared with the experimental results later.

Experimental

Figure 2 shows a schematic diagram of the experimental apparatus. The detail of the disk is shown in Fig. 3. A liquid sample is supplied by a pump from a constant-temperature bath through a flow meter and a nozzle to the center of a rotating disk. The liquid removed from the disk is collected in a vessel. The rotational speed is changed from 200 to 9000 rpm by a stepless variable-speed control, and is determined by a stroboscope.

Figure 4 shows a schematic diagram of a measuring apparatus of the liquid film thickness on the disk. A needle, movable vertically and horizontally, is installed as detector on the stage. The displacement is measured by two micrometers. The needle is connected with copper leads through a metal block in an acrylic tube.

The thickness of liquid film at any position on the disk is obtained from the difference between the surface position of the liquid film and that of the disk. The liquid surface was determined by observing the standing waves when the head of the needle just reached the free surface. The disk surface was determined by using such an electric circuit connecting the needle with the stationary disk through an electric cell which can detect a current generated when the needle has touched the surface of the disk. However, it is necessary to check whether the surface level of the stationary disk is the same as that of the rotating disk. To make it clear, the vibration of the rotating disk was measured by a dial gauge. The amplitude of the vibration was about 30μ at maximum. The errors caused by vibration were less than ten percent at most for the measurement of film thickness.

Fig. 5 Comparison between theoretical and experimental results

Table 2 Experimental results of liquid film thickness

r [cm]	ν [cS]	N [r.p.m.]	Q [cm ³ /sec]	z_0 [μ]	ξ_0 [—]	Q^+ [—]
2	9.61	250	5.3	485	0.800	0.133
5	9.61	250	5.3	251	0.414	0.0213
2.5	9.61	250	10.2	489	0.807	0.164
4	9.61	250	20.1	454	0.749	0.126
2	9.61	500	10.2	395	0.920	0.181
3	9.61	500	10.2	271	0.631	0.0806
5	9.61	500	10.2	190	0.443	0.0290
2.5	9.61	800	5.3	171	0.505	0.0475
5	9.61	800	5.3	113	0.334	0.0119
3	9.61	800	20.1	255	0.753	0.125
4	9.61	800	20.1	206	0.608	0.0704
3	9.61	1000	31.2	272	0.898	0.174
4	9.61	1000	31.2	207	0.683	0.0980
2	24.0	500	4.99	390	0.576	0.0561
5	24.0	500	4.99	204	0.301	0.0090
3	24.0	500	19.6	483	0.713	0.0979
3	24.0	500	31.0	586	0.865	0.155
4	24.0	1000	4.99	151	0.315	0.0099
2	24.0	1000	10.2	308	0.643	0.0810
4	24.0	1000	10.2	178	0.372	0.0202
5	24.0	1000	10.2	164	0.343	0.0130
2.5	24.0	1000	31.0	409	0.854	0.158
2	58.3	250	7.0	899	0.602	0.0713
5	58.3	250	7.0	469	0.314	0.0114
2	58.3	250	20.7	1439	0.964	0.211
3	58.3	250	20.7	1003	0.672	0.0938
5	58.3	250	20.7	696	0.466	0.0338
2.5	58.3	250	31.0	1447	0.969	0.202
5	58.3	250	31.0	820	0.549	0.0506
2	58.3	500	6.68	586	0.549	0.0477
4	58.3	1500	6.68	171	0.278	0.0069
2	58.3	1500	20.0	392	0.637	0.0824
3	58.3	1500	20.0	286	0.464	0.0366
5	58.3	1500	20.0	214	0.348	0.0132
5	58.3	1500	29.9	242	0.393	0.0197

The density and viscosity of liquid were measured by an ordinary hydrometer and a viscometer of Ubellode type, respectively.

Results and Discussion

The thickness of liquid film observed in this experiment was more than a few hundred microns. Representative observed values are presented in Table 2 together with the experimental conditions.

Various previous theoretical solutions are compared with the experimental results in Fig. 5.

It is seen that all theoretical solutions show good agreement with the experimental results at small values of ξ_0 . There is no difference between the various theoretical results. An increase of the value of ξ_0 makes a little difference between some theoretical results and the experimental data. The polynomial approximation of order up to fourth degree or the higher degree, Bruin's result and the numerical solution show very good agreement with the experimental results in a wide range of ξ_0 , as shown in Fig. 5.

Conclusion

Various theoretical solutions for the thickness of a viscous liquid film on a rotating disk were summarized. They were compared with the experimental data. As a result, it is seen that the solution of the numerical analysis and Bruin's solution give the most accurate information on film thickness, and that even the polynomial approximation of order up to the fourth degree is accurate enough for predicting film thickness.

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Nomenclature

a	= coefficient of polynomials	[—]
a_1	= coefficient of polynomials	[—]
b_1	= coefficient of polynomials	[—]
F	= dimensionless radial velocity	[—]
Fr	= Froude number	[—]
G	= dimensionless tangential velocity	[—]
H	= dimensionless axial velocity	[—]
m	= dimensionless rotational parameter	[—]
Q	= flow rate	[cm ³ /sec]
Q^+	= dimensionless flow rate	[—]
r	= radial position	[cm]
r_0	= radius of feed nozzle	[cm]
Re_0	= Reynolds number	[—]
u	= radial velocity	[cm/sec]

w_0	= feed velocity by nozzle	[cm/sec]
z	= axial coordinate	[—]
z_0	= film thickness	[cm]
β	= half angle at apex of cone	[degree]
ξ	= dimensionless coordinate	[—]
ξ_0	= dimensionless film thickness	[—]
ν	= kinematic viscosity of liquid	[cm ² /sec]
ω	= angular velocity of disk	[radian/sec]

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A STUDY OF MASS TRANSPORT IN TURBULENT FLOW *

SHIRO ITO AND KOHEI OGAWA

Department of Chemical Engineering, Tokyo Institute of Technology, Tokyo, Japan

Considering the analogical equation for mass transfer with the kinematic energy equation, a new eddy diffusivity is defined in much the same way as the eddy viscosity presented in the previous paper⁵⁾. Being different from the conventional eddy diffusivity, the newly defined eddy diffusivity can be determined uniquely on the notice point from a single mean concentration gradient to any direction at that point.

At the transitional region from laminar to turbulent flow of liquid in a circular pipe, further, experimental values of the newly defined eddy diffusivity are obtained by simultaneous measurement of the liquid velocity fluctuations and the concentration fluctuations of injected tracer by electro-chemical techniques.

Introduction

The general equation of mass transfer for turbulent flow for constant density and the coefficient of molecular diffusion is obtained by substituting the concentration and velocity components, which are decomposed into their time-mean parts and fluctuating parts, into the general equation of mass transfer for non-turbulent flow and averaging with respect to time as follows (using Cartesian tensor notation).

$$\frac{\partial \bar{C}}{\partial t} + \bar{U}_i \frac{\partial \bar{C}}{\partial x_i} = \frac{\partial}{\partial x} \left(D \frac{\partial \bar{C}}{\partial x_i} - \overline{u_i c} \right) \quad (1)$$

where C denotes the concentration, U_i is the velocity, D is the coefficient of molecular diffusion, and the bars denote the time-mean parts and the small letters denote the fluctuating parts. Hereupon, the conventional eddy diffusivity, ε_c , has been defined analogically, with the

definition of the conventional eddy viscosity as the following equation.

$$-\overline{u_i c} = \varepsilon_c \left(\frac{\partial \bar{C}}{\partial x_i} \right) \quad (2)$$

But this concept brings complexity when the above-defined eddy diffusivity is applied to practical study of turbulent mixing phenomena. In a word, the plural values of the eddy diffusivity are decided at one notice point, and they complicate the analysis of mixing phenomena just as does the conventionally defined eddy viscosity.

It is thus the purpose of this paper to define a new eddy diffusivity which is determined uniquely on the notice point from a single mean concentration gradient to any direction at that point, in much the same way as the definition of the eddy viscosity reported before⁵⁾.

Theoretical Consideration

Multiplication of mass transfer equation by C and summation with respect to the index i will yield after some transformation an equation which corresponds to

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〒152 東京都目黒区大岡山2-12-1
東京工業大学工学部化学工学科 小川浩平