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SHORT COMMUNICATIONS

NOTES ON DETERMINING THE MOMENTS OF THE IMPULSE RESPONSE FROM THE BASIC TRANSFORMED EQUATIONS*

MOTOYUKI SUZUKI

Institute of Industrial Science, University of Tokyo, Tokyo, Japan

Dynamic testing methods have been widely used for determining process parameters. Jeffreson¹⁾ showed the universal relations between both step and frequency responses and the moments of the impulse response. N-th absolute moment, μ_n , of the impulse response is given by

$$\mu_n \equiv \int_0^\infty t^n c(t) dt / \int_0^\infty c(t) dt$$

$$= \left[\lim_{p \rightarrow 0} \left(-\frac{d}{dp} \right)^n \bar{c}(p) \right] / \bar{c}(0) \quad (1)$$

where $\bar{c}(p)$ is a transfer function (Laplace transform of $c(t)$) and the central moment (moment about the mean residence time) is defined as

$$\mu'_n \equiv \int_0^\infty (t - \mu_1)^n c(t) dt / \int_0^\infty c(t) dt = \sum_{i=1}^n \binom{n}{i} \mu_i (-\mu_1)^{n-i} \quad (2)$$

N-th cumulant, κ_n , of $c(t)$ is also related to $\bar{c}(p)$ by the equation

$$\kappa_n = \lim_{p \rightarrow 0} \left(-\frac{d}{dp} \right)^n (\log \bar{c}(p)) \quad (3)$$

The relations between the moments and the cumulants are shown in the column I in **Table 1**. Therefore the moments are easily related to process parameters

once the solution in the Laplace domain is obtained. However this procedure is sometimes tedious since it involves the step of solving the basic differential equations in Laplace domain and the step of differentiation. Lees⁵⁾ extended the method proposed by Klinkenberg³⁾, which is based on the idea that the application of Eq.(1) to the basic transformed equations gives a set of relations among the moments that can then be solved by normal methods. This technique is advantageous for stagewise processes or differential processes which do not involve higher derivatives.

The present note proposes an alternate method applicable to differential processes which offers a way to estimate moments of an impulse response rapidly. Examples are shown to illustrate the usefulness of the method.

Basic Method

The basic differential equation describing the system is transformed with respect to time into Laplace domain

$$\sum_{n=0}^m A_n(p) \frac{d^n \bar{c}}{dz^n} = 0 \quad (4)$$

where $A_n(p)$ is generally an arbitrary function of p , which is then expanded near $p=0$.

$$A_n(p) = a_{n0} + a_{n1}p + a_{n2}p^2 + \dots \quad (5)$$

The transfer function $\bar{c}(p)$ is also expanded in the

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〒106 東京都港区六本木7-22-1

東京大学生産技術研究所 鈴木基之

Table 1 Relations among moments, cumulants and coefficients b_i and a_{ij}

(I)	(II)	(III)	
$\mu_1 = \kappa_1$	$\kappa_1 = -b_1 z$	$b_1 = -\frac{a_{01}}{a_{10}}$	(13)
$\mu'_2 = \kappa_2$	$\kappa_2 = 2b_2 z$	$b_2 = -\frac{1}{a_{10}} \{a_{20}b_1^2 + a_{11}b_1 + a_{02}\}$	(14)
$\mu'_3 = \kappa_3$	$\kappa_3 = -6b_3 z$	$b_3 = -\frac{1}{a_{10}} \{a_{30}b_1^3 + a_{21}b_1^2 + 2a_{20}b_1b_2 + a_{11}b_2 + a_{03}\}$	(15)
$\mu'_4 = \kappa_4 + 3\kappa_2^2$	$\kappa_4 = 24b_4 z$	$b_4 = -\frac{1}{a_{10}} \{a_{40}b_1^4 + a_{31}b_1^3 + 3a_{30}b_1^2b_2 + a_{22}b_1^2 + 2a_{21}b_1b_2 + a_{20}b_2^2 + 2a_{20}b_1b_3 + a_{13}b_1 + a_{12}b_2 + a_{11}b_3 + a_{04}\}$	(16)

Table 2 Characteristic equation for $m=4$

$$\begin{aligned}
 & (a_{00} + a_{10}b_0 + a_{20}b_0^2 + a_{30}b_0^3 + a_{40}b_0^4) \\
 & + p[a_{01} + (a_{11}b_0 + a_{10}b_1) + (a_{21}b_0^2 + 2a_{20}b_0b_1) + (a_{31}b_0^3 + 3a_{30}b_0^2b_1) + (a_{41}b_0^4 + 4a_{40}b_0^3b_1)] \\
 & + p^2[a_{02} + (a_{10}b_2 + a_{11}b_1 + a_{12}b_0) + \{a_{20}(b_1^2 + 2b_0b_2) + 2a_{21}b_0b_1 + a_{22}b_0^2\} + \{a_{30}(3b_0^2b_2 + 3b_0b_1^2) + 3a_{31}b_0^2b_1 + a_{32}b_0^3\} + \{a_{40}(4b_0^3b_2 + 6b_0^2b_1^2) \\
 & + 4a_{41}b_0^3b_1 + a_{42}b_0^4\}] \\
 & + p^3[a_{03} + (a_{10}b_3 + a_{11}b_2 + a_{12}b_1 + a_{13}b_0) + \{a_{20}(2b_0b_3 + 2b_1b_2) + a_{21}(b_1^2 + 2b_0b_2) + 2a_{22}b_0b_1 + a_{23}b_0^2\} + \{a_{30}(3b_0^2b_3 + 6b_0b_1b_2 + b_1^3) \\
 & + a_{31}(3b_0^2b_2 + 3b_0b_1^2) + 3a_{32}b_0^2b_1 + a_{33}b_0^3\} + \{a_{40}(4b_0^3b_3 + 12b_0^2b_1b_2 + 4b_0b_1^3) + a_{41}(4b_0^3b_2 + 6b_0^2b_1^2) + 4a_{42}b_0^3b_1 + a_{43}b_0^4\}] \\
 & + p^4[a_{04} + (a_{10}b_4 + a_{11}b_3 + a_{12}b_2 + a_{13}b_1 + a_{14}b_0) + \{a_{20}(2b_0b_4 + 2b_1b_3 + b_2^2) + a_{21}(2b_0b_3 + 2b_1b_2) + a_{22}(b_1^2 + 2b_0b_2) + 2a_{23}b_0b_1 + a_{24}b_0^2\} \\
 & + \{a_{30}(3b_0^2b_4 + 6b_0b_1b_3 + 3b_0b_2^2 + 3b_1^2b_2) + a_{31}(3b_0^2b_3 + 6b_0b_1b_2 + b_1^3) + a_{32}(3b_0^2b_2 + 3b_0b_1^2) + 3a_{33}b_0^2b_1 + a_{34}b_0^3\} + \{a_{40}(4b_0^3b_4 + 12b_0^2b_1b_3 \\
 & + 6b_0^2b_2^2 + 12b_0b_1^2b_2 + b_1^4) + a_{41}(4b_0^3b_3 + 12b_0^2b_1b_2 + 4b_0b_1^3) + a_{42}(4b_0^3b_2 + 6b_0^2b_1^2) + 4a_{43}b_0^3b_1 + a_{44}b_0^4\}] \\
 & + p^5[a_{05} + \dots] + \dots \\
 & = 0
 \end{aligned} \tag{11}$$

form of Paynter's generalized transfer function⁶⁾

$$\bar{c}(p) = \exp\left(\sum_{i=0}^{\infty} B_i p^i\right) \tag{6}$$

In the case when the following relation holds for any z_1 ($0 < z_1 < z$)

$$\frac{\bar{c}_z}{c_0} = \frac{\bar{c}_z}{c_{z_1}} \cdot \frac{\bar{c}_{z_1}}{c_0} \tag{7}$$

and when the system is uniform, B_i becomes proportional to z and then Eq.(6) is rewritten as

$$\bar{c}(p) = \exp\left(\sum_{i=0}^{\infty} b_i p^i\right) z \tag{8}$$

Then the application of Eq.(3) to Eq.(8) gives the relation between the cumulants and b_i as shown in the second column in Table 1 with

$$\exp(b_0 z) = c(0) \tag{9}$$

By substituting Eq.(9) into Eq.(4), the following relation is obtained as a characteristic equation

$$\sum_{n=0}^m A_n(p) \left(\sum_{i=0}^{\infty} b_i p^i\right)^n = 0 \tag{10}$$

Since m , the highest order of the derivatives in Eq.(4), seldom becomes greater than 4, $m=4$ is taken here for the sake of convenience. Then substituting Eq.(5) into Eq.(10) and rearranging the terms in the ascending order of p gives the equation shown in **Table 2**.

Since Eq.(11) holds near $p=0$, each coefficient of p^i ($i \geq 0$) should be zero. Obviously, the constant term (the coefficient of p^0) in Eq.(11) is a characteristic equation of the steady-state version of the original equation. By choosing simple boundary conditions such as doubly infinite condition and using non-reaction (no sink or source) system for dynamic testing,

the sole solution to the steady state characteristic equation gives

$$b_0 = 0 \tag{12}$$

Introducing Eq.(12) into Eq.(11) and equating the coefficients of p^1 , p^2 , p^3 and p^4 to zero, b_1 , b_2 , b_3 and b_4 are determined successively as shown in the third column of Table 1.

Eqs.(13-16) are readily used for determining b_1 - b_4 , which then give μ_1 , μ'_2 , μ'_3 and μ'_4 by means of the second and first columns of Table 1.

Discussion

The biggest assumption adopted here is the application of Eqs.(7) and (8). Similar assumption was used in the recent work by Kim *et al.*²⁾ for the reduction of thermal frequency response data in a packed bed. They showed that the assumption was reasonable in the range of their study.

Another assumption of Eq.(12) is not critical since it is possible, in principle, to extend this technique to a reacting system where $b_0 \neq 0$. Apparently, in this case, the resulting equations would be more complicated.

The moments of the impulse response are easily determined by straightforward calculations once the coefficients of the basic transformed equation are given and expanded if necessary. Two examples are given to illustrate the method proposed here.

Example (A)

Adsorption in packed beds to porous particles:
Basic equations are given as

Table 3 (a) The coefficients a_{ij} for Example A

$$\begin{aligned}
 a_{01} &= -\left\{1 + \frac{(1-\alpha)}{\alpha} \beta \delta\right\} \\
 a_{02} &= \frac{(1-\alpha)}{\alpha} \beta \left\{ \frac{\rho_p}{\beta} \frac{K_a^2}{k_a} + \left(\frac{1}{5} + \frac{1}{Bi}\right) \frac{T_d}{3} \delta^2 \right\} \\
 a_{03} &= -\frac{(1-\alpha)}{\alpha} \beta \left\{ \frac{\rho_p}{\beta} \frac{K_a^3}{k_a^2} + \left(\frac{1}{5} + \frac{1}{Bi}\right) \frac{2T_d}{3} \delta \frac{\rho_p}{\beta} \frac{K_a^2}{k_a} \right. \\
 &\quad \left. + \left(\frac{2}{35} + \frac{2}{5Bi} + \frac{1}{Bi^2}\right) \frac{T_d^2}{9} \delta^3 \right\} \\
 a_{04} &= \frac{(1-\alpha)}{\alpha} \beta \left\{ \frac{\rho_p}{\beta} \frac{K_a^4}{k_a^3} + \left(\frac{1}{5} + \frac{1}{Bi}\right) \frac{T_d}{3} \frac{\rho_p}{\beta} \frac{K_a^3}{k_a^2} (2 + 3 \frac{\rho_p}{\beta} K_a) \right. \\
 &\quad \left. + \left(\frac{2}{35} + \frac{2}{5Bi} + \frac{1}{Bi^2}\right) \frac{T_d^2}{3} \frac{\rho_p}{\beta} \frac{K_a^2}{k_a} \right. \\
 &\quad \left. + \left(\frac{3}{175} + \frac{54}{175Bi} + \frac{3}{5Bi^2} + \frac{1}{Bi^3}\right) \frac{T_d^3}{27} \delta^4 \right\} \\
 a_{20} &= E_z, a_{10} = -u \text{ and the other } a_{ij} = 0 \ (i+j \leq 4) \\
 \text{where } \delta &= 1 + \frac{\rho_p}{\beta} K_a
 \end{aligned}$$

Table 3 (b) Moments for Example A

$$\begin{aligned}
 \mu_1 &= \frac{z}{u} \left\{ 1 + \frac{(1-\alpha)}{\alpha} \beta \left(1 + \frac{\rho_p}{\beta} K_a \right) \right\} \\
 \mu_2' &= \frac{2z}{u} \left[\frac{(1-\alpha)}{\alpha} \beta \frac{\rho_p}{\beta} \frac{K_a^2}{k_a} + \frac{R^2 \beta \left(1 + \frac{\rho_p}{\beta} K_a \right)^2}{3De} \left(\frac{1}{5} + \frac{1}{Bi} \right) \right. \\
 &\quad \left. + \frac{E_z}{u^2} \left\{ 1 + \frac{(1-\alpha)}{\alpha} \beta \left(1 + \frac{\rho_p}{\beta} K_a \right) \right\}^2 \right] \\
 \mu_3' &= \frac{6z}{u} \left[\frac{(1-\alpha)}{\alpha} \beta \left\{ \frac{\rho_p}{\beta} \frac{K_a^3}{k_a^2} + \frac{\rho_p}{\beta} \frac{K_a^2}{k_a} \frac{2R^2 \beta \left(1 + \frac{\rho_p}{\beta} K_a \right)}{3De} \left(\frac{1}{5} \right. \right. \right. \\
 &\quad \left. \left. + \frac{1}{Bi} \right) + \left(\frac{R^2 \beta}{De} \right)^2 \left(1 + \frac{\rho_p}{\beta} K_a \right)^3 \frac{1}{9} \left(\frac{2}{35} + \frac{2}{5Bi} + \frac{1}{Bi^2} \right) \right\} \right. \\
 &\quad \left. + \frac{2E_z}{u^2} \left\{ 1 + \frac{(1-\alpha)}{\alpha} \beta \left(1 + \frac{\rho_p}{\beta} K_a \right) \right\} \frac{(1-\alpha)}{\alpha} \beta \left\{ \frac{\rho_p}{\beta} \frac{K_a^2}{k_a} \right. \right. \right. \\
 &\quad \left. \left. + \frac{R^2 \beta \left(1 + \frac{\rho_p}{\beta} K_a \right)^2}{3De} \left(\frac{1}{5} + \frac{1}{Bi} \right) \right\} \right. \\
 &\quad \left. + \frac{2E_z^2}{u^4} \left\{ 1 + \frac{(1-\alpha)}{\alpha} \beta \left(1 + \frac{\rho_p}{\beta} K_a \right) \right\}^3 \right] \\
 \mu_4' &= \frac{24z}{u} \left[\frac{(1-\alpha)}{\alpha} \beta \left\{ \frac{\rho_p}{\beta} \frac{K_a^4}{k_a^3} + \frac{\rho_p}{\beta} \frac{K_a^3}{k_a^2} \left(2 + 3 \frac{\rho_p}{\beta} K_a \right) \frac{R^2}{De} \left(\frac{1}{5} \right. \right. \right. \\
 &\quad \left. \left. + \frac{1}{Bi} \right) + \frac{\rho_p}{\beta} \frac{K_a^2}{k_a} \left(1 + \frac{\rho_p}{\beta} K_a \right)^2 \left(\frac{R^2}{De} \right)^2 \frac{1}{3} \left(\frac{2}{35} + \frac{2}{5Bi} \right. \right. \\
 &\quad \left. \left. + \frac{1}{Bi^2} \right) + \left(1 + \frac{\rho_p}{\beta} K_a \right)^4 \left(\frac{R^2 \beta}{De} \right)^3 \frac{1}{27} \left(\frac{3}{175} + \frac{54}{175Bi} + \frac{3}{5Bi^2} \right. \right. \\
 &\quad \left. \left. + \frac{1}{Bi^3} \right) + \frac{2E_z}{u^2} \left(1 + \frac{(1-\alpha)}{\alpha} \beta \left(1 + \frac{\rho_p}{\beta} K_a \right) \right) \frac{(1-\alpha)}{\alpha} \beta \left\{ \frac{\rho_p}{\beta} \frac{K_a^3}{k_a^2} \right. \right. \\
 &\quad \left. \left. + \frac{\rho_p}{\beta} \frac{K_a^2}{k_a} \left(1 + \frac{\rho_p}{\beta} K_a \right) \frac{R^2 \beta}{De} \frac{2}{3} \left(\frac{1}{5} + \frac{1}{Bi} \right) \right\} \right. \\
 &\quad \left. + \left(1 + \frac{\rho_p}{\beta} K_a \right)^3 \left(\frac{R^2 \beta}{De} \right)^2 \frac{1}{9} \left(\frac{2}{35} + \frac{2}{5Bi} + \frac{1}{Bi^2} \right) \right\} \right. \\
 &\quad \left. + \frac{6E_z^2}{u^4} \left(1 + \frac{(1-\alpha)}{\alpha} \beta \left(1 + \frac{\rho_p}{\beta} K_a \right) \right)^2 \frac{(1-\alpha)}{\alpha} \beta \left\{ \frac{\rho_p}{\beta} \frac{K_a^2}{k_a} \right. \right. \right. \\
 &\quad \left. \left. + \left(1 + \frac{\rho_p}{\beta} K_a \right)^2 \frac{R^2 \beta}{De} \frac{1}{3} \left(\frac{1}{5} + \frac{1}{Bi} \right) \right\} \right. \\
 &\quad \left. + \frac{5E_z^3}{u^6} \left(1 + \frac{(1-\alpha)}{\alpha} \beta \left(1 + \frac{\rho_p}{\beta} K_a \right) \right)^4 \right]
 \end{aligned}$$

$$E_z \frac{\partial^2 c}{\partial z^2} - u \frac{\partial c}{\partial z} + \frac{3(1-\alpha)}{\alpha R} N_o = \frac{\partial c}{\partial t} \quad (A1)$$

$$\frac{De}{\beta} \left(\frac{\partial^2 c_i}{\partial r^2} + \frac{2}{r} \frac{\partial c_i}{\partial r} \right) - N_i = \frac{\partial c_i}{\partial t} \quad (A2)$$

$$N_o = k_f (c - c_i) \Big|_{r=R} = -De \frac{\partial c}{\partial r} \Big|_{r=R} \quad (A3)$$

Table 4 (a) The coefficients a_{ij} for Example B

$$\begin{aligned}
 a_{40} &= \frac{K_s K_f}{uHC}, a_{30} = -\frac{K_s}{HC}, a_{21} = -\frac{K_s/C + K_f}{uH} \\
 a_{20} &= -\frac{K_s + K_f}{u}, a_{11} = \frac{1}{H}, a_{10} = 1, \\
 a_{02} &= \frac{1}{uH}, a_{01} = \frac{1+C}{u} \text{ and the other } a_{ij} = 0
 \end{aligned}$$

Table 4 (b) Moments for Example B

$$\begin{aligned}
 \mu_1 &= \frac{z}{u} (1+C) \\
 \mu_2' &= \frac{2z}{u} \left\{ \frac{C}{H} + \frac{(1+C)^2}{u^2} (K_s + K_f) \right\} \\
 \mu_3' &= \frac{6z}{u} \left[\frac{C}{H^2} + \frac{2(1+C)}{u^2 H} \{ (2C+1)K_s + CK_f \} \right. \\
 &\quad \left. + \frac{(1+C)^3}{u^4} (K_s + K_f)^2 \right] \\
 \mu_4' &= \frac{24z}{u} \left[\frac{C}{H^3} + \frac{1}{u^2 H^2} \{ (10C^2 + 12C + 3)K_s + (3C^2 + 6C + 4)K_f \} \right. \\
 &\quad \left. + \frac{(1+C)^2}{u^4 H} \left\{ \frac{15C^2 + 10C + 1}{C} K_s^2 + 4(5C+2)K_s K_f + 6CK_f^2 \right\} \right. \\
 &\quad \left. + \frac{5}{u^6} (1+C)^4 (K_s + K_f)^3 \right]
 \end{aligned}$$

$$N_i = \frac{\rho_p}{\beta} k_a \left(c_i - \frac{n}{\frac{\rho_p}{\beta} K_a} \right) \quad (A4)$$

Eliminating c_i and n from the Laplace transform of the above equations gives the following equation of $\bar{c}$.

$$E_z \frac{\partial^2 \bar{c}}{\partial z^2} - u \frac{\partial \bar{c}}{\partial z} + A_0(p) \bar{c} = 0 \quad (A5)$$

where

$$A_0(p) = -p - \frac{3(1-\alpha)}{\alpha T_d} Bi \left(1 - \frac{Bi}{\Phi(p) \coth \{ \Phi(p) \} + Bi} \right) \quad (A6)$$

$$\Phi(p) = \sqrt{T_d} \left\{ p + \frac{\rho_p}{\beta} \frac{k_a K_a p}{k_a + K_a p} \right\}^{1/2} \quad (A7)$$

$$T_d = \beta R^2 / De, \quad Bi = k_f R / De$$

Then from Eq.(A5) and expansion of $A_0(p)$ around $p=0$, a_{ij} are determined as shown in **Table 3(a)**, which finally give the moments shown in **Table 3(b)** by the application of Table 1. First and second moments coincide with the results shown by Kubin⁴.

Example (B)

Heat transfer in packed beds:

Heat balances in solid and gas phases are given as

$$k_{es} \frac{\partial^2 T_s}{\partial z^2} - h_p a (T_s - T_f) = C_{ps} \rho_s (1-\alpha) \frac{\partial T_s}{\partial t} \quad (B1)$$

$$\begin{aligned}
 k_{ef} \frac{\partial^2 T_f}{\partial z^2} - C_{pf} \rho_f u_o \frac{\partial T_f}{\partial z} + h_p a (T_s - T_f) \\
 = C_{pf} \rho_f \alpha \frac{\partial T_f}{\partial t}
 \end{aligned} \quad (B2)$$

Eliminating $\bar{T}_s$ from the Laplace transform of the above equations gives

$$\begin{aligned}
 \frac{K_s K_f}{uHC} \frac{\partial^4 \bar{T}_f}{\partial z^4} - \frac{K_s}{HC} \frac{\partial^3 \bar{T}_f}{\partial z^3} - \left\{ \frac{K_s/C + K_f}{uH} p \right. \\
 \left. + \frac{K_s + K_f}{u} \right\} \frac{\partial^2 \bar{T}_f}{\partial z^2} + \left(\frac{p}{H} + 1 \right) \frac{\partial \bar{T}_f}{\partial z}
 \end{aligned}$$

$$+\left\{\frac{1}{uH}p^2+\frac{1+C}{u}p\right\}\bar{T}_f=0 \quad (\text{B3})$$

where

$$K_s=k_{es}/C_{pf}\rho_f\alpha, \quad K_f=k_{ef}/C_{pf}\rho_f\alpha, \quad u=u_o/\alpha, \\ C=C_{ps}\rho_s(1-\alpha)/C_{pf}\rho_f\alpha, \quad H=h_p a/C_{ps}\rho_s(1-\alpha)$$

By comparing Eq.(B3) with Eq.(5), a_{ij} become as shown in **Table 4(a)**, which give the resulting moments shown in **Table 4(b)**.

THE PENETRATION THEORY WITH HIGH MASS FLUX AND VARIABLE PROPERTIES*

KORETSUNE UHEYAMA AND JUN-ICHI HATANAKA

Department of Chemical Engineering, University of Osaka Prefecture, Sakai, Japan

Introduction

The purpose of the present note is to modify the penetration theory so as to allow for the effects of high mass flux and variable transport properties on mass transfer rate. More particularly, two liquid phases partially miscible with each other come into contact as in **Fig. 1**, in which the upper phase is of component *A* saturated with the component *B* and the lower phase is of pure *B*, and then binary diffusion occurs only in the lower phase. In such case the mass transfer mechanism is usually described by the simple penetration theory. But it is a more complex problem if we take into account the fact that transport properties vary with concentration of *A* and that diffusional flow is not neglected. Duda and Vrentas³⁾ have shown that in the case of a uniform phase, variation of mass density has an important effect on experimental determination of diffusivity.

Analysis

It is assumed that the diffusion in the lower liquid phase is isothermal and one-dimensional. Furthermore, we assume that there are no interfacial disturbances or interfacial resistances, and that mass density of the solution decreases monotonically with the concentration of component *A*, and hence there is no free convection. The coordinate system is shown in **Fig. 1**.

The governing equation and its boundary conditions are written as

$$\rho \frac{\partial \omega}{\partial t_1} + \rho v_1 \frac{\partial \omega}{\partial x_1} = \frac{\partial}{\partial x_1} \left(\rho D \frac{\partial \omega}{\partial x_1} \right) \quad (1)$$

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〒591 堺市百舌鳥梅町4-804

大阪府立大学工学部化学工学教室 畑中淳一

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$\omega(x_1=x_s)=\omega_s, \omega(t_1=0)=\omega_o, \partial\omega/\partial x_1(x_1=0)=0$ (2)
where x_s is the distance between the bottom of the cell and the interface. Introducing new variables, Eqs.(1) and (2) are rewritten as

$$\left. \begin{array}{l} x_1=x_s(t)-x \\ t_1=t \end{array} \right\} \quad (3)$$

$$\rho \frac{\partial \omega}{\partial t} + \rho(v-v_*) \frac{\partial \omega}{\partial x} = \frac{\partial}{\partial x} \left(\rho D \frac{\partial \omega}{\partial x} \right) \quad (4)$$

$$\omega(x=0)=\omega_s, \omega(t=0)=\omega_o, \partial\omega/\partial x(x=x_s)=0 \quad (5)$$

where v_* is the velocity of the interface and $v=-v_1$. Eqs.(4) and (5) show that the governing equation can be reduced to the equation with a static interface by using the relative velocity to the interface. The over-all continuity equation is written as

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho(v-v_*)}{\partial x} = 0 \quad (6)$$

$$\rho(v-v_*) - \rho_s(v_s-v_*) = - \int_0^x \frac{\partial \rho}{\partial t} dx \quad (7)$$

Ratio of mass flux relative to the velocity of the interface, v_* , is defined as

$$\omega_T \equiv [\rho_{As}(v_{As}-v_*)]/[\rho_s(v_s-v_*)] \quad (8)$$

Mass flux of *A* relative to mass average velocity is expressed as

$$\rho_A(v_A-v) = -D\rho\partial\omega/\partial x \quad (9)$$

Combination of Eqs.(8) and (9) gives

$$\rho_s(v_s-v_*)(\omega_T-\omega_s) = -\rho_s D_s \partial\omega/\partial x|_s \quad (10)$$

Substituting Eqs.(7) and (10) into Eq.(4), we obtain

Fig. 1 Co-ordinate system