Data Driven Synthesis of Three Term Digital Controllers

Lee H. Keel *, Sandipan Mitra **, and Shankar P. Bhattacharyya ***

Abstract : This paper presents a method for digital PID and first order controller synthesis based on frequency domain data alone. The techniques given here first determine all stabilizing controllers from measurement data. In both PID and first order controller cases, the only information required are frequency domain data (Nyquist-Bode data) and the number of open-loop RHP poles. Specifically no identification of the plant model is required. Examples are given for illustration.

Key Words: digital control, first order control, model free, PID control.

1. Introduction

In much of industrial practice, controllers are designed without an analytical model of the plant. Instead, design is based on other available information about the plant. Typical information available for the plant is time or frequency response of the plant [1]. For stable plants, engineers use a sinusoidal input to obtain the response of a linear time-invariant system. For unstable plants, the plants are stabilized first, then a sinusoidal input is given to obtain the frequency response. Indeed, this approach is as popular as the plant model based approaches in classical control design.

In modern/postmodern control theory a different concept of control design was introduced based on a state space model. The technique parametrizes all controllers that stabilize the given plant. An optimal controller is then selected by searching the space of stabilizing controllers [2]. One of the severe drawbacks of this otherwise elegant approach is that it tends to produce excessively high order controllers. Thus, controller order reduction and subsequent re-validation of the reduced order controller are often required. In Hara, Shikata, and Iwasaki [3], the generalization of the KYP lemma designed to be valid over prescribed frequency ranges was developed to deal with fixed order controller synthesis. Henrion et al.[4] have advocated a relaxation approach to the design of fixed order controllers. Haddad, Hwang, and Bernstein [5] have discussed the design of fixed order controllers in the discrete-time case. Iwasaki and Skelton [6] have studied design of H∞ controllers of fixed order. Dorato [7] has advocated the use of quantifier elimination (QE) techniques to deal with the fixed order controller design problem. Gryazina and Polyak [8] have revisited Neimark’s D-Decomposition technique [9],[10] to design fixed order controllers. There has been a number of papers addressing the fixed order controller design problem using LMI techniques [11]. New results that completely determine the entire set of PID and first order stabilizing controllers for a given LTI plant were also introduced in [12]-[14]. Using this stabilizing set, smaller sets that satisfy additional closed-loop performance and/or robustness requirements were also found [15]. These techniques also require some form of mathematical model of the plant.

Recently, a model free synthesis approach based on time series data has been proposed by Ikeda [16],[17]. Design of control based on input output data has also been advocated by Skelton and Shi [18], Furuta [19], Chan [20]. These approaches to find the entire sets of PID and first order stabilizing controllers (three term controllers) that satisfy prescribed performance specifications were also extended to deal with plants without analytical models [21],[22]. The approaches are based on frequency domain responses of an LTI plant. It has been shown that the performance attainment problem can also be cast as a problem of stabilizing a family of complex plants.

In the present paper we consider digital three term controllers to be designed for plants where the only information available is the frequency response data. We show how complete sets achieving stability and performance can be determined from this data alone without identifying the plant model. Examples are included for illustration. These techniques have been implemented in Labview and an interactive computer aided design tool has been developed. This paper also includes a demonstration of this computer aided control design tool.

2. Notation and Preliminaries

Consider the rational function

\[ Q(z) = \frac{P_1(z)}{P_2(z)} \]  

where \( P_i(z), i = 1,2 \) are real polynomials, which are assumed to be devoid of zeros on the unit circle. Let \( i_e \) \( (i_p) \) denote the numbers of zeros and poles of \( Q(z) \) located inside the unit circle. Also let \( \Delta_\theta^e \) \( Q(e^{j\theta}) \) denote the net change in phase of \( Q(e^{j\theta}) \) as \( \theta \) runs from 0 to \( \pi \). Then we have

\[ \Delta_\theta^e \Rightarrow Q(e^{j\theta}) = \pi (i_e - i_p). \]  

This follows from the fact that each zero (each pole) inside the unit circle contributes \( +\pi \) (\( -\pi \)) to the net phase change whereas zeros (poles) outside the unit circle do not contribute to the net phase change. We now define the signature of \( Q(z) \) as

\[ \sigma(Q) := i_e - i_p. \]
To evaluate $Q(z)$ on the unit circle we use the Tchebyshev decomposition $[13],[23].$ Set

$$u := -\cos \theta, \quad v := \sqrt{1 - u^2}$$  \hfill (4)

and

$$z = e^{j\theta} = -u + jv.$$  \hfill (5)

Then

$$z^k = e^{jk\theta} = \cos k\theta + j \sin k\theta$$  \hfill (6)

where

$$\cos k\theta := c_k(u) \quad \text{and} \quad \frac{\sin k\theta}{\sin \theta} = \frac{\sin k\theta}{v} := s_k(u)$$  \hfill (7)

and $c_k(u)$ and $s_k(u),$ the Tchebyshev polynomials are real polynomials in $u$ with

$$s_k(u) = \frac{1}{k!} \frac{d^k}{du^k} (u^k - 1), \quad k = 1, 2, \cdots$$

and

$$c_{k+1}(u) = -uc_k(u) - v^2 s_k(u), \quad k = 1, 2, \cdots.$$  

Let

$$P_i(z)|_{z=-u+jv} = R_i(u) + jvT_i(u), \quad i = 1, 2$$

denote the Tchebyshev representations of $P_i(z), i = 1, 2$ where $R_i(u)$ and $T_i(u)$ are real polynomials in $u.$ Then it is easy to see that

$$Q(z)|_{z=-u+jv} = R_q(u) + jvT_q(u) =: \dot{Q}(u)$$  \hfill (8)

where

$$R_q(u) = \frac{R(u)}{D(u)}, \quad T_q(u) = \frac{T(u)}{D(u)}$$  \hfill (9)

and

$$R(u) = R_1(u)R_2(u) + v^2 T_1(u)T_2(u)$$

$$T(u) = T_1(u)R_2(u) - R_1(u)T_2(u)$$

$$D(u) = R_2^2(u) + v^2 T_2^2(u).$$  \hfill (10)

Since $D(u) > 0$ for all $u \in [-1, 1],$ the zeros of $T_q(u)$ are identical to those of $T(u).$ Let $\text{sgn}[\cdot]$ denote the usual sign function. Then $\text{sgn}[R_q(.)] = \text{sgn}[R(.)].$ Suppose that $T(u)$ has $p$ zeros at $u = -1$ and let $t_1, \cdots, t_p$ denote the real distinct zeros of $T(u)$ of odd multiplicity ordered as follows:

$$-1 < t_1 < t_2 < \cdots < t_p < 1.$$

We now state the following signature formula for a rational function.

**Theorem 1** Let $Q(z)$ be a real rational function with $i_c$ and $i_p$ being zeros and poles, respectively, inside the unit circle $C$ and no poles or zeros on the unit circle. Then

$$\sigma(Q) := i_c - i_p = \frac{1}{2} \text{sgn} \left[ T_q^{(p)}(-1) \right] \left[ \text{sgn}[R_q(-1)] \right]$$

$$+ 2 \sum_{j=1}^{p} (-1)^j \text{sgn}[R_q(t_j)] + (-1)^{j+1} \text{sgn}[R_q(+1)].$$

**Proof:** Note that

$$\Delta_{-1}^p Q(e^{j\theta}) = \pi \text{sgn}(Q)$$  \hfill (11)

and

$$\Delta_{-1}^p Q(e^{j\theta}) = \Delta_{-1}^{p+1} \dot{Q}(u),$$

$$= \Delta_{-1}^p \dot{Q}(u) + \Delta_{-1}^{p+1} \dot{Q}(u) + \cdots + \Delta_{-1}^{1} \dot{Q}(u).$$

We also have

$$\Delta_{-1}^{p+1} \dot{Q}(u) = \frac{\pi}{2} \text{sgn} \left[ T_q(t_j^*) \right] \left( \text{sgn}[R_q(t_j)] - \text{sgn}[R_q(t_{j+1})] \right),$$

$$\text{for } i = 0, 1, \cdots, k$$

$$\text{sgn}[T_q(t_j^*)] = -\text{sgn}[T_q(t_{j+1})],$$

$$\text{for } i = 0, 1, \cdots, k$$

$$\text{sgn}[T_q(-1^*)] = \text{sgn}[T_q^{(p)}(-1)].$$

Thus,

$$\sigma(Q) = \frac{1}{2} \text{sgn} \left[ T_q(-1^*) \right] \left( \text{sgn}[R_q(-1)] - \text{sgn}[R_q(t_1)] \right)$$

$$+ \frac{1}{2} \text{sgn} \left[ T_q(t_2^*) \right] \left( \text{sgn}[R_q(t_1)] - \text{sgn}[R_q(t_2)] \right)$$

$$+ \cdots + \frac{1}{2} \text{sgn} \left[ T_q(t_p^*) \right] \left( \text{sgn}[R_q(t_{p-1})] - \text{sgn}[R_q(+1)] \right)$$

$$= \frac{1}{2} \text{sgn} \left[ T_q^{(p)}(-1) \right] \text{sgn}[R_q(-1)]$$

$$+ 2 \sum_{j=1}^{p} (-1)^j \text{sgn}[R_q(t_j)] + (-1)^{j+1} \text{sgn}[R_q(+1)].$$

The above signature formula is the key to constructing a set of linear inequalities representing stabilizing regions in the controller parameter space. In subsequent sections, we first develop a way to use the signature formula in Theorem 1 without knowledge of any form of analytical models to develop a computation of all digital PID controllers stabilizing the plant. The design of first order controllers without analytical models will also be dealt with similarly.

Let the plant transfer function be the rational function

$$P(z) = \frac{N(z)}{D(z)}$$  \hfill (13)

**Assumption 1**

1. The plant is controllable and observable, that is, the polynomials $N(z)$ and $D(z)$ are coprime.

2. The plant has no poles on the unit circle.

3. We assume that the only information available for the plant to the designers are:

A. Knowledge of the frequency response magnitude and phase, i.e.,

$$P(e^{j\theta}), \quad \text{for } \theta \in [0, 2\pi],$$

B. Knowledge of the number of unstable poles of the plant, that is, poles of the plant located outside the unit circle.

C. Knowledge of the relative degree $r$ of the plant.
Consider a stable plant shown in Fig. 1.

\[ u(t) = \sin \omega t \]

Fig. 1 Stable linear time-invariant discrete-time system.

Then it is easy to see that the steady state value of the discrete time output is

\[ y_s(kT) = \left| P(e^{j\omega T}) \right| \sin (k\omega T + \angle P(e^{j\omega T})). \]

The frequency response of the plant can be thus determined from the measurement:

\[ P(e^{j\omega T}) = Me^{j\theta} \]  

where

\[ M := \left| P(e^{j\omega T}) \right| \quad \text{and} \quad \theta := \angle P(e^{j\omega T}). \]

Clearly this amounts to knowledge of the complex function

\[ P(z)\big|_{z \rightarrow -e^{j\omega T}} = R_p(u) + jvT_p(u) := P_i(u), \]

that is, knowledge of the rational functions \( R_p(u) \) and \( T_p(u) \) evaluated for \( n \in [-1, +1]. \)

3. PID Controllers for Discrete-time Systems

Consider the feedback system shown in Fig. 2.

![Unity feedback system](image)

Fig. 2 Unity feedback system.

Consider the general formula of a discrete-time PID controller

\[ C(z) = K_P + \frac{K_I T_i}{z - 1} + \frac{K_D(z - 1)}{T_D z} = \frac{K_2 z^2 + K_1 z + K_0}{z - 1} \]

where

\[ K_P = -K_1 - 2K_0, \quad K_I = \frac{K_0 + K_1 + K_2}{T}, \quad K_D = K_0 T. \]

The closed-loop characteristic polynomial is

\[ \delta(z) := z(z - 1)D(z) + \left( K_0 z^2 + K_1 z + K_0 z \right)N(z). \]

Closed-loop stability requires that \( \sigma(\delta) = n + 2. \)

With

\[ \Pi(z) := \frac{\delta(z)}{D(z)} = z(z - 1) + \left( K_0 z^2 + K_1 z + K_0 z \right)P(z), \]

closed-loop stability requires that

\[ \sigma(\Pi) = n + 2 - i_p \]

where \( i_p \) is the number of poles of \( P(z) \) located inside the unit circle. Finally, introduce

\[ \nu(z) = z^{-1}P(z^{-1})\Pi(z) = (z - 1)P(z^{-1}) + \left( K_0 z^{-1} + K_1 + K_2 z \right)P(z)P(z^{-1}). \]

It will turn out that the solution to the PID stabilization problem can be conveniently obtained in terms of the signature of \( \nu(z) \) because the property of parameter separation in the real and imaginary parts.

Lemma 1

The net change of phase of \( P(e^{j\theta}) \) for \( \theta \in [0, \pi] \) is:

\[ \Delta \nu e^{j\theta}P(e^{j\theta}) = -\pi \left[ r + (\alpha_p - \alpha_f) \right] \]

where \( \alpha_f \) and \( \alpha_p \) are the numbers of zeros and poles of \( P(z) \) located outside the unit circle, respectively and \( r \) is the relative degree of the plant \( P(z). \)

Proof: Let \( m \) and \( n \) be the degrees of \( N(z) \) and \( D(z) \), respectively. Then

\[ \pi (i_p - i_f) = \pi \left[ (m - \alpha_f) - (n - \alpha_p) \right] = -\pi \left[ r + (\alpha_p - \alpha_f) \right], \]

(21)

Theorem 2

Let \( P(z) \) be the plant with relative degree \( r \). Let the PID controller be

\[ C(z) = \frac{K_P z^2 + K_I z + K_0}{z - 1}. \]

Then the closed-loop system is stable if and only if

\[ \sigma(\nu) = r + \alpha_z + 1. \]

Proof: Note the fact that

\[ P(z^{-1}) = \frac{N(z^{-1})}{D(z^{-1})} = z^{-m}P(z) = \frac{z^{-m}P(z)}{\pi(\nu)} \]

where \( m \) and \( n \) are degrees of \( N(z) \) and \( D(z) \), respectively, and \( N_r(z) \) and \( D_r(z) \) are reverse polynomials of \( N(z) \) and \( D(z) \), respectively. Also note that

\[ \sigma(P_i) = \alpha_z - \alpha_p \]

\[ \sigma(P(z^{-1})) = \sigma(z^{-1}P_i) = r + \alpha_z - \alpha_p \]

(22)

(23)

where \( \alpha_z \) and \( \alpha_p \) are the numbers of zeros and poles of \( P(z) \) located outside the unit circle. Then

\[ \sigma(\nu) = \sigma \left[ z^{-1}P(z^{-1})\Pi(z) \right] = -1 + r + \alpha_z - \alpha_p + n + 2 - i_p \]

\[ = r + \alpha_z + 1 + n - (\alpha_f + i_p) \]

\[ = r + \alpha_z + 1. \]

(24)

In what follows, we show that it is possible to find all \( K_0, K_1, \) and \( K_2 \) values that satisfy the signature (stability) condition directly from the raw data and the transfer function of the system is not required.

To evaluate \( \sigma(\nu) \), write

\[ P(z)\big|_{z \rightarrow -e^{j\omega T}} = \frac{N(z)}{D(z)} = \frac{R_X(u) + \nu T_X(u)}{R_D(u) + \nu T_D(u)} \]

\[ = \frac{R_D(u)R_D(u) + \nu^2 T_X(u)T_D(u)}{R_D(u) + \nu T_D(u)} \]

\[ = \frac{R_D(u) + \nu T_D(u)}{R_D(u)} \]

\[ = R_D(u) + \nu T_D(u). \]
Consequently,
\[ P(z^{-1})|_{z=a+uj} = R_p(u) - jνT_p(u). \]

Now
\[ v(z)|_{z=a+uj} = z^{-1}P(z^{-1}) \Pi(z)|_{z=a+uj} = \left[ (z-1)P(z^{-1}) + (K_0z^{-1} + K_1 + K_0z) P(z)P(z^{-1}) \right]_{z=a+uj} \]
\[ = (-u - 1 + jν)(R_p(u) - jνT_p(u)) \]
\[ + \left[ K_0(-u - jν) + K_1 + K_2(-u - jν) \right] m(u) \]
where
\[ m(u) = |P(z)|_{z=a+uj}. \]
Then
\[ v(z) = R_i(u, K_0, K_1, K_3) + jνT_v(u, K_3) \]
where
\[ K_3 := K_2 - K_0 \]
\[ R_i(u, K_0, K_1, K_3) = -(u + 1)R_p(u) + (1 - u^2)T_p(u) + K_1m^2(u) - u(2K_0 + K_3)m^2(u) \]
\[ T_v(u, K_3) = R_p(u) + (1 + 1)T_p(u) + K_3m^2(u). \]

For fixed \( K_3 = K_3^* \), we may solve
\[ K_3^* = \frac{-R_p(u) - (u + 1)T_p(u)}{m^2(u)} =: g(u) \tag{25} \]
determine the real roots \( t_1, t_2, \ldots \) of odd multiplicities of eq. \( (25) \) contained in the open interval \((-1, +1)\):
\[ t_0 = -1 < t_1 < t_2 < \cdots < t_i < t_{i+1} = +1 \]
and develop linear inequalities corresponding to stability as follows. Let
\[ I^i = \{ i_0^i, i_1^i, \cdots, i_k^i, i_{k+1}^i \} \]
denote a string where
\[ i_j^i \in \{0, 1, -1\} \]
and let \( k \in \{+1, -1\} \) such that
\[ \frac{k}{2} \left( i_0^i - 2i_1^i - 2i_2^i - \cdots - (-1)^{i_{k+1}^i} \right) = r + \sigma_i + 1. \tag{26} \]

For each string \( I^i \) satisfying eq. \( (26) \), we have the set of inequalities
\[ \text{sgn} \left[ T_v(-1^i) \right] \cdot k > 0 \tag{27} \]
\[ \text{sgn} \left[ R_i(u_t, K_0, K_1, K_3^*) \right] i_j^i > 0 \tag{28} \]
which is a set of linear inequalities in \( K_0, K_1 \) space for fixed \( K_3^* \). By constructing these inequalities for each string satisfying eq. \( (26) \), we obtain the stabilizing set for \( K_3 = K_3^* \). By sweeping over \( K_3 \) we can generate the complete set. The range of \( K_3 \) to be swept is determined by the requirement that eq. \( (25) \) should have \( r + \sigma_i \) roots at least.

We now illustrate that the above signature relationship corresponding to stability can be computed without knowing the plant transfer function \( P(z) \) but from knowledge of the frequency response.

**Example 1.** To illustrate, we take a set of frequency data points from the example used in [13]. Let us assume that the following information is available.

**Available Information:**

1. Frequency domain data
\[ P(e^{jωT}) := \left\{ P(e^{jωT}), ω = \frac{2π}{T} \text{ sampled every } T = 0.01 \right\}. \]
2. The plant is stable. In other words, the number of unstable poles of the plant is 0, that is, \( σ_p = 0 \).
3. The relative degree of the plant is 2, that is, \( r = 2 \).

The Nyquist plot of the plant \( P(z) \) is shown in Fig. 3.

![Nyquist Diagram](image)

**Fig. 3** Nyquist plot of \( P(e^{jω}) \).

From Fig. 3 and Lemma 1,
\[ Δ_0^T P(e^{jω}) = -π \left\{ r + (σ_1 - σ_p) \right\} := -2π. \]

Therefore,
\[ σ_1 = 2 - r + σ_p = 2 - 2 + 0 = 0. \]

Using Theorem 2, the stability requirement is equivalent to \( σ(ω) = 2 + 0 + 1 = 3 \).

Then Theorem 1 requires that
\[ \frac{1}{2} \text{sgn} [T(-1)] [\text{sgn} [R(-1)] - 2\text{sgn} [R(t_1)] + 2\text{sgn} [R(t_2)] - \cdots \text{sgn} [R(1)]] = 3 \]
where \( t_i \) are the zeros of \( g(u) \) in \( (25) \) for fixed \( K_3 \). It is easy to see that at least two zeros \( t_i \) are required and also that the only feasible string of sign sequences is:
\[ \text{sgn of } T(-1) R(-1) R(t_1) t_2 R(1) \]

\[ \begin{array}{cccc}
1 & 1 & -1 & 1 \\
1 & 1 & -1 & -1 \\
\end{array} \]

The feasible range of \( K_3 \) values is that corresponding to the requirement of two zeros in \( T(u) \). We now plot the right hand side of eq. \( (25) \). Using the relationship \( u = -\cos θ \), we have the set of Nyquist data points in \( u \) axis.

\[ \left. P(e^{jωT}) \right|_{u=-\cos θ} = P(e^{jω}) = R_p(u) + jνT_p(u). \]

Using eq. \( (25) \), we now plot the following (see Fig. 4).

\[ K_3^* = \frac{1}{m^2(u)} (-R_p(u) - (1 + u)T_p(u)). \]

Using eq. \( (28) \), we construct the set of linear inequalities for
Then the set of linear inequalities corresponding to parameter regions shown in Fig. 5 that is identical to the region $u$ accurate data of the system is needed up to the frequency where the procedure shown in Fig. 4 also tells us that ac-

For each $K_3$ value. For example, at $K_3 = 1.3$, it is found from Fig. 4 that $t_1 = -0.4736$, $t_2 = -0.0264$.

Then the set of linear inequalities corresponding to $K_3 = 1.3$ is

$$T(-1) = 1$$

$$R(-1) = -2.311 + 1.7777K_1 + 3.5556K_2 > 0$$

$$R(-0.4736) = -0.6939 + 0.7473K_1 + 0.7078K_2 < 0$$

$$R(-0.0264) = 0.7226 + 0.6403K_1 + 0.0338K_2 > 0$$

$$R(1) = -0.3556 + 1.7777K_1 - 3.5556K_2 < 0$$

By sweeping $K_3$ over (-0.7, 1.4), we have the stabilizing PID parameter regions shown in Fig. 5 that is identical to the region obtained in [13].

**Remark 1** The procedure shown in Fig. 4 also tells us that accurate data of the system is needed up to the frequency where $u = 0.5$, that is,

$$\omega = \cos^{-1} u \frac{T}{\lambda}$$

To obtain the measurement data, it may not be necessary to excite the system beyond this frequency.

### 4. First Order Controllers for Discrete-time Systems

Let the plant and controller transfer functions be

$$P(z) = \frac{N(z)}{D(z)}$$

$$C(z) = \frac{x_1z + x_3}{z + x_3}$$

(29)

In [14], it is shown that the entire set of first order stabilizing controllers for a given discrete-time LTI plant can be characterized in $(x_1, x_2, x_3)$ space by at most two straight lines and one curve. The analytic expressions of the two straight lines and the curve are obtained in terms of the plant transfer function coefficients. In this section, we give new set of expressions that are equivalent but rely only on the frequency domain data points of the plant instead of its analytical model. Unlike the PID controller case described in the previous section, the assumption regarding the knowledge of the relative degree of the plant is not required in this case. All other assumptions listed in Assumption 1 hold.

Consider the frequency response of the discrete-time plant $P$:

$$P(z)_{\text{eq}_{u=\omega}} = R_p(u) + \nu T_p(u).$$

(30)

Note that $R_p(u)$ and $T_p(u)$ for $-1 \leq u \leq 1$ are immediately available from the given frequency response data points provided by $P(e^{i\theta})$ for $\theta \in [0, \pi]$.

Consider the real rational function

$$F(z) = (z + x_1) + (z_1x_1 + x_2) P(z).$$

(31)

**Theorem 3** Let $P(z)$ be the plant with the number of unstable poles being $o_p$. Let the first order controller be

$$C(z) = \frac{x_1z + x_3}{z + x_3}.$$  

Then the closed-loop system is stable if and only if

$$\sigma(F) = o_p + 1.$$

**Proof:** Let $n$ be the order of the plant $P(z)$ and $i_p$ be the number of stable poles of $P(z)$. Then closed-loop stability requires that

$$\sigma(F) = n + 1 - i_p$$

$$= (o_p + i_p) + 1 - i_p$$

$$= o_p + 1.$$

(32)

$$F(z)_{\text{eq}_{u=\omega}} = (-u + x_3 + \nu) + \left(-ux_1 + x_2 + \nu x_3\right)$$

$$R_p(u) + \nu T_p(u)$$

$$= (x_3 - u) + R_p(u)x_2$$

$$- \left(\frac{R_p(u)}{T_p(u)}\right) x_1$$

$$+ \nu \left(\frac{R_p(u)}{T_p(u)}\right)^{-1} x_2$$

For complex root crossing, $\nu \neq 0$ and we have the expression for the stability boundary in $(x_1, x_2)$ space as:

$$\begin{bmatrix}
- \left(\frac{R_p(u)}{T_p(u)}\right) & R_p(u) \\
R_p(u) - uT_p(u) & T_p(u)
\end{bmatrix}^{-1} x_1(u)$$

$$A(u)$$

$$x_2(u).$$

(33)

Since

$$\det[A(u)] = - \left(\frac{R_p(u)}{T_p(u)}\right) - \nu^2 T_p(u)$$
the solution of the above is
\[
\begin{bmatrix}
  x_1(u) \\
  x_2(u)
\end{bmatrix} = -\frac{1}{|P(e^{i\theta})|^2} \begin{bmatrix}
  A_1 \\
  A_2
\end{bmatrix}
\]
where

\[
A_1 = - (x_1 - u) T_p(u) + R_p(u)
\]
\[
A_2 = (x_3 - u) R_p(u) - u T_p(u)
\]
\[
+ (u R_p(u) + v^2 T_p(u))
\]

and
\[
\begin{bmatrix}
  x_1(u) \\
  x_2(u)
\end{bmatrix} = \frac{(x_3 - u) T_p(u) - R_p(u)}{|P(e^{i\theta})|^2}
\]
\[
- \left(-u x_3 + u^2 + v^3\right) T_p(u) + x_3 R_p(u)
\]
\[
= -\frac{1}{|P(e^{i\theta})|^2} \begin{bmatrix}
  (u - x_3) T_p(u) + R_p(u) \\
  (1 - u x_3) T_p(u) + x_3 R_p(u)
\end{bmatrix}.
\]

The two straight lines representing the real root crossings can be obtained from the expression of \( F(z) \) by letting \( u = -1 \) and \( u = 1 \), equivalently letting \( \theta = 0 \) and \( \theta = \pi \).

\[
(x_3 - 1) + P(e^{i\theta})(x_2 - x_1) = 0 \quad (34)
\]
\[
(x_3 - 1) + P(e^{i\theta})(x_3 - x_1) = 0. \quad (35)
\]

**Example 2** To illustrate the method, we take a set of frequency domain data points from the plant used in [14].

**Available Information:**

1. Frequency domain data

\[
P(e^{i\theta}), \omega = \frac{\pi}{T} \text{ sampled every } T = 0.01.
\]

2. The plant is stable, i.e., the number of poles outside the unit circle is 0, that is, \( \alpha_p = 0 \).

At \( x = 0.75 \), Fig. 6 is obtained. Note that each separated region in Fig. 6 represents a set of controller parameters that gives a fixed number of unstable poles of the closed-loop system. To identify the stabilizing region, we arbitrarily select a point from each region and plot the corresponding Nyquist plot, that is,

\[
P(e^{i\theta}) \frac{x_1 z + x_2}{z + x_3}
\]

Figure 7 shows the Nyquist plots with selected controllers from the four specified regions.

The Nyquist plot with a controller from Region 1 shows that the encirclement around \(-1\) point is 2\(\pi\), i.e., \( N = 2 \). Since \( \alpha_p = 0 \), the corresponding closed-loop system will have 2 poles outside the unit circle. Similarly, corresponding closed-loops system with controllers from Region 3 and 4 will have 2 and 3 poles outside the unit circle, respectively. This test led us to the conclusion that the region 2 is the only stabilizing controller parameter region. By sweeping over \( x_3 \), we have the entire set of first order stabilizing controllers for the given plant shown below in Fig. 8.

![Fig. 6 Root invariant regions with first order controller.](image)

![Fig. 7 Nyquist plots with selected controllers.](image)

![Fig. 8 All stabilizing first order controllers.](image)

**5. Computer Aided Design Using Labview**

The algorithm for the design of a discrete time PID Controller from the frequency response data of a stable system has been programmed in LabVIEW due to its user-friendly graphical environment. The Virtual Instrument (VI) has a front panel that is displayed to the user and a block diagram, where the computations are performed. The inputs to the LabVIEW program are the frequency response data of the stable system, the sampling time and the number of samples to be considered to design the controller. Given these inputs, the entire range of \( K_3 \) that can stabilize the system is displayed, and as the user scrolls through the stabilizing range of \( K_3 \), entire stabilizing ranges of \( K_1 \) and \( K_2 \) are displayed. When some value of controller param-
eters are chosen, the performance parameters like gain margin, phase margin of the open loop and rise time, overshoot, peak time and pole zero placement of the closed loop system can be displayed. Additionally the entire 3-D stabilizing set can also be displayed. Furthermore, when some performance constraints are specified, the subset achieving the desired performance criteria can also be displayed. Once a particular value of controller parameters are chosen based on performance criteria, it can be converted to $K_p$, $K_i$ and $K_d$ values through a simple linear transformation. The two examples described below illustrate the above capabilities.

Example 3 The file containing the frequency response data of a stable system is fed into the program through the file path box located at the top left hand side of the VI as shown in Fig. 9. When the number of samples and sampling time are selected, the stabilizing range of $K_3$ is displayed. On selecting a particular value of $K_3$, (=0.5 in the example), the corresponding stabilizing region in $K_1$-$K_2$ space is obtained.

On choosing particular values, $K_1 = 0.35$ and $K_2 = -0.6$ corresponding to the chosen value of $K_3$, the performance parameters are displayed as shown in Fig. 10.

Note that all the performance indicators are so arranged that higher values correspond to better performance. This helps in better understanding of how the system behaves when browsing through $K_1$-$K_2$ values. The pole-zero position corresponding to the above controller parameters is also shown in Fig. 11.

The entire 3-D stabilizing region is shown in Fig. 12. On specifying some performance criteria like Gain Margin > 4db, Phase Margin > 25° and overshoot < 50%, the subset achieving the required criteria is shown in Fig. 13.
Example 4 The frequency response of another stable system as shown in Fig. 14 is obtained from a file. Carrying out similar steps as in the previous example, the $K_1 - K_2$ stabilizing set is obtained for $K_3 = 1.5$.

The subset satisfying the criteria Gain Margin $> 2\,\text{db}$ is as shown in Fig. 15. Further when the constraint Overshoot $< 50\%$ is imposed, the set shrinks as shown in Fig. 16. When a third condition Phase Margin $> 14^\circ$ is imposed, the set further shrinks as shown in Fig. 17. This illustrates the fact that as more and more conditions are imposed, the resultant set achieving all specifications is the subset of the previous set.

6. Concluding Remarks

The method given here shows that complete stabilizing regions can be constructed without analytical state space or transfer function models provided we know the frequency response measurements and the number of unstable poles for three term controllers. The inclusion of performance requirements leads to complex stabilization problems for families of plants which can be solved in like manner.

We note that an alternative to the results given here is to apply a model based control design to a mathematical model identified from the frequency domain data. Thus, under the assumption of perfect identification, availability of frequency domain data is equivalent to that of a mathematical model. However, identification involves approximation and assumptions on the system order. Because of this, designs by the two approaches are in general not equivalent and the resulting controllers will have different properties. These issues are subject to further investigation. Thus, we believe that the proposed method is a good complement to the existing model based design methods. The extension of these concepts to higher order and MIMO controllers is a topic of future research.

References


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### Lee H. Keel

Received the B.S. degree in Electronic Engineering from Korea University, Seoul, Korea in 1978, and the M.S. and Ph.D degrees in Electrical Engineering from Texas A&M University, College Station, Texas in 1983 and 1986, respectively. Since then, he has been with the Department of Electrical & Computer Engineering and also with the Center of Excellence in Information Systems at Tennessee State University where he is now Professor. His research interests include robust control, system identification, structure and control, and computer aided design. He has authored and coauthored over 150 technical papers in the field of control systems and three books. He is a senior member of IEEE and AIAA.

### Sandipan Mitra

Received his B.E. degree in Electrical and Electronics Engineering from National Institute of Technology Karanataka, Surathkal, India in 2004 and the M.S. in Electrical Engineering from Texas A&M University, College Station, Texas in 2007. From 2004 to 2005, he worked as a software developer at Infosys Technologies Ltd, India. In 2007, he joined The MathWorks, Inc. where he is presently working. His research interests include Computer-aided design and synthesis of PID Controllers.

### Shankar P. Bhattacharyya

Was born in Yangon, Myanmar on June 23, 1946. He obtained the B. Tech. degree from IIT Bombay in 1967 and the M.S. and Ph.D degrees from Rice University in 1969 and 1971. He is presently the Robert M. Kennedy Professor of Electrical Engineering at Texas A&M University. His contributions to control theory span 40 years and include the first solution of the linear servomechanism problem, the theory of robust and unknown input observers, a pole assignment algorithm based on Sylvester’s equation, the computation of the parametric stability margin, a generalization of Kharitonov’s theorem, the demonstration of the fragility of optimal and high order controllers, the synthesis of PID and fixed order controllers and most recently an approach to model free, data driven controller synthesis. These are documented in 5 books and over 200 journal and conference papers authored or coauthored by him. Bhattacharyya is also a performing artist and has played concerts of Indian Classical music on the Sarode, in several countries.