Coordinated Rhythmic Motion by Uncoupled Neuronal Oscillators with Sensory Feedback

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Abstract: This paper explores the potential of biological oscillators as a basic unit for feedback control to achieve rhythmic motion of locomotory systems. Among those properties of biological control systems that are useful for engineering applications, we focus on decentralized coordination, that is, the ability of uncoupled neuronal oscillators to coordinate rhythmic body movements to achieve locomotion with the aid of local sensory feedback. We will consider the reciprocal inhibition oscillator (RIO) as a candidate for the basic control unit, and show that uncoupled RIOs can achieve decentralized coordination of a prototype mechanical rectifier (PMR) that captures essential dynamics underlying animal locomotion by a simple arm-disk configuration. Optimality of the induced locomotion is studied in comparison with analytical results we derive for statically optimal PMR locomotion.

Key Words: animal locomotion, central pattern generator, oscillation, decentralized control.

1. Introduction

Most animal locomotions such as walking, swimming, crawling, and flying, are achieved by periodic motion of the body, coupled with the surrounding environment to generate propelling force. For instance, a snake crawls by undulating its body on the ground, while a bird flies by flapping its wings in the air. The coupled body-environment system may be conceived as a “mechanical rectifier” that converts a periodic motion input to a biased forward velocity [1], analogously to the electrical rectifier that converts an AC signal to a DC signal. On the other hand, rhythmic animal motions are known to be achieved by certain neuronal circuits called the central pattern generators (CPGs). The CPG is an autonomous nonlinear oscillator that can exhibit, even when isolated from the body, a specific oscillation profile (e.g. frequency and phase) of the neuronal membrane potentials similar to the kinematic rhythm. During locomotion, the CPG acts as the feedback controller, receiving sensory inputs and activating the muscle contractions. Thus, animal locomotion is realized by effective coupling of a CPG and a mechanical rectifier.

Figure 1 summarizes our view of the animal locomotion system, based on the above observations. The dashed box is the mechanical rectifier formed by the interaction of the physical body and the surrounding environment. The (force) input $u$ and the locomotion variable $y$ (such as translational velocity) go through the body dynamics to generate some rhythmic motion pattern $x$ which interacts with the environment to yield the locomotion $y$. The input $u$ is generated by the CPG based on the sensory feedback of body motion $x$ and the descending control signal $r$. The desired locomotion profile is encrypted in the high-level signal $r$, determined by the brain based on the information on the locomotion environment $z$ and other factors $w$. The boxes marked as A and S stand for the actuation (e.g. muscle) and sensing (e.g. receptor neuron), respectively.

CPGs have been extensively studied for a wide variety of vertebrates and invertebrates, and their mathematical models have been developed and validated through carefully designed experiments [2]–[5]. The CPG consists of a group of interconnected neurons. The cell membrane potentials of the neurons oscillate at a certain frequency with specific phase relations, generating a pattern for the muscle activation. The oscillation pattern is predominantly determined by the topological architecture of neural interconnections. For instance, the body waves traveling from head to tail in the swimming motion of lampreys or leeches are generated by CPGs formed by weakly coupled segmental oscillators in a chain [6]–[8]. With sensory feedback, the CPG modifies its oscillation pattern to conform with the biomechanical and environmental constraints. The role of sensory feedback in the CPG-based control mechanism has been studied [9]–[14] but still remains largely unknown and awaits further investigations.

The communication among local oscillators through neuronal connections has been believed to be essential to coordinate oscillations of various body parts (see e.g. [15] for a sur-
vey). However, a recent study [16] has shown that the communication through the nerve cord is not a prerequisite for coordination. It is experimentally examined how the swimming motion of leeches is affected when the nerve cord is severed in the middle. Surprisingly, the coordinated undulatory locomotion continued to be observed afterward, albeit with greater than normal phase lags. This means that the information necessary for coordination flows through the body as mechanical signals, and each of anterior and posterior CPGs is capable of extracting the information to determine an appropriate phase. Thus, CPGs can achieve coordination without directly communicating to each other. This observation suggests possibility for decentralized control of mechanical rectifiers to achieve coordinated rhythmic movements, which could have a significant impact on engineering applications for modular and robust designs [17],[18].

The main objective of this paper is to show that a CPG-based controller with a decentralized structure can achieve coordinating the motion of a mechanical rectifier to achieve locomotion, without the descending signal from the brain in the framework of Fig. 1. An implication of the result is that CPGs may provide promising basic architectures for nonlinear control systems to achieve stable, coordinated oscillations. To this end, we shall first develop a prototype mechanical rectifier (PMR) that captures essential dynamics of the coupled body-environment system for animal locomotion. The PMR consists of a two-link arm and a rotating disk, and this simple structure allows us to see how the rectification occurs through the interaction between the body (arm) and the environment (disk). We will then develop a CPG-based controller with a decentralized structure using the reciprocal inhibition oscillator (RIO) [19]–[21] as a basic unit. The behavior of the closed-loop CPG-PMR system is studied via numerical experiments.

Our first result confirms that locomotion of the PMR (sustained unidirectional rotation of the disk) can be achieved by the decentralized CPG for a range of parameter values. We will then investigate optimality of the PMR locomotion, motivated by the premise that biological systems have evolved to optimize certain dynamical properties. The question is: what would be the cost function, if any, optimized by the decentralized CPG-based controller? We have attempted to answer this question by developing analytical characterizations of statically optimal locomotion with respect to some criteria, and comparing observations from numerical experiments with the optimal behavior. It turned out that the decentralized CPG did not seem to optimize the energy efficiency (not reported here; see [22]), but could potentially minimize the (speed and extent of) body movements by proper choices of the parameters.

2. Prototype Mechanical Rectifier

2.1 Overview

Consider a mechanical system composed of a two-link arm and a rotating disk as shown in Fig. 2. The pivot $O$ of the arm is fixed to the inertial frame, and the two joints $O$ and $A$ are equipped with actuators to generate torques $u_1$ and $u_2$ and sensors to measure the angular displacements $\theta_1$ and $\theta_2$. The tip of the arm $B$ is not fixed to, but just touches, the disk, and exerts friction force on the disk. The center of the disk is fixed to the inertial frame through a bearing and the disk can rotate around its center $C$.

The intended operation of the system is basically to swing the arm to make the disk rotate. The motion would be analogous to human legs propelling a bicycle. Since the tip $B$ of the arm is not fixed to the disk, there are three degrees of freedom instead of one, and the dynamics is more complex than riding on a bike. The system captures the essential dynamics of many forms of animal locomotion. The key is that oscillatory motion of the body (arm) can generate locomotion (steady rotation of the disk) if and only if the oscillations of body parts ($\theta_1$ and $\theta_2$) are appropriately coordinated. This property motivates us to call the system a PMR.

2.2 Basic Model

The differential equations governing the motion of the PMR are presented below. For simplicity, we impose the following:

Assumption 1

(a) Each link has a uniform mass distribution and both links have the same mass density and an identical shape (i.e., proportional dimensions).

(b) There are frictions at the disk bearing $C$ and the contact point $B$ between the disk and arm, and their magnitudes are proportional to the angular velocity of the disk and the relative velocity between the arm tip and point $B$ on the disk, respectively.

Under these assumptions, the equations of motion are given by

\[ J_1 \ddot{\theta}_1 + G_o \dot{\theta}_1^2 + c R o^2 (R_o \dot{\theta}_1 + r_o \dot{\theta}_2) = Bu_1, \]
\[ J_2 \ddot{\theta}_2 + J_o r_o (R_o \dot{\theta}_1 + r_o \dot{\theta}_2) + \dot{\theta}_o = 0, \]

where the time axis has been scaled so that the derivatives are with respect to the new time variable $\tau := c_b \ell_1 / J_1$,

\[ \theta := \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}, \quad u := \begin{bmatrix} J_1 \\ J_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad c := \frac{c_r \ell_2^2}{c_o}, \]
\[ \alpha := \frac{\ell_2}{\ell_1}, \quad \alpha := \frac{J_0}{J_1}, \quad \alpha := \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}, \quad \beta := \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}, \quad \gamma := \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix}, \quad \delta := \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix}, \]
\[ C_\theta := \begin{bmatrix} \cos \theta_1 & 0 \\ 0 & \cos \theta_2 \end{bmatrix}, \quad S_\theta := \begin{bmatrix} \sin \theta_1 & 0 \\ 0 & \sin \theta_2 \end{bmatrix}. \]
\[ \Omega_0 := \begin{bmatrix} \cos \theta_1 & \cos \theta_2 \\ \sin \theta_1 & \sin \theta_2 \end{bmatrix}, \quad J_0 := \Omega_0 L, \quad R_0 := \Omega_0 \ell + z_c, \]

\[ J_\theta := J + S \theta H \theta + C_\theta H C_\theta, \quad G_\theta := S \theta H C_\theta - C_\theta H \theta, \]

\[ J := \begin{bmatrix} 1 & 0 & \alpha^3 \\ 0 & 1 & \alpha^3 \end{bmatrix}, \quad H := \begin{bmatrix} 1 + 4 \alpha^3 & 2 \alpha^4 \\ 2 \alpha^4 & \alpha^5 \end{bmatrix}, \quad L := \begin{bmatrix} 1 & 0 & \alpha \\ 0 & 1 & \alpha \end{bmatrix}, \quad \ell := \begin{bmatrix} 1 \\ \alpha \end{bmatrix}, \quad B := \begin{bmatrix} 1 & -1 \end{bmatrix}, \]

and the variables and parameters are summarized in Table 1. The equations of motion have been derived through applications of the first principle, and the details can be found in [22].

| \( \theta_0 \) | rotational angle of disk |
| \( \theta_1, \theta_2 \) | link angles from downward vertical |
| \( u_1, u_2 \) | torque inputs |
| \( \ell_1, \ell_2 \) | link lengths |
| \((x_c, y_c)\) | coordinate of disk center (\( O \) is the origin) |
| \( c_b \) | friction coefficient at disk bearing |
| \( e_c \) | friction coefficient at arm/disk contact |
| \( J_o, J_1, J_2 \) | moments of inertia for disk and links |

For the study of the PMR motion, we have chosen the PMR parameters to yield a generic configuration. First, we may assume \( \ell_1 = 1 \) and \( x_c = 0 \) without loss of generality by choosing an appropriate spatial coordinate system. The two links of the arm are assumed to have the same length (\( \alpha = 1 \)). The disk center is located at the three-quarter of the total arm length from the pivot (\( y_c = -1.5 \)). The moment of inertia for the disk is several times larger than the link (\( J_1 = 5 \)). We take \( J_l = 1 \) and \( c_b = 1 \) so that the scaling factors for input \( u \) and time \( \tau \) are both unity. The variation in the dynamical behavior of the PMR-CPG system due to these factors will be captured later by the feedback gain and intrinsic frequency of the CPG-based controller. Unless otherwise noted, all the parameters are fixed to these nominal values (summarized in the Appendix).

Clearly, the first equation in (1) is for the arm, while the second for the disk. These equations are basically standard for e.g. multilink robotic manipulators [23], but the terms associated with the friction at the arm-disk contact point (those containing the parameter \( c \)) are somewhat special. Indeed, the effect of rectification is embedded in the friction term of the second equation. To see this, let \( \rho(t) \in \mathbb{R}^2 \) be the vector from the disk center to the arm tip, scaled by \( \ell_1 \). Then it can be verified that

\[ \rho = S r_\theta, \quad \dot{\rho} = R_\theta \dot{\theta}, \quad S := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \]

Thus \( r_\theta \) is the vector obtained by rotating \( \rho \) counterclockwise by 90 degrees, and \( R_\theta \dot{\theta} \) is the velocity of the arm tip. Note that the velocity of the point on the disk at which the arm tip contacts is \(-r_\theta \theta_c\). Hence, the term \( R_\theta \theta + r_\theta \theta_c \) in the equations of motion represents the velocity of arm tip relative to the disk, and its product with \( c \) is the friction force. Using \( \rho \) as the new variable replacing \( \theta \), the disk equation can be rewritten as

\[ J_\theta \dot{\omega}_\theta + (1 + c|\dot{\rho}|^2) \omega_\theta + c \rho^T S \dot{\rho} = 0 \quad (2) \]

where \( \omega_\theta := \dot{\theta}_c \) is the angular velocity of the disk. This equation describes how the arm tip motion \( \rho \) drives the disk through the frictional contact. When the arm goes through a periodic motion, the last term in (2) can be nonzero on average due to nonintegrability so that the disk angle \( \theta_c \) travels in one direction every cycle, resulting in what could be perceived as locomotion.

In particular, if the time constant of the \( \omega_\theta \) dynamics (or \( J_\theta \)) is much larger than the cycle period \( T \), then \( \omega_\theta \) can be regarded as constant over a cycle and the standard averaging technique [24] yields

\[ \omega_\theta \approx \frac{2cA}{\int_0^T (1 + c|\dot{\rho}|^2) dt}, \quad A := -\frac{1}{2} \int_0^T \rho^T \dot{S} \dot{\rho} dt = \frac{1}{2} \int_0^T (\rho_2 \dot{\rho}_1 - \rho_1 \dot{\rho}_2) dt. \]

Note that \( A \) is the area enclosed by the closed orbit of the tip trajectory \( \rho(t) \), which is defined positive when the orbit goes clockwise. Thus, for a generic periodic orbit \( \rho(t) \), the area \( A \) is nonzero and the disk angle would be displaced in the same direction at the end of every cycle. If we define this phenomenon as “locomotion,” then our hypothesis — a decentralized CPG can achieve locomotion — would just mean phase locking between PMR and CPG. This definition is not very meaningful because the phase locking can readily be accomplished by driving the PMR with a generic CPG without sensory feedback. For our purpose, the PMR is said to achieve locomotion if the disk velocity \( \omega_\theta(t) \) is periodic and its average value is significantly larger than its amplitude. Such locomotion would require not just phase locking but in addition a proper phase coordination. The meaning of this rather qualitative definition of locomotion will become clearer later when we discuss our results of numerical experiments.

A block diagram of the PMR can be drawn as the dashed box in Fig. 1 where “BODY” and “ENV” correspond to the arm and the disk, respectively. The torque input \( u \) generates periodic motion \( x := \theta \) of the arm which in turn interacts with the disk to yield its rotation with velocity \( y := \omega_\theta \). Thus, the PMR represents typical dynamics found in animal locomotion. For instance, a mathematical model of an \( n \)-link robotic snake [25],[26] also has this exact structure where \( u \), \( x \) and \( y \) stand for the joint torques, joint angles, and velocity vector of the center of gravity for the whole body. It should be emphasized that the interaction between the arm and disk captures the fundamental mechanism of rectification. Indeed, equation (2) is essentially equivalent to the simplest, canonical form of rectifying dynamics studied by Brockett [1],[27] in the context of nonholonomic systems control.

### 2.3 Statically Optimal Locomotion of PMR

In this section, we shall study how the PMR works by formulating a simple optimization problem. Specifically, we consider the class of all periodic arm motions (or gaits) that yield a given constant locomotion speed \( \omega_\theta \). Among such motions, we will first characterize those that minimize the arm tip speed \( |\dot{\rho}| \) at every time instant, and then find the one that also minimizes the tip speed on average. The cost function \( |\dot{\rho}(t)| \) captures the size of rhythmic body (arm) movement in terms of its frequency and amplitude. Let us formalize the optimality criterion as follows.

**Definition** 1 A function \( \theta(t) \) is called a gait of the PMR if it is periodic and satisfies (1) for some \( u(t) \) and \( \theta_0(t) \). A gait \( \theta(t) \) of
the PMR is said to be \textit{statically optimal} with locomotion speed $\omega_c$ if it possesses the following properties:

(a) $\theta_0(t) \equiv \omega_c$ for all $t$;
(b) At every time instant, the arm tip speed $\|\dot{r}(t)\|$ is minimum among those gaits having property (a).
(c) The average value of $\|\dot{r}(t)\|$ over one cycle is minimum among those gaits having properties (a) and (b).

If property (b) were absent, we have a standard definition for a (dynamically) optimal gait that minimizes the average tip speed while maintaining a given constant disk speed. Property (b) imposes a static requirement that the tip speed is minimal not only on average but also at every time instant. This additional requirement may result in a larger value of the average tip speed than that for the dynamically optimal gait, but will make the optimal gait problem tractable, allowing us to obtain a simple and insightful solution in an analytical fashion. Moreover, it turns out that the PMR locomotion achieved by our CPG-based decentralized control is similar to the statically optimal one defined here. We will elaborate on this point in a later section.

The problem is to determine the statically optimal gait with a given locomotion speed $\omega_c \neq 0$. Below, we assume $\ell_1 = 1$ for simplicity but the other parameter values are arbitrary. Recall that the cost function can be expressed as $\|\vec{r}\| = \|\dot{R}_0\|$. So we should minimize this expression subject to (1) and $\theta_0(t) \equiv \omega_c$. Noting that $B$ in (1) is invertible and $u$ does not appear explicitly in the cost function $\|\dot{R}_0\|$, we may ignore the first equation in (1) and concentrate on the trajectory optimization based on the second equation. This is because any smooth trajectory $\theta$ can be generated by an appropriate choice of $u$ due to invaritibility of $B$. Hence we formulate the following problem:

$$\min_{\theta} \|\dot{R}_0\| \quad \text{subject to} \quad cr_0^2(R_0\dot{\theta} + r_0\omega_0) + \omega_0 = 0,$$

where we substituted $\dot{\theta}_0(t) \equiv 0$ and $\dot{\theta}_0(t) \equiv \omega_c$ in the second equation in (1) to obtain the constraint equation. The problem can also be stated in a simpler form using the arm tip vector $\rho$ as the variable:

$$\min_{\rho} \|\dot{\rho}\| \quad \text{subject to} \quad (1 + c\|\rho\|^2)\omega_0 + c\|\rho\|^2\dot{S}\rho = 0. \quad (3)$$

This is a static optimization problem in which we choose the best $\rho(t)$ based on the current state $\rho(t)$. A general solution will be of the form $\rho = f(\rho)$ for some function $f$. The corresponding solution in terms of $\theta$ can then be obtained as $\dot{\theta} = R_0^{-1} f(S \dot{r}_0)$, provided $R_0$ is invertible on the statically optimal trajectory (which turns out to be the case if $c$ is sufficiently large as shown below). We shall call $\rho(t)$ statically optimal if the corresponding gait $\theta(t)$ exists and satisfies properties (a), (b) and (c) of Definition 1.

First note that (3) is a standard linear algebra problem of finding the minimum norm solution $X$ to the matrix equation $AX = B$. The solution is unique and is given by $X = A^\dagger B$ where $(\cdot)\dagger$ is the Moore-Penrose inverse [28]. Using this result, the solution to (3) is found as

$$\dot{\rho} = \omega S \rho, \quad \omega := \left(1 + \frac{1}{c\|\rho\|^2}\right)\omega_0, \quad (4)$$

where we have noted that $\rho(t) \neq 0$ for all $t$ because otherwise the constraint in (3) implies $\omega_0 = 0$ which violates the supposition. Since $S$ is a skew-symmetric matrix, we have

$$\frac{d\|\rho\|^2}{dt} = 2\rho^T \dot{\rho} = 2\omega^2 S \rho = 0.$$

Hence $\|\rho(t)\|$ is constant over time, that is, the statically optimal trajectory $\rho(t)$ is circular with radius $r := \|\rho(0)\|$. Since $\omega$ is constant, the differential equation (4) can readily be solved for

$$\rho(t) = \begin{bmatrix} r \sin(\omega t + \psi) \\ r \cos(\omega t + \psi) \end{bmatrix},$$

where the amplitude $r$ and phase $\psi$ are constant parameters. Thus, properties (a) and (b) in Definition 1 have resulted in a periodic trajectory although the periodicity was not explicitly required. Note that the initial condition $\rho(0)$ determines the phase $\psi$ and amplitude $r$ of the circular orbit, the latter of which in turn specifies the frequency $\omega$ through (4).

We now optimize the initial condition $\rho(0)$ to satisfy property (c). From (4), we have

$$\|\dot{\rho}(t)\| = \left(r + \frac{1}{c r}\right)\omega_0.$$

Thus the tip speed is constant for the statically optimal trajectory, and its value depends on the amplitude $r$, but not on the phase $\psi$. In fact, the value of $\psi$ can be chosen to be zero without loss of generality as it contributes only to the shift of the time axis. On the other hand, a simple algebra verifies that the optimal value of $r$ that minimizes $\|\dot{\rho}\|$ is given by $r = 1/\sqrt{c}$. Therefore, the statically optimal trajectory of the arm tip $\rho(t)$ is the circular orbit with center at $C$ in Fig. 2 and radius $r = 1/\sqrt{c}$.

We now turn our attention to existence of the gait $\theta(t)$ corresponding to the statically optimal tip trajectory $\rho(t)$. The corresponding gait exists if and only if $\rho - S \Omega S = S \Omega_S$ is solvable for $\theta$. The condition means that the circular orbit $\rho(t) - S \Omega S$ is contained in the region reachable by the arm tip $\{S \Omega_S : \theta \in \mathbb{R}^2\}$, or equivalently,

$$|1 - \alpha| \leq \|\rho(t) - S \Omega_S\| \leq 1 + \alpha$$

for all $t \geq 0$. This condition holds for the circle $\rho(t)$ of radius $1/\sqrt{c}$ if and only if

$$|1 - \alpha| + 1/\sqrt{c} \leq \|\Omega_S\| \leq 1 + \alpha - 1/\sqrt{c}. \quad (6)$$

Note that, in order for the statically optimal trajectory to exist, the PMR must be configured such that $|1 - \alpha| < \|\Omega_S\| < 1 + \alpha$ holds, i.e., the distance $OC$ is between the shortest and the longest of $OB$ in Fig. 2. In this case, the existence is guaranteed whenever the friction coefficient $c$ is sufficiently large.

The result can now be summarized as follows.

\textbf{Theorem 1} The statically optimal trajectory $\rho(t)$ of the PMR arm tip with locomotion speed $\omega_c \neq 0$ is uniquely given by

$$\rho(t) = \begin{bmatrix} r \sin(\omega t) \\ r \cos(\omega t) \end{bmatrix}, \quad r := 1/\sqrt{c}, \quad \omega := 2\omega_0$$

with an appropriate shift of the time axis, provided (6) holds. Moreover, the tip speed is constant and is given by

$$\|\dot{\rho}(t)\| = 2\omega_0/\sqrt{c}.$$

The above result states that the arm tip should rotate in a circular orbit around the disk center, with radius $r = 1/\sqrt{c}$ and frequency $\omega = 2\omega_0$, if the tip speed is statically minimized. This description of the statically optimal trajectory matches with our
intuition; (i) the tip trajectory $\rho(t)$ should encircle the disk center to yield a reasonable locomotion, (ii) the orbital amplitude $r$ would be larger if the arm-disk contact is more slippery (i.e. if $c$ is smaller), and (iii) the cycle frequency $\omega$ increases linearly with the locomotion speed $\omega_c$. Note that the last two properties can be violated by a generic family of circular trajectories parameterized by $c$ and $\omega_c$. In particular, the torque balance equation (2) becomes

$$(1 + cr^2)\omega_c = cr^2 \omega$$

if the circular orbit of radius $r$ and frequency $\omega$ is assumed for $\rho(t)$. For given $(c, \omega_c)$, there are infinitely many combinations of $(r, \omega)$ satisfying this equation. We may expect from our intuition that the driving frequency $\omega$ should be adjusted linearly with respect to the desired locomotion speed $\omega_c$. In this case, the torque balance implies that the radius $r$ should be proportional to $1/\sqrt{c}$. This is exactly the result of the optimization performed here.

Finally, let us comment on how the statically optimal gait $\theta(t)$ can be characterized. As mentioned earlier, the differential equation for $\theta(t)$ can be obtained from that for $\rho(t)$ using their kinematic relationship. In particular, noting that the statically optimal $\rho(t)$ satisfies

$$\dot{\rho} = 2\alpha_0 S \rho, \quad ||\rho(0)|| = 1/\sqrt{c},$$

we have

$$\dot{\theta} = -2\alpha_0 R_0^2 \theta_0, \quad ||\theta(0)|| = 1/\sqrt{c}, \quad (7)$$

where $\theta_0(0)$ denotes the value of $\theta_0$ at $t = 0$. Thus, a statically optimal $\theta(t)$ can be generated by integrating the differential equation in (7) from an initial condition $\theta(0)$ satisfying $||\theta(0)|| = 1/\sqrt{c}$. It is easy to verify that $R_0$ is nonsingular if and only if $\theta_1 - \theta_2 \neq n\pi$ holds for any integer $n$, i.e., the two-link arm does not become straight. This is the case for all points on the statically optimal trajectory if and only if the circular orbit $\rho(t)$ satisfies (5) with strict inequalities as the singularity occurs on the boundaries. Hence, the statically optimal gait $\theta(t)$ exists, does not encounter the singularity of $R_0$, and is well defined by (7), if and only if (6) holds with strict inequalities. In this case, for any initial tip location specified by $r_0(0)$ satisfying $||r_0(0)|| = 1/\sqrt{c}$, there corresponds an initial link angle $\theta(0)$. In fact, there are two configurations of the arm corresponding to a single $r_0(0)$, depending on which side (right or left) of the line $OB$ the point $A$ lies in Fig. 2. Clearly, these two configurations of $\theta(0)$ lead to different but symmetric arm motions, and the resulting tip trajectories are the same.

### 3. CPG-based Decentralized Control

We consider the control system shown in Fig. 1 where the PMR presented in the previous section is substituted for the dashed box. Our objectives are to (i) design a CPG-based controller with a decentralized structure, based on knowledge from computational neuroscience, (ii) assess whether the controller can achieve locomotion of the PMR, and, if so, (iii) study optimality properties of CPG-induced PMR locomotion. The issue of descending control is out of the scope of this paper and hence we do not consider the outer feedback loop through “BRAIN.” In the following subsections, we shall address objective (i) and develop a CPG-based controller. Objectives (ii) and (iii) will be addressed through numerical experiments in the sections that follow.

#### 3.1 Neuron Model

The electrical dynamics of the neuronal cell membrane can be modeled as an input-output system $v = N(i)$ from the current injection input $i$ to the membrane potential output $v$ [5],[29],[30]. In this paper, we adopt the following model for neuronal dynamics [13]

$$v = \mathcal{A}(q), \quad q = \mathcal{F}(i)$$

where $\mathcal{A}$ represents the dynamics of action potential generation at the axon, $\mathcal{F}$ captures the dynamics associated with the synapse and soma, and $q$ is the current flowing from the soma to the axon. Specifically, the dynamical mapping $\mathcal{F}$ is assumed linear and described by the following transfer function

$$F(s) = \frac{ks}{(1 + \tau_1s)(1 + \tau_2s)},$$

while the mapping $\mathcal{A}$ is given by

$$\tau_q \dot{v} = \psi_1(v) - w + q, \quad \tau_q \dot{w} = \psi_2(v) - w \quad (8)$$

where functions $\psi_k (k = 1, 2)$ are defined by

$$\psi_1(v) = c\phi(v) + q_o - bv, \quad \psi_2(v) = \phi(d(v + v_o)), \quad \phi(x) := \frac{1}{1 + e^{2-x}}. \quad (9)$$

The parameter values are summarized in the Appendix.

The transfer function $F(s)$ captures the time lag and adaptation effect in the signal processing at the synapse and soma. In particular, if a step input $i(t)$ is applied, the output $q(t)$ will first rise to respond to the input, and then decays to zero due to adaptation. The time constants $\tau_1$ and $\tau_2$ are related to the speed of response (rise time) and the speed of adaptation (decay rate), respectively. Typically, $\tau_2$ is much larger than $\tau_1$ so that adaptation is slower than the initial response [31]. Adaptation is a fundamental neuronal property that is important for generating a stable pattern in a certain class of CPGs [20]. In our model, the adaptation property is realized by the zero at the origin, i.e., $F(0) = 0$, as suggested in [32]. In the literature, there is another mechanism for adaptation based on the negative feedback of the membrane potential [31]. A recent study [13] has shown that the two mechanisms do not lead to radically different dynamical properties in the context of CPG-based feedback control of a simple pendulum, but the former makes the theoretical analysis simpler.

The axon model in (8) is a standard two-variable model for excitable nerve membranes [33]–[36]. Typically, the isocline functions $-\psi_1$ and $\psi_2$ are “N-shaped” and monotonically increasing, respectively. An additional feature embedded in our model through (9) is that the isocline functions are realized by linear transformations of a single sigmoid function $\phi$. With this structure, every trajectory of (8) is always bounded whenever the input $q$ is bounded, and the existence of oscillations has been formally proven [13] for the type of CPG adopted in the present paper, which is discussed in the next section.

#### 3.2 Reciprocal Inhibition Oscillator

The reciprocal inhibition oscillator is the simplest CPG consisting of two neurons with mutually inhibitory synaptic connections [20]. The block diagram of the RIO is shown in Fig. 3 where two neurons $N$ are connected via inhibitory synapses with strength $\sigma$. Note that the RIO has inputs $r_i$ and outputs $v_i$ ($i = 1, 2$).
call that the PMR is actuated through the two torque inputs that the relative angles intrinsic frequency in the RIO parameters are fixed as in the Appendix except for the PMR using the RIO described in the previous section. All 3.3 Decentralized Control Architecture system described in the next section.

Figure 4 shows the time responses of an RIO when a short pulse input is applied to \( r_2 \). As shown, the typical response is given by alternating bursts of action potentials in \( v_1 \) and \( v_2 \) with 180 degree phase difference. In this simulation, the pulse is of magnitude 1 and duration 1, and the model parameters are taken from [37],[38] with a modification that the time axis and the membrane potential variable \( v_i \) are scaled so that the period of the RIO oscillation is \( 2\pi \) and the integral of \( v_1 \) over one period is 2 (which is the area under the sinusoidal curve \( \sin t \) over the duration \( 0 \leq t \leq \pi \)). The parameter values are summarized in the Appendix. This RIO is bistable when the inputs are zero, with a stable equilibrium and a stable limit cycle (spike train). The pulse input pushes the states from the equilibrium into the domain of attraction of the limit cycle, initiating the out-of-phase oscillations.

By scaling the time axis for the RIO model, the intrinsic frequency \( \sigma \) of its oscillation can be set to any desired value. Specifically, if the RIO parameters associated with the time constants are divided by \( \sigma \) as shown in the Appendix, then the resulting frequency is \( \sigma \). We shall use the intrinsic frequency \( \sigma \) as the only design parameter for the CPG in the PMR control system described in the next section.

3.3 Decentralized Control Architecture

We now propose a CPG-based feedback control system for the PMR using the RIO described in the previous section. All the RIO parameters are fixed as in the Appendix except for the intrinsic frequency \( \sigma \) which remains as a design parameter. Recall that the PMR is actuated through the two torque inputs \( u_i \) (i = 1, 2) at the pivot and joint of the two-link arm. We assume that the relative angles are measured and conditioned through nonlinearities

\[ \varphi_i(\phi) := a_i \tanh(b_i(\phi - \phi_{io})) \]

where \( a_i \) and \( b_i \) are positive constants and \( \phi_{io} \) are the nominal relative angles. We shall drive each input \( u_i \) by an RIO based on the sensory signal through the feedback gain \( k_i \) as shown in Fig. 5. Note that this is essentially a positive feedback where the positive/negative \( \phi_1 - \phi_{io} \) tends to generate positive/negative \( u_i \).

We use an identical system for both RIO-1 and RIO-2, which is given by the block diagram in Fig. 3 where the neuron model \( N \) is as described in Section 3.1 with parameters in the Appendix. The actuator dynamics are ignored and the input torques \( u_i \) to the PMR are simply proportional to the RIO outputs with constant gains \( \kappa_i \). The effect of the actuator bandwidth limitation has been examined in [22]. For simplicity, we consider the case \( \kappa_1 = \kappa_2 = \kappa \). The nominal PMR configuration is defined by alignment of the arm tip and the disk center. Thus the nominal relative angles are given by \( 2\theta_{io} = -\phi_{2o} = 82.8^\circ \). The other parameters \( a_i \) and \( b_i \) in the nonlinear function \( \varphi_i \) are roughly determined through a few trial and error (\( a_1 = a_2 = 0.1, b_1 = 6, b_2 = 2 \)). The saturation effect turns out to be not very important in the steady state locomotion as the sensory system operates in the (almost) linear region of the tanh function. The gain values are such that the effect of the actuator bandwidth limitation is not very important in the steady state locomotion as the sensory system operates in the (almost) linear region of the tanh function. The gain values are such that the effect of the actuator bandwidth limitation is not very important in the steady state locomotion as the sensory system operates in the (almost) linear region of the tanh function.

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from the sensory feedback should be enough to coordinate the RIO oscillations. In the next section, we will first verify that this conjecture is true by numerical experiments. Optimality of the PMR-CPG system will then be investigated for various values of the PMR friction coefficient $c$ and the essential parameters of the CPG-based controller — the feedback gain $\kappa$ and the intrinsic frequency $\omega$ of the RIO.

4. Decentralized Coordination Results

This section first examines feasibility of the coordinated PMR locomotion via the CPG-based control architecture developed in the previous section. In particular, we will show, through numerical experiments, that there exists a set of controller parameters $(\omega, \kappa)$ that yield locomotion of the PMR. The set of feasible parameters is found by an exhaustive search on the $(\omega, \kappa)$ plane, and the resulting PMR locomotions are quantitatively analyzed in terms of optimality properties.

4.1 Feasibility

We conduct numerical experiments of the control system in Fig. 5. The impulse input $\delta(t)$ is applied to the RIOs through $r_{12}$ and $r_{21}$ channels to initiate the RIO oscillations when the system is at rest in the nominal configuration. These choices of input channels lead to out-of-phase oscillations of the two RIOs if there were no sensory feedback. We will see that the phase relationship is properly modified through the sensory feedback, and the steady state behavior is insensitive to the particular initialization.

We fix the parameters $c = 50$ and $\sigma = 1$ and adjust the gain $\kappa$ to look for the PMR locomotion. If the gain $\kappa$ is small, the connection between the PMR and the CPG is weak. In this case, it is observed (not shown) that the two RIOs oscillate at slightly (roughly by 5%) different frequencies near $\omega$. In particular, the RIOs do not phase lock to each other, and the disk velocity $\omega$ oscillates with several times larger amplitude than its average, with a large period (roughly 120 time units) which corresponds to the difference of the two RIO frequencies. When the gain $\kappa$ is increased above a threshold value, the phase locking starts to occur, resulting in locomotion of the PMR. That is, the average value of the disk velocity $\omega$ is much larger than its oscillation amplitude. Such locomotion is observed when $1.6 \leq \kappa \leq 3.1$. If the gain is further increased, the arm goes “wild” and a chaotic motion results.

Figure 6 shows the time responses for the case $\kappa = 3$. The PMR is driven by the CPG without sensory feedback during $0 \leq t < 150$. Due to the impulse inputs to $r_{12}$ and $r_{21}$ at $t = 0$, the RIOs enter the out-of-phase oscillations with frequency $\omega = 1$ as shown at the bottom left of Fig. 6. Accordingly, the relative angles of the two-link arm are also out of phase in the steady state (middle left figure). In this case, the PMR makes locomotion with the average speed $\text{ave}(\omega) = 0.104$ (top figure, $100 < t < 150$). The profile of the achieved oscillation is summarized in the first row of Table 2. It should be noted, however, that this apparent locomotion is just a coincidence. In particular, the phase difference between the RIOs depends on how they are initiated, and the PMR locomotion is highly sensitive to the initialization process. For instance, if the impulse inputs are given to $r_{12}$ and $r_{21}$, then the RIOs become completely in-phase, and the average value of $\omega$ in the steady state is $\text{ave}(\omega) = 0.0031$ with the amplitude of fluctuation around 0.013, exhibiting no locomotion.

In the numerical experiment, the sensory feedback is turned on at $t = 150$. The effect appears immediately as seen in the top of Fig. 6. Specifically, the PMR enters, after a transient, the steady state where a faster locomotion (ave($\omega$) = 0.338) is achieved with a very small fluctuation. We see from the middle/bottom right figures that the phases are coordinated between the two RIOs, and also between the two joint angles. The phase lag of $\omega_2$ with respect to $\omega_1$ is 93.5°, while that of $\phi_2$ with respect to $\phi_1$ is 150°, calculated from the locations of peaks. It is worth noting that the cycle frequency is also modified through the sensory feedback from $\omega = \sigma = 1$ to $\omega = 0.878$. This frequency reduction may be considered as a result of the fact that the arm achieved a better grip of the disk through the proper phase coordination. The oscillation profile is summarized in the second row of Table 2.

The corresponding steady state trajectory of the arm tip is shown in Fig. 7 as the two ellipse-like orbits near the disk center. The larger and smaller orbits are for the cases with and without sensory feedback, respectively. In both cases, the arm tip moves clockwise. The smaller orbit is thin and does not properly enclose the disk center. On the other hand, the larger orbit is more circular (i.e., the ratio of the long and short axes of the ellipse-like orbit is closer to one), and the disk center is

![Figure 6 Decentralized RIOs achieve coordination via sensory feedback.](image)
in the middle of the orbit. Based on our intuition, we may view this sensory modification as an improvement of the locomotion quality. This potentially subjective judgment will be justified in the next section through a quantitative study of optimality with respect to a certain criterion.

![Fig. 7 PMR and the tip trajectories with and without sensory feedback. The trajectory with sensory feedback appears more reasonable as it goes around the disk center (indicated by *) in a more circular, balanced orbit.](image)

In summary, the CPG with the decentralized structure is capable of coordinating the arm motion through the sensory feedback to induce the PMR locomotion, as expected from the experimental study of leech swimming.

4.2 Optimality of CPG-PMR System

In this section, we shall discuss optimality of the PMR locomotion achieved by the decentralized CPG with sensory feedback. Reasonable ways to define the notion of optimality are not unique. For instance, it may be defined by minimality of the average power $\hat{\theta} \hat{B} u$ over a cycle supplied through the torque input $u$ to achieve a given locomotion speed $\omega_0$. Alternatively, the average value of $||u||^2 - \omega_0^2$ in the steady state may be used as the cost function. Our interest here is to find a particular cost function such that the locomotion achieved by our CPG-based controller is close to the optimum with respect to the chosen criterion. In other words, we look for a criterion with which a CPG-based controller seems to optimize the locomotion.

Among the criteria we have considered, including the kinetic energy and the energy loss due to the frictions, the average value of the arm tip speed $||\rho||$ during locomotion at a given speed $\omega_0$ turns out to be the best candidate for the cost function minimized by the CPG-based controller. Below, we shall discuss the optimality of the locomotion in comparison with the statically optimal gait introduced in Section 2.3. It is worth mentioning that the PMR locomotion induced by the decentralized CPG has been found not optimal with respect to the energy efficiency, consuming roughly three times more energy, at best, than the optimum to achieve a given locomotion speed. The details can be found in [22].

Recall that there are two basic design parameters for the CPG-based controller — the feedback gain $\kappa$ and the intrinsic frequency $\sigma$ of the RIO. We have gridded the parameter plane $(\sigma, \kappa)$ and identified the region where the controller achieves locomotion. The friction coefficient is fixed as $c = 100$ (for different values of $c$ roughly in the range $50 \leq c \leq 200$, one can obtain similar results to those presented below), and the other parameter values are as given in the Appendix. The intensity $p$ of the initial impulse inputs $p(t)$ is chosen slightly above the threshold above/below which the RIO oscillation does/does not occur.

For each grid point on the $(\sigma, \kappa)$ plane, the average values of the arm tip speed $||\rho||$ and the locomotion speed $\omega_0$ are calculated whenever locomotion is achieved. Figure 8(a) shows the feasible region with contours of $\text{ave}(\omega_0)$. Note that the vertical axis has been scaled by $\sigma$ to emphasize that $\sigma$ and $\kappa$ are roughly proportional in the region. Each data point is marked by a dot in Fig. 8(b) based on the achieved average values of $(\omega_0, ||\rho||)$. The straight line in the figure shows the $\omega_0$-$||\rho||$ relation for the statically optimal locomotion ($||\rho|| = 2\omega_0/\sqrt{c}$). The lower right boundary of the set of the dots (marked by ‘*’) gives the Pareto-optimal points on which no other choice of the controller parameters exists to achieve the same locomotion speed with a smaller value of $\text{ave}||\rho||$. The Pareto-optimal points are fairly close to the statically optimal line (within 5% of the optimal $||\rho||$ for each $\omega_0$). Moreover, all the data points are not too far away from the optimal line. Therefore, the class of CPG-based decentralized controllers seems to provide a reasonable framework for suboptimal feedback control designs in the sense that a given locomotion speed is achieved without introducing excessively large/fast motion of the arm.

The parameter values corresponding to the Pareto-optimal points are marked by ‘*’ in Fig. 8(a). We see that the best feedback gain, for a given intrinsic RIO frequency $\sigma$, is given by the maximum value of $\kappa$ for which locomotion occurs. This indicates the lack of robustness of the Pareto-optimal controllers with respect to the gain perturbation. Figure 8(c) shows the relation between the arm cycle frequency $\omega$ and the locomotion speed $\omega_0$. The Pareto-optimal cases indicated by ‘*’ are well aligned with the statically optimal relation $\omega = 2\omega_0$. Figure 8(d) shows the orbit of the arm tip during locomotion for different values of $\kappa$ where $\sigma$ is fixed as $\sigma = 2$. As the gain $\kappa$ gets larger, the orbit becomes larger, and just before losing locomotion, the orbit is of size comparable to the statically optimal orbit indicated by the lightly colored circle of radius $1/\sqrt{c} = 0.1$.

In summary, we have found that (i) the Pareto-optimal feedback gain $\kappa$ is always at the upper boundary of the parameter region for guaranteed locomotion on the $(\kappa, \sigma)$ plane, and (ii) for such $\kappa$, the CPG-based decentralized controller can achieve the performance close to the static optimum as measured by the average speed of the arm tip. Finally, we remark that the Pareto-optimal controllers are fragile because a small perturbation causing larger $\kappa$ and/or smaller $\sigma$ may make the locomotion disappear.

5. Conclusion

Motivated by the intriguing experimental result on the phase coordination in the leech swimming with severed nerve cord [16], we have examined the property of the CPG-based feedback controller with a decentralized structure in the context of

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1 The frequency $\omega$ is always found close to $0.9 \sigma$ regardless of the feedback gain $\kappa$. 
Fig. 8 Optimality of CPG-induced PMR locomotion. (a) The feasible region of the controller parameters $\sigma$ and $\kappa$ for which locomotion is achieved. The contours indicate that the locomotion speed is larger if $\sigma$ and $\kappa$ are larger. (b) The relation between the average values of the arm tip speed $|v|$ and the locomotion speed $\omega\omega$. The data points are generated by different combinations of $(\sigma, \kappa)$ in the feasible region. The points * are Pareto-optimal within the set of CPG-induced locomotions in that a given $\omega\omega$ is achieved with the minimum $|v|$. The straight line indicates the statically optimal relation. (c) The relation between the average locomotion speed $\omega\omega$, and the cycle frequency $\alpha$ of the arm. The statically optimal relation is $\omega = 2\omega\omega$. (d) The steady state tip orbits for different values of $\kappa$ with fixed $\sigma = 2$ and $c = 100$. As $\kappa$ is made larger, the tip orbit becomes larger and gets closer to the statically optimal orbit indicated by the circle with radius $1/\sqrt{2}$.

driving a PMR that captures the essential dynamics of animal locomotion. The main findings in this paper can be summarized as follows.

- It is possible to coordinate the rhythmic motion of the PMR to achieve locomotion through the CPG-based control consisting of two uncoupled oscillators with local sensory feedbacks.
- The PMR locomotion is observed for a range of the controller parameters; the intrinsic RIO frequency $\sigma$ and the feedback gain $\kappa$, each of which is positively correlated with the locomotion speed $\omega\omega$.
- The set of PMR locomotions generated by the CPG-based control is quantitatively close to the statically optimal locomotion, where the arm tip speed is minimized at every time instant subject to a given constant locomotion velocity.

In addition, it has been found that the RIO-based controller exhibits some degree of robustness in sustaining PMR locomotion with respect to the actuator bandwidth limitation, sensor noises, and change in the environmental factor dictated by the contact friction. These results are not reported here due to space limitation, but can be found in [22].

Overall, our results suggest that the following control framework may be viable for coordinating rhythmic motions of general multilink systems to achieve mechanical rectification (i.e., locomotion): an RIO with a local sensory feedback drives each joint through a positive loop gain. Such RIO-based control has been shown to achieve entrainment to the resonance frequency of a simple pendulum, maintaining the natural oscillation with efficiency [13],[14]. On the other hand, our result regarding the optimality of the decentralized CPG control should be understood carefully. It is the arm tip speed that tends to be optimized; it has been observed that the locomotion is far from optimum with respect to the energy efficiency [22].

While the direct communication between the RIOs has been found not essential for achieving the PMR locomotion, such communications may be important if the system has many degrees of freedom. Moreover, communications among RIOs may play a crucial role in coordinating complex motions and in optimizing a given desired criterion. How to design an appropriate connectivity architecture among RIOs would be an interesting open problem that needs to be addressed. Our results suggest that such biologically inspired control mechanism would be (or can be made) robust against the failure of the communication paths in maintaining a reasonable locomotion profile.

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Appendix A Nominal Parameter Values

PMR parameters

- $\alpha = 1$, $J_1 = 5$, $c = 50$, $\ell_1 = 1$, $J_1 = 1$, $c_0 = 1$, $\tau_c = \begin{bmatrix} -1.5 \\ 0 \end{bmatrix}$.

RIO parameters

- $\sigma = 1$, $\sigma = 2.176$, $\tau_v = 0.1121/\sigma$, $\tau_\theta = 0.0816\tau_v$, $\tau_1 = 3\tau_v$, $\tau_2 = 30\tau_v$, $k = 3\tau_v$, $q_0 = -0.2$, $v_0 = -1.287$, $a = 0.4896$, $b = 0.8160$, $c = 2.200$, $d = 1.360$.

Sensor/actuator parameters

- $\phi_1 = 41.42^\circ$, $\phi_2 = -82.82^\circ$, $\kappa_1 = \kappa_2 = 3$, $a_0 = 0.1$, $b_1 = 6$, $b_2 = 2$.

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