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Abstract: This paper presents a feasibility study of iterative learning control for a class of redundant multi-joint robotic systems when a desired motion trajectory is specified in task-space with less dimension than that of joint space. First, it is shown that if the desired trajectory described in task-space for a time interval \( t \in [0, T] \) is twice continuously differentiable then a unique control signal describable in task-space exists despite of the system joint-redundancy. Second, a learning control update law is constructed through transpose of the Jacobian matrix of task-space coordinates with respect to joint coordinates by using measured data of motion trajectories in task-space. Third, the convergence of trajectory trackings through iterative learning is proved theoretically on the basis of original nonlinear robot dynamics in joint space.

Key Words: iterative learning, redundant robot, trajectory tracking, transpose jacobian.

1. Introduction

When infants of 3 to 5 months old start to learn reaching for an object and grasping, they gaze at the object but do not see any motion of the arm as pointed out in many observations obtained in experimental and developmental psychology [1],[2]. Even in the case of well matured adults, the first trial in learning to reach a target point and write a circle on a black board is primarily a problem of control of the arms through haptic and proprioceptive information. Later, in repetition of the exercises, they rely increasingly on vision to refine motion trajectories in the task space and focus to adjust the arm endpoint (maybe, a white chalk) to their envisioned target. During such fine-tuning processes, they do not watch movements of the arm joints such as wrist, elbow, and shoulder joints.

Accurate trajectory tracking can be achieved theoretically in the case of robot arms by using feedforward model-based control called the computed torque method. However, this methodology may produce unsatisfactory results when parametric uncertainties in the arm dynamics are present to some extent and/or unmodeled dynamics of actuators and payloads are supposed to exist. To cope with this parameter-uncertainty problem, an appealing approach of trajectory tracking for robotic systems was proposed in 1984 [3] (see Fig.1), later called the iterative learning control (ILC) and widely investigated not only in trajectory tracking of robotic systems [4] but also in control of repetitive tasks for mechatronics systems (see Fig.1). However, in most papers of the literature on ILC except the paper by De Luca and Mataloni [5], only a family of objective dynamics whose number of DOFs is equal to the dimension of task space, that is, non-redundant dynamics, have been treated as far as nonlinear robot dynamics is concerned, though it has been believed at least among roboticists that surplus DOFs in robotic systems may offer advantages in executing dexterous tasks. The reason is that redundancy in DOFs of objective dynamics may incur illposedness of inverse kinematics and make the control problems more sophisticated. It was very recent that even in the case of redundant arms a task space PD feedback for multijoint reaching produces satisfactory skilled motions without calculating the pseudo-inverse of a Jacobian matrix or introduce any cost function to determine the inverse kinematics uniquely [6]–[8]. In the problem of Point-to-Point (PTP) reaching, there is no need of planning a trajectory in the task space in advance. On the other side, there is a vast literature of research works on PTP reaching for redundant robots which are based upon planning an optimized trajectory in joint space by introducing an artificial performance index such as “manipulability,” “kinetic energy,” and “quadratic functions of acceleration,” “jerk,” or “torque,” etc. Once the optimal trajectory in some sense is determined in joint space, the problem is of application of the computed torque to tracking control in the joint level. Thus, very recently, Nakanishi et al.[9] showed interesting results on comparative and quantitative studies on performances of task space trajectory tracking among such optimization techniques for redundant robots.

This paper extends the conventional ILC scheme for joint space motion-trajectory tracking to that for task space motion-trajectory tracking in the case that the number of physical variables (or dimension) necessary and sufficient for description of a given task in task-space is less than the total DOFs of an objective robot arm. Differently from De Luca et al.’s approach [5],[10] based upon frequency-domain, our proposed learning update law is quite of a simple form and constructed in.
time-domain in such a way that the next control signal is composed in task-space of a linear sum of the present control signal and the task-space velocity-error signal with a constant coefficient (see Fig.2). When this control input is applied for making the next trial, it is exerted at arm joints in a feedforward manner through transpose of the Jacobian matrix of physical variables in task-space with respect to arm joint variables.

The most crucial problem underlying this ILC scheme is whether, for a desired endpoint movement described by \( x_d(t) \) for \( t \in [0,T] \) in task-space, there exists uniquely in task-space a control \( v_d(t) \) realizing faithful endpoint trajectory tracking exists regardless of joint-redundancy of the system. In this paper, we denote a pose of the robotic arm by joint vector \( q \) and its endpoint by \( x(q) \) that is called the forwarded kinematics from joint space to endpoint task-space. Since there are an infinite number of possible poses \( q(0) \) that realize \( x(q(0)) = x_d(0) \) for a given endpoint \( x_d(0) \) at the initial time \( t = 0 \) because of joint-redundancy, it is necessary to assume that one of such initial poses is chosen and fixed the same at any initialization during trials of iterative learning. Then, this paper first shows that, for a desired endpoint movement \( x_d(t) \) for \( t \in [0,T] \), there exists uniquely a control \( v_d(t) \) in task-space at least for some local interval \( t \in [0,\alpha] \) with some positive \( \alpha > 0 \) regardless of joint-redundancy of the system. Further, it is possible to prove that \( v_d \) belongs to the image space of Jacobian matrix \( J(q) \) defined as \( \partial x(q)/\partial q \), and if there is no control input in the kernel space of \( J(q) \) then the desired joint vector \( q_d(t) \) is determined uniquely so that \( x(q_d(t)) = x_d(t) \). Furthermore, if there does not arise any finite escape-time within the time interval \( [0,T] \) for a key nonlinear differential equation determining such a unique pose \( q_d(t) \) in joint space, then \( v_d(t) \) and \( q_d(t) \) can exist and be determined for all \( t \in [0,T] \). Non-existence of such a finite escape-time in \( [0,T] \) depends on the choice of \( x_d(t) \), and usually it is expected that, unless \( x_d(t) \) enforces difficult continuous movements of the arm pose or as far as the magnitude of joint velocity \( \|q_d(t)\| \) remains bounded uniformly in \( t \in [0,T] \), the control signal \( v_d(t) \) can be extended until \( t \) reaches the terminal time \( T \). Based on this result, we show that, at the k-th trial of ILC, the input-output pair \( \{\Delta u_k = v_k - v_d, \Delta x_k = x_k - x_d\} \) of nonlinear error dynamics concerning \( \Delta q_k = q_k - q_d \) satisfies passivity and output-dissipativity, from which convergences of \( \Delta x_k, \Delta x_d, \Delta q_k, \) and \( \Delta q_d \) to zero as \( k \to \infty \) can be proved straightforwardly. This result corresponds to an extension of equivalence relations among “learnability”, “output-dissipativity”, and “strict positive realness with extra condition” for a class of linear time-invariant systems [11],[12] (see Fig.2).

2. Dynamics of Multi-Joint Movement

In order to gain a physical insight into the problem, we consider dynamics of a planar multi-joint system whose motions are confined to a horizontal plane described in \( O-xy \) coordinates as shown in Fig.3. Lagrange’s equation of motion of such a planar multi-joint system is described by the formula (see [13])

\[
H(q)\ddot{q} + \left( \frac{1}{2}H(q) + S(q, q') \right) = u
\]

(1)

where \( q = (q_1, q_2, q_3, q_4)^T \) denotes the vector of joint angles, \( H(q) \) the inertia matrix, and \( S(q, q') \) the gyroscopic force term including centrifugal and Coriolis forces. It is well known that the inertia matrix \( H(q) \) is symmetric and positive definite and there exists a positive constant \( h_0 \) together with a positive definite constant diagonal matrix \( H_0 \) such that

\[
h_0H_0 \leq H(q) \leq H_0
\]

(2)

for any \( q \). It should be also noted that \( S(q, q') \) is skew symmetric and linear and homogeneous in \( \dot{q} \). More in detail, the \( ij \)-th entry of \( S(q, \dot{q}) \) denoted by \( s_{ij} \) can be described in the form [13]:

\[
s_{ij} = \frac{1}{2} \left( \frac{\partial^2}{\partial q_i} \sum_{k=1}^n q_k \dot{H}_{jk} - \frac{\partial}{\partial q_i} \sum_{k=1}^n q_k \dot{H}_{jk} \right)
\]

(3)

from which it follows apparently that \( s_{ij} = -s_{ji} \).

Now, suppose that a desired motion over a finite time interval \( t \in [0,T] \) is specified as a vector-valued function \( x_d(t) \) (= \( (x_d(t), y_d(t)) \)) in task-space in terms of \( O-xy \) coordinates as shown in Fig.3. Throughout this paper, we assume for convenience of simplifying the argument that \( x_d(t) \) is twice continuously differentiable and the first and second time-derivatives are zero at \( t = 0 \), i.e., \( x_d(0) = 0 \) and \( \dot{x}_d(0) = 0 \). Further, we treat the case that the control input in eq.(1) is designed by a combination of task-space PD feedback with damping shaping in joint space and a feedforward control in task space such that \( u = u_1 + u_2 \),

\[
\begin{align*}
\begin{cases}
1 & \text{if } \dot{C}q - J^T(q) \left( k\Delta x(t) + \zeta \sqrt{k} \Delta \dot{x}(t) \right) \\
2 & \text{if } J^T(q)\dot{w}_2(t)
\end{cases}
\end{align*}
\]

(4)

(5)

where \( C \) denotes a positive definite and diagonal damping coefficient matrix (i.e., \( C = \text{diag}(c_1, \ldots, c_4) \) where \( c_i > 0 \)), \( J(q) \) is the Jacobian matrix of task coordinates \( x \) in joint coordinates \( q \), \( \Delta x(t) = x(t) - x_d(t) \), and \( k \) and \( \zeta \) are positive constant gain parameters. The feedforward signal \( v_2(t) \) is determined by the following learning update law:

\[
v_2(t) = \begin{cases}
0 & \text{for } k = 1 \\
\phi \Delta x_{k-1}(t) & \text{for } k > 1
\end{cases}
\]

(6)
where \( k \) denotes the trial number and \( \Phi \) is a positive constant parameter for ILC (see Fig.2). Then, by substituting \( u = u_1 + u_2 \) into eq.(1), we obtain the closed-loop dynamics at the \( k \)-th trial as follows:

\[
H(q_k)\ddot{q}_k + \left(\frac{1}{2}H(q_k) + S(q_k, \dot{q}_k) + C\right)q_k + J^T(q_k)\left[\xi \Delta x_k + \zeta^T \sqrt{\gamma} \Delta \dot{x}_k\right] = J^T(q_k)w
\]

In order to gain a physical insight into the problem of convergence of \( \Delta x_k \) to zero with the increase of \( k \), it is convenient to transform \( \ddot{q}_k \) into

\[
\ddot{q}_k = \left(J^*_k(q_k), \alpha P_k\right)
\]

where \( \alpha \) is an appropriate positive parameter,

\[
J^*_k = \left(J_k J_k^T\right)^{-1}
\]

and \( P_k \) is a \( 4 \times 2 \)-matrix orthogonal to \( J_k^T(q_k) \), i.e., \( P_k^T J_k^* = 0 \), with a property that \( P_k = (p_1, p_2) \), \( p_i \) denotes a \( 4 \times 1 \) column vector with \( \|p_i\| = 1 \) and \( p_i^T p_2 = 0 \). Then, it is easy to see that if we define

\[
Q_k = (J_k^*, \alpha P_k), \quad Q_k^{-1} = \left(\frac{J_k}{\alpha^{-1} p_i^T}\right)
\]

\[
H_k = Q_k^T H(q_k) Q_k, \quad C_k = Q_k^T C Q_k
\]

\[
S_k = Q_k^T S Q_k - \frac{1}{2}Q_k^T H Q_k + \frac{1}{2}Q_k^T H Q_k
\]

then, by substituting eq.(8) into eq.(7) and multiplying the resultant equation by the transpose of \( Q_k \) from the left-hand, we have

\[
H_k \left(\frac{\ddot{x}_d}{\eta_d}\right) + \frac{1}{2} H_k + S_k + C_k \left(\frac{\dot{x}_d}{\eta_d}\right) + \zeta^T \sqrt{\gamma} \Delta \dot{x}_k = \left(\frac{0}{0}\right)
\]

3. Unique Existence of a Control Signal in Task-Space

If we assume that there exists a control signal \( v_d(t) \) over \( t \in [0, T] \) in task-space that realizes faithful task-space trajectory tracking, i.e., \( \Delta x(t) = 0 \) for \( t \in [0, T] \), then it must satisfy the following equation together with some \( \eta_d(t) \) and \( q_d(t) \):

\[
H_d \left(\frac{\ddot{x}_d}{\eta_d}\right) + \frac{1}{2} H_d + S_d + C_d \left(\frac{\dot{x}_d}{\eta_d}\right) = \left(\frac{0}{0}\right)
\]

where \( Q_d = (J_d^*(q_d), \alpha P(q_d)), \quad H_d = Q_d^T H(q_d) Q_d, \quad S_d = Q_d^T S(q_d, q_d) Q_d - \frac{1}{2} Q_d^T H Q_d + \frac{1}{2} Q_d^T H Q_d, \quad C_d = Q_d^T C Q_d \). Note that \( S_d \) is again skew-symmetric. For convenience, we define

\[
B_d = \frac{1}{2} H_d + S_d + C_d
\]

Then, it follows from eq.(14) that

\[
H_{22} \eta_d + B_{22} \eta_d = -H_{12}^T \ddot{x}_d - B_{21} \ddot{x}_d \]

where

\[
B_d = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}
\]

Equation (16) means that if \( \dot{x}_d \) and \( q_d \) are given then \( \eta_d \) is determined uniquely from solving the differential equation of eq.(16) under the initial condition \( \eta_d(0) = 0 \). Now, multiplying eq.(16) by \( H_{22}^{-1} \) and accompanying this with eq.(8) (in this case, we rewrite the suffix "k" into "d") yield

\[
\begin{align*}
\ddot{q}_d &= J^*(q_d) \dot{x}_d + \alpha P(q_d) \eta_d \\
\dot{\eta}_d &= -H_{12}^T(q_d) \ddot{x}_d + H_{12}(q_d) \dot{x}_d + H_{22}^{-1} B_{22} \ddot{x}_d \\
\end{align*}
\]

This couple implies a set of six simultaneous differential equations of 1st order concerning six variables \( \{q_d, \dot{q}_d, \eta_d\} \) though the right hand side of eq.(19) contains \( \dot{q}_d \) and hence it is an implicit expression. However, substituting eq.(18) into the \( \dot{q}_d \) in eq.(19), we obtain a set of six differential equations of 1st order concerning the six variables in the explicit form

\[
\begin{align*}
\ddot{q}_d &= J^*(q_d) \dot{x}_d + \alpha P(q_d) \eta_d \\
\dot{p}_d &= -H_{12}^T(q_d) \ddot{x}_d + J \dot{x}_d + \alpha P \ddot{p}_d \\
\end{align*}
\]

where we set \( p_d = \eta_d \). The right hand side is nonlinear in \( q_d \) and \( p_d \), but it is Lipschitz continuous in \( q_d \) and \( p_d \) locally. Therefore, for given \( \dot{x}_d \) and \( \dot{x}_d \), there exists a unique solution \( \{q_d(t), p_d(t)\} \) for an interval \( t \in [0, a] \) with some \( a > 0 \) satisfying \( q_d(0) = q(0) \) and \( p_d(0) = 0 \), where \( q(0) \) signifies an initial posture satisfying \( x(q(0)) = x_d(0) \). Further, it is possible to show that, if \( \eta_d(t) \) is uniformly bounded over \( t \in [0, T] \) (that is, there does not arise a finite escape-time regarding a solution to the system of eqs.(18) and (19)), then existence of the solution \( \{q_d(t), p_d(t)\} \) of eq.(20) can be extended further to cover the remaining interval \([a, T] \) beyond \( t = a \). The details of the mathematical proof will be given in Appendix A under the following assumption:

**Assumption 1**

For any \( t \in [0, T] \), any pose \( q \) at time \( t \) satisfying \( x(q) = x_d(t) \) does not come close to any singular pose that satisfies \( q_i = n \pi \) with integer \( n \) for \( i = 2, 3, 4 \).

It is obvious that the Jacobian matrix \( J(q) = \partial x(q)/\partial q \) becomes degenerate if and only if \( q_2 = q_3 = q_4 = n \pi \), which corresponds to such a singular pose that all the links make a straight line. Therefore, if the desired task-space trajectory \( x_d(t) \) is carefully given by enough avoiding the singular poses, it is possible to assume that there exists a positive constant \( \lambda > 0 \) such that \( J(q) J^T(q) \geq \lambda I \) for any \( t \in [0, T] \) and any \( q \) such that \( x(q) = x_d(t) \). This assumption implies the existence of another positive constant \( \Delta_t > 0 \) so that, for any \( t \in [0, T] \) and any \( q \) with \( x(q) = x_d(t) \), it holds

\[
\|J(q)\| \leq \Delta_t
\]

Further, throughout the paper, we reasonably assume that the joint damping matrix \( C \) is chosen to be of order of the square root of the matrix \( H_0 \) that is defined as in eq.(1). This reason is discussed in detail in the previous papers [6,7].

Now, it is possible to state the main result of the paper:

**Proposition 1**

For a given twice continuously differentiable endpoint trajectory \( x_d(t) \) satisfying Assumption 1 or eq.(21) with an initial pose \( q_d(0) \) that satisfies \( x(q_d(0)) = x_d(0) \) and \( q_d(0) = 0 \), there exists a unique joint trajectory \( q_d(t) \) together with a desired control signal \( v_d(t) \) for \( t \in [0, T] \) such that \( v_d(t) \) belongs to the image space of \( J(q(t)) \) for \( t \in [0, T] \) and this \( v = v_d \) and \( q = q_d \) satisfies uniquely the equation:

\[
H(q) \ddot{q} + \left(\frac{1}{2} H(q) + S(q, q) + C\right) \dot{q} + J^T(q) \left[\xi \Delta x + \zeta^T \sqrt{\gamma} \Delta \dot{x}\right] = J^T(q)v
\]
where $\Delta x = x - x_d$ and $x(t) = x_d$.

In order to confirm the existence of such a desired control signal $\nu_d$ that belongs to the image space of $J(q(t))$, we show a simulation result based on the 4-DOF planar robot arm shown in Fig.3 with physical parameters given in Table 1. The values for length, mass, and inertia moment of the first link correspond to those of an upper arm of average human adult (male), and the values for the second link corresponds to those of a lower arm. The third link corresponds to a hand palm and the fourth an index finger. The desired task is to write a handwritten character "a" on the xy-plane, which is given as a dotted locus of the endpoint (see Fig.4 (a)). More explicitly, this desired trajectory is given by the equation:

$$ x_d(t) = \begin{bmatrix} 0.00 \\ 0.30 \end{bmatrix} \left[ 0.075 \cos \omega(t) \\ 0.100 \cos 1.5\omega(t) \right] $$

(23)

where $\omega(t) = 2.0\pi \left\{ -15 \left( \frac{t}{T} \right)^4 + 6 \left( \frac{t}{T} \right)^5 + 10 \left( \frac{t}{T} \right)^3 \right\}$

(24)

and $T = 2.0$ s. The initial pose of the arm is given in Table 2 and the control parameters are chosen as in Table 2. For reference, initial values of the inertia matrix $H(q(0))$ are calculated as in Table 3. Based on this initial pose and given $x_d(t)$ together with $x_d(t)$ and $\dot{x}_d(t)$, the system of differential equations of eqs.(18) and (19) are numerically solved by using the Runge-Kutta method and the control signal $\nu_d$ is obtained by numerically calculating the left-hand side of eq.(14). The obtained solution $\nu_d(t)$ is plotted as in Fig.5.

4. Convergence of the ILC Scheme

We now consider the iterative learning control scheme introduced on the basis of the learning update law defined by eq.(6). When $q = q_d$, eq.(22) can be reduced to eq.(14) by using the transformation

$$ \dot{q}_d = \left( J_d^T(q_d) \alpha P_d(q_d) \right) \begin{bmatrix} \dot{x}_d \\ \dot{\eta}_d \end{bmatrix} $$

(25)

where $H_d$ and $S_d$ are defined below eq.(14). Then, subtraction of eq.(14) from eq.(13) yields
Then, eq.(28) is reduced to
\[
\|q_t(t) - q_d(t)\| \leq \lambda_1 \|\Delta x_k\| + \alpha \|\Delta \hat{h}_k\|
\]
\[
+ (\lambda_2 \|\Delta \hat{h}_k\| + \alpha \lambda_1 \|\hat{h}_k\|) \|q_t(t) - q_d(t)\|
\]
(30)

To simplify the argument, we introduce the notations:
\[
\begin{align*}
\|q_t(t) - q_d(t)\| &= f_1(t) \\
\lambda_1 \|\Delta x_k(t)\| + \alpha \lambda_1 \|\hat{h}_k(t)\| &= L(t) \\
\int_0^t \left[\lambda_1 \|\Delta x_k(\tau)\| + \alpha \|\Delta \hat{h}_k(\tau)\|\right] d\tau &= g_1(t)
\end{align*}
\]
(31)

Next, note that it follows from eq.(30) that
\[
f_1(t) = \|q_t(t) - q_d(t)\| = \left\| \int_0^t (q_t(\tau) - q_d(\tau)) d\tau \right\|
\leq \int_0^t \|q_t(\tau) - q_d(\tau)\| d\tau
\]
(32)

Then, substituting eq.(30) into the integrand of the above equation and referring to eq.(31), we obtain
\[
f_1(t) \leq g_1(t) + \int_0^t L(\tau) f_1(\tau) d\tau
\]
(33)

This can be regarded as an integral inequality concerning the function \(f_1(t)\). Then, by applying Gronwall-Bellman’s lemma to eq.(33), it follows that
\[
f_1(t) \leq g_1(t) + \int_0^t g_1(\tau) L(\tau) \exp \left\{ \int_\tau^t L(u) du \right\} d\tau
\]
(34)

In this paper, we implicitly assume that the terminal time \(T\) of the time interval \([0, T]\) is at most a few seconds. Then, a part of the integrand of eq.(34) is upper-bounded in the following way:
\[
L(\tau) \exp \left\{ \int_\tau^t L(u) du \right\} \leq l(\tau)
\]
(35)

where \(l(\tau)\) is independent of \(t\). At this stage we set \(\alpha = \lambda_1\) and denote the maximum of \(l(\tau)\) over \(\tau \in [0, T]\) by \(l_M\). Then eq.(34) can be rewritten in the form
\[
\|q_t(t) - q_d(t)\| \leq g_1(t) + l_M \int_0^t g_1(\tau) d\tau
\]
(36)

because \(g_1(\tau)\) is monotonously increasing in \(\tau \in [0, T]\).

Now, the norm of the nonlinear term \(h_k\) of eq.(27) can be upper-bounded in such a way that
\[
\|h_k\| \leq \|H_k(q_k) - H_k(q_d)\| L_M
\]
\[
+ \left\{ \frac{1}{2} \|H_k - H_d\| + \|S_k - S_d\| + |C_k - C_d| \right\} L_m
\]
\[
\leq \|q_t(t) - q_d(t)\| (\gamma M + \gamma_m L_m)
\]
\[
+ \|q_t(t) - q_d(t)\| \gamma_0 L_m
\]
(37)

where \(L_M\) denotes the maximum of the norm \(\|\Delta x_k, \hat{h}_k\|\) over \(t \in [0, T]\), \(L_m\) the maximum of \(\|\Delta x_k, \hat{h}_k\|\), and \(\gamma, \gamma_m, \text{ and } \gamma_0\) can be selected as appropriate constants. Further, this inequality can be rewritten by using eqs.(30) and (36) in the following way:
\[
\|h_k\| \leq \gamma \|q_t - q_d\| + \gamma_0 \|q_k - q_d\|
\]
\[
\leq \gamma (1 + l_M T) g_1(t) + \gamma_0 g_1(t)
\]
\[
\leq (\gamma + \gamma_0 l_M) T (1 + l_M T) g_1(t) + \gamma_0 g_1(t)
\]
(38)

where we set
\[
\tilde{\gamma} = (\gamma M + \gamma_m L_m), \quad \tilde{\gamma}_0 = \gamma_0 L_m
\]
(39)

and use the expression
\[
\dot{g}_1(t) = \lambda_1 \|\Delta x_k(t)\| + \|\Delta \hat{h}_k(t)\|
\]
(40)

Next, taking an inner product between eq.(26) and the \(n\)-dimensional vector \((\Delta \hat{x}_k, \hat{h}_k)\) yields
\[
\Delta \hat{x}_k^T \Delta \hat{v}_k = \frac{d}{dt} E_k + \zeta_1 \sqrt{\|q_t\|} \|\Delta \hat{x}_k\|^2
\]
\[
+ \left( \Delta \hat{x}_k^T, \Delta \hat{h}_k \right) C_k \left( \Delta \hat{x}_k^T, \Delta \hat{h}_k \right)^T
\]
\[
+ \left( \Delta \hat{x}_k^T, \hat{h}_k \right) \hat{h}_k
\]
(41)

where
\[
E_k(t) = \frac{1}{2} \left( \Delta \hat{x}_k^T, \Delta \hat{h}_k \right) B_k \left( \Delta \hat{x}_k^T, \hat{h}_k \right)^T
\]
\[
+ \frac{k}{2} \|\Delta \hat{x}_k\|^2
\]
(42)

The last term of eq.(41) is reduced to
\[
(\Delta \hat{x}_k^T, \Delta \hat{h}_k) \hat{h}_k \geq - \|\Delta \hat{x}_k\| \|\Delta \hat{h}_k\| \|\hat{h}_k\|
\]
\[
\geq -\lambda_1 \gamma_0 \|\Delta \hat{x}_k\|^2 + \|\Delta \hat{h}_k\|^2
\]
\[
-(\bar{\gamma} + \bar{\gamma}_0 l_M) T (1 + l_M T) \frac{d}{dt} \left\{ \frac{1}{2 \lambda_1} (\tilde{g}_1)^2 \right\}
\]
(43)

Since it follows from Schwartz’s inequality that
\[
\left| \int_0^t \frac{d}{dt} \left\{ \frac{1}{2 \lambda_1} (\tilde{g}_1)^2 \right\} d\tau \right| = \frac{1}{2 \lambda_1} \left\{ \int_0^t (\tilde{g}_1(t))^2 d\tau \right\}
\]
\[
\leq \frac{\lambda_1}{2} \left\{ \int_0^t \|\|\Delta \hat{x}_k\| + \|\Delta \hat{h}_k\|\| d\tau \right\}^2
\]
\[
\leq \frac{\lambda_1 T}{2} \int_0^t \|\Delta \hat{x}_k\|^2 + \|\Delta \hat{h}_k\|^2 d\tau
\]
\[
\leq \lambda_1 T \int_0^t \|\Delta \hat{x}_k\|^2 + \|\Delta \hat{h}_k\|^2 d\tau
\]
(44)

the integral of eq.(41) over \(t \in [0, T]\) is reduced to
\[
\int_0^T \Delta \hat{x}_k^T \Delta \hat{v}_k d\tau
\]
\[
\geq E_k(t) - E_k(0) + \int_0^T \left\{ \zeta_1 \sqrt{\|q_t\|} \|\Delta \hat{x}_k\|^2
\]
\[
+ \left( \Delta \hat{x}_k^T, \Delta \hat{h}_k \right) C_k \left( \Delta \hat{x}_k^T, \Delta \hat{h}_k \right)^T \right\} d\tau
\]
\[
- \lambda_1 \left\{ \tilde{\gamma}_0 T (\bar{\gamma} + \bar{\gamma}_0 l_M) (1 + l_M T) \right\}
\]
\[
\times \int_0^t \left( \|\Delta \hat{x}_k\|^2 + \|\Delta \hat{h}_k\|^2 \right) d\tau
\]
(45)

For convenience, we define
\[
\gamma = \tilde{\gamma}_0 T (\bar{\gamma} + \bar{\gamma}_0 l_M) (1 + l_M T)
\]
(46)

Thus, if the coefficient \(\zeta_1 \sqrt{\gamma}\) for damping in task space and the damping matrix \(C\) in joint space are selected adequately large enough ensuring the following inequality for any \(k\) and \(t \in [0, T]\):
control signal $v$

when $k$

to the convergence of task-space control signals by the endpoint trajectory. From the second trial, feedforward can be expressed in the form

$J^*(q)\ddot{q} + \left(\frac{1}{2}H(q) + S(q,q)\right)\dot{q} + g(q) = u$

where $g(q)$ stands for the gravity term that can be regarded as a gradient vector of a potential function $G(q)$ with respect to $q$, that is, $g(q) = \delta G(q)/\delta q$. Similarly to the previous argument, we introduce the Jacobian matrix $J(q)$ of the endpoint $x(q)$ in $q$ and its pseudo-inverse $J^*(q)$. Then, we split $g(q)$ into

$g(q) = J^*(q)J(q)g(q) + (J_n - J^*(q)J(q))g(q)$

$= g_n(q) + g_2(q)$

Next, denote the dimension of the task space $x$ by $m$ for a planar case and define the $n \times m$ matrix $P(q) = \{p_1, \ldots, p_m\}$ with $m$ column vectors $p_i$ with $\|p_i\| = 1$ and $p_i^Tp_j = 0$ for $i \neq j$ and satisfying

$P^T(q)J^*(q) = 0$

In a similar way to introduction of eq.(10), we define

$Q(q) = (J^*(q), \alpha P(q)).$  $Q(q)^{-1} = \left(\frac{J(q)}{\alpha^{-1}P^T(q)}\right)$

For a given desired endpoint trajectory $x_d(t)$ for $t \in [0,T]$, we consider the control signal

$u = -C\ddot{q} - J^*(q)\left[k\Delta x + \zeta_1\sqrt{\kappa}\Delta x + J^*(q)\right]v$

where $\Delta x = x(t) - x_d(t)$. Substituting this into eq.(49) and using the transform

$\ddot{\bar{q}} = (J^*(q), \alpha P(q))\left(\begin{array}{c} \dot{x} \\ \eta \end{array}\right)$

we obtain

$H_d\left(\begin{array}{c} \ddot{x}_d \\ \ddot{\eta}_d \end{array}\right) + \left(\frac{1}{2}H_d + S_d + C_d\right)\left(\begin{array}{c} \dot{x}_d \\ \dot{\eta}_d \end{array}\right) + \left(\frac{J(q)}{\alpha^{-1}P^T(q)}\right)g_d = \left(\begin{array}{c} v_d \\ 0 \end{array}\right)$

where $\eta$ denotes an $(n - m)$-dimensional vector and $g_d = g(q_d)$. Similarly to the derivation of eq.(20), we can obtain the following equations from eqs.(54) and (55):

$\ddot{q}_d = (J^*(q_d))\ddot{x}_d + \alpha P(q_d)p_d$

$p_d = -H_dB_d\ddot{x}_d + H_dC_d + \alpha P^T(q_d)g_d$

where we put $p_d = \dot{\eta}_d$. Since the right-hand side of this set of differential equations are also locally Lipschitz-continuous, there exists a unique solution $[q_d(t), p_d(t)]$ for the initial condition $q_d(0) = q(0)$ and $p_d(0) = 0$ in an interval $t \in [0, \alpha]$ with some $\alpha > 0$. Then, the control signal $v_d(t)$ existing in the image space of Jacobian matrix $J(q_d(t))$ can be expressed as

$v_d = (J^*(q_d))^T[H(q_d)\ddot{q}_d + \left(\frac{1}{2}H(q_d) + S(q_d, q_d) + C\right)\dot{q}_d + g(q_d)]$

5. Simulation Results

In order to confirm the theoretical analysis of convergence of the iterative learning control scheme defined by eq.(6), we show a computer simulation result for a given desired task-space trajectory $x_d(t)$ and $\ddot{x}_d(t)$, with continuous $x_d(t)$ and $\ddot{x}_d(t)$ as given by eqs.(23) and (24). In this simulation, the planar robot model with four DOFs as shown in Fig.1 is used, which has physical parameters given in Table 1. At the first trial when $k = 1$, the control signal $v_1(t)$ is set as $v_1(t) = 0$ for $t \in [0,T]$. Therefore, the dynamics of the robot arm can be regarded as a closed-loop system with a task-space PD feedback and without the feedforward term. Then, the trajectory $x_1(t)$ in the task space ($xy$-plane) is shown in Fig.4 (a) as specified by the endpoint trajectory. From the second trial, feedforward terms should be constructed by the learning update law defined by eq.(6). As seen in Fig.4 (b), (c), and (d), the endpoint trajectories converge quickly with increase of the trial number $k$. We also show the convergence of task-space control signals $v_k(t)$ when $k$ increases as in Fig.6. When $k = 10$, the trajectory of control signal $v_{10}(t)$ is almost coincident with the task-space control signal $v_d(t)$ as shown in Fig.6, that uniquely exists in the image space of Jacobian matrix $J(q_d)$.

6. Extension to Robot Dynamics under the Existence of Gravity

Most of the previous results on unique existence of a task-space control and convergence of the iterative learning control scheme can be extended to the case that robot dynamics are subject to the effect of gravity. In this case, the robot dynamics can be expressed in the form[13]:

$H(q)\ddot{q} + \left(\frac{1}{2}H(q) + S(q,q)\right)\dot{q} + g(q) = u$

Fig. 6 Transient responses of the ILC term $v_k = (v_{k1}, v_{k2})^T$.
7. Iterative Learning Control for Redundant Robots under the Gravity Effect

When we apply this ILC scheme to a robot arm under the effect of gravity, it is important to select the joint damping matrix $C$ adequately so that the task-space position and velocity errors $\Delta x_k$ and $\Delta \dot{x}_k$ at the first trial $k=1$ do not deviate far away from zeros. At the same time, it is necessary to assume implicitly that the terminal time $T > 0$ of time duration $[0, T]$ of maneuvering the robot must be set small enough to maintain motion of the arm endpoint within a prescribed range around its target position $x_T(t)$ without much affection from the gravity effect. This shows the necessity of more extensive researches on applicability of the ILC schemes for a more general class of robot dynamics that are subject to the gravity effect and/or constrained to other physical conditions such as contacts with the environment. In the previous papers [8,14], an ILC scheme for robots that are subject to the gravity effect is treated, but it uses a direct compensation for the gravity term.

8. Conclusions

As for the proposed learning control scheme with a learning update law in task-space for a class of redundant robots, it is shown that a feedforward control signal exists in task-space and is determined uniquely. Furthermore, this control signal achieving the desired endpoint trajectory in task-space can be acquired through the proposed iterative learning control scheme. The proof of convergence of both trajectory tracks in task-space and joint space is presented when there is no effect of gravity or the gravity term is compensated directly.

References


Appendix A

Let us partition the $n \times n$ matrices $S_d$ and $C_d$ into

$$ S_d = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}, \quad C_d = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} $$

(A. 1)

Then, both submatrices $S_{11}$ and $S_{22}$ are skew-symmetric and $S_{12}^T = -S_{21}$. More precisely, it follows that

$$ \begin{pmatrix} C_{11} & (J^T)^T C J \alpha & C_{12} \alpha \frac{J^T}{2} C P_d \\ C_{21} & C_{22} \alpha & C_{21} C_{22} \alpha \frac{J^T}{2} C P_d \end{pmatrix} $$

(A. 2)

Since eq.(16) can be written in the form

$$ H_{22}\dot{\eta}_d + \left( \frac{1}{2} H_{22} + S_{22} + C_{22} \right) \dot{\eta}_d = -H_{22}\ddot{x}_d - \left( \frac{1}{2} H_{21} + S_{21} + C_{21} \right) \dot{x}_d $$

(A. 3)

an inner product between this equation and $\dot{\eta}_d$ yields

$$ \frac{d}{dt} \left( \frac{1}{2} \eta_d^T H_{22} \ddot{\eta}_d \right) + \eta_d^T C_{22} \dot{\eta}_d = -\eta_d^T \frac{1}{2} H_{22} \ddot{x}_d - \eta_d^T \left( \frac{1}{2} H_{21}^T + S_{21} + C_{21} \right) \dot{x}_d $$

(A. 4)

Since $H_{12}$ and $S_{21}$ are linear and homogeneous in $\ddot{x}_d$ and $\dot{\eta}_d$, there exist positive constants $\beta_1$ and $\beta_2$ such that

$$ -\eta_d^T \frac{1}{2} H_{22} \ddot{x}_d \leq \beta_1 \| \ddot{x}_d \|_2 + \beta_2 \| \dot{x}_d \|_2 $$

(A. 5)

It should be also noted that for an arbitrary $\beta_1 > 0$

$$ -\eta_d^T \frac{1}{2} H_{22} \ddot{x}_d \leq \beta_1 \| \ddot{x}_d \|_2 + \frac{1}{2} \beta_2 \| \dot{x}_d \|_2 $$

(A. 6)

It is also easy to see that from eq.(A.2)

$$ -\eta_d^T C_{21} \ddot{x}_d = a \eta_d^T P_{d}^T C J \alpha \ddot{x}_d $$

$$ \leq \frac{1}{2} \eta_d^T P_{d}^T C P_d \ddot{\eta}_d + \frac{1}{4} \eta_d^T (J^T)^T C J \alpha \eta_d $$

$$ \leq \frac{1}{2} \eta_d^T C_{22} \ddot{\eta}_d + \frac{1}{2} \eta_d^T C_{21} \ddot{x}_d $$

(A. 7)
Substituting eqs. (A.5) to (A.6) into eq. (A.4) yields

$$\frac{d}{dt} \left( \frac{1}{2} \eta_d^T H_{22} \dot{\eta}_d \right) \leq -\frac{1}{2} \dot{\eta}_d (C_{22} - (\beta_3 + 2\beta_1) I_2) \dot{\eta}_d + \frac{1}{2} \dot{x}_d^T C_1 \dot{x}_d + \frac{1}{2\beta_3} \|H_{12} \dot{x}_d\|^2 + \beta_2 \|\dot{x}_d\|^2$$  \hspace{1cm} (A.8)

Since $\beta_3$ can be arbitrarily chosen, we choose it such that $\beta_3 = \beta_1$. Next, note that the original damping matrix $C$ should be chosen so that $C \geq c_0 H_{22}^{1/2}$ and $\alpha$ can be chosen as $\alpha = \lambda_1$. Then, it is possible to show that

$$C_{22} - (\beta_3 + 2\beta_1) I_2 = C_{22} - 3\beta_1 I_2 > \gamma_0 H_{22}$$  \hspace{1cm} (A.9)

and $\gamma_0 > 0$ is at least of numerical order of $O(1)$ in the physical unit $[s^{-1}]$. Therefore, eq. (A.8) is reduced to

$$\frac{d}{dt} \eta_d(t) \leq -\gamma_0 \eta_d(t) + \xi_d(t)$$  \hspace{1cm} (A.10)

where we put

$$\begin{align*}
\eta_d(t) &= \frac{1}{2} \eta_d^T H_{22} \dot{\eta}_d(t) \\
\xi_d(t) &= \frac{1}{2} \dot{x}_d^T C_1 \dot{x}_d + \frac{1}{2\beta_1} \|H_{12} \dot{x}_d\|^2 + \beta_2 \|\dot{x}_d\|^2
\end{align*}$$  \hspace{1cm} (A.11)

Note that $\xi_d(t)$ is uniformly bounded in $t \in [0, T]$. Clearly eq. (A.10) implies

$$\eta_d(t) \leq \int_0^t e^{-\gamma_0(t-\tau)} \xi_d(\tau) d\tau$$  \hspace{1cm} (A.12)

which is uniformly bounded in $[0, T]$. 

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