Parametric Design of a Dual-Rate Controller for Improvement in Steady-State Intersample Response

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Abstract : The present paper proposes a new design method for a dual-rate system, in which a plant in continuous time is controlled by updating a control input in discrete time, in which the sampling interval of the plant output differs from the update interval of the control input. A dual-rate control law that stabilizes a discrete-time closed-loop system is extended. One of the advantages of such an extension is that the dual-rate control law can be redesigned to be independent of the discrete-time closed-loop system. Consequently, the intersample response can be improved by making it independent of the sampled response using the newly introduced design parameters. The present paper proposes a design method for design parameters to improve the intersample response in the steady state. Numerical examples are presented to confirm the effectiveness of the proposed method.

Key Words : dual-rate system, continuous-time system, discrete-time system, sampled behavior, intersample behavior, sampled-data system, zero-order hold, steady-state behavior.

1. Introduction

The present paper discusses a sampled-data control system, in which a continuous-time plant is controlled using a digital controller in discrete time. In the sampled-data control system, a control input is updated using a sampled output and is sent to a controlled plant through a holder. Numerous design methods have been proposed for a single-rate system, in which the sampling interval of the plant output is the same as the update interval of the control input. However, in many applications, the interval is constrained by, for example, the structure of the controlled plant or the properties of a sensor or holder. Hence, the sampling interval is not always the same as the update interval, and such systems are referred to as multirate systems [1]–[5].

The present paper deals with a multirate system, in which the sampling interval is an integer multiple of the update interval. If the sampling interval cannot be shortened to be the same as the update interval, then it is not possible to obtain a fast-rate single-rate system with a sampling interval that is equivalent to the update interval. Thus, in order to design a single-rate system, the update interval has to be set to be the same as the sampling interval. However, the obtained single-rate system becomes a slow-rate system, and its performance decreases relative to that of a multirate system. It is therefore desirable to set the update interval to be as short as possible. Generally, the design of a multirate system is more difficult than that of a single-rate system, but a multirate system can be designed to be the same as a single-rate system by using techniques such as lifting [6].

However, multirate systems have the associated problem of ripples emerging between sampling instants when a sampled output converges from the controller to a controlled plant through a holder. Numerous design methods have been proposed for a single-rate system, in which the e error exists [13]. In the multirate system, if the steady-state gains of a controller are constant, intersample ripples are prevented [12]. However, this condition [12] is not sufficient because it does not consider overall intersample behavior [20]. In sampled-data control theory, intersample ripples can be handled because a discrete-time controller is designed on the basis of continuous-time behavior. Furthermore, a design method using a generalized hold has been proposed and its application has been reported [17]–[19]. Nevertheless, the condition [12] is straightforward and can be used to suppress intersample ripples in steady state using a conventional D/A converter that employs zero-order hold. The present paper proposes a new design method for a multirate system without an integrator. Since a dual-rate control law can be redesigned independent of a closed-loop system using new design parameters, the intersample response can be improved independent of the sampled response. In the present paper, new design parameters are designed to suppress intersample ripples in the steady state using the condition [12] and zero-order hold. Consequently, while the transient response is not always improved, ripples do not emerge between sampling instants when a sampled output converges to a reference input. Furthermore, output response is not deteriorated by an integrator. The simulation results indicate the advantages of the proposed method.

In the present paper, $z_1^{-1}$ denotes the one-step backward shift operator, $z_1^{-1}y[k] = y[k-1]$ and $z_2^{-1} = z_1^{-1}$. A polynomial is

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described as $A[z^{-1}]$, and a polynomial vector and a polynomial matrix are described as $A[z^{-1}]$.

### 2. Problem Statements

A controlled plant is assumed to be a linear time-invariant single-input single-output continuous-time:

$$
\begin{align*}
\dot{x}(t) &= A_s x(t) + B_s u(t) \\
y(t) &= C_s x(t)
\end{align*}
$$

(1)

(2)

where $x(t)$, $u(t)$, and $y(t)$ are the state vector, the plant output and the control input, respectively. The dimensions of matrices $A_s$, $B_s$, and $C_s$ are $n \times n$, $n \times 1$, and $1 \times n$, respectively. The present paper describes how a continuous-time plant is controlled by a discrete-time controller. In the design of a single-rate sampled-data control system, a discrete-time controller calculates a control input using a sampled plant output obtained at sampling instants, and the calculated control input is then held between sampling instants using a hold器. Using interval $T_s$ and

$$
\begin{align*}
u(t) &= u[k] \\
(kT_s \leq t < (k+1)T_s, \quad k = 0, 1, 2, \cdots)
\end{align*}
$$

the continuous model is transformed into the following discrete-time model:

$$
\begin{align*}
A_1[z^{-1}] y[k] &= B_1[z^{-1}] u[k-1] \\
A_1[z^{-1}] &= 1 + a_1 z^{-1} + \cdots + a_n z^{-n} \\
&= (1-a_1 z^{-1}) \cdots (1-a_n z^{-n}) \\
B_1[z^{-1}] &= b_{10} + b_1 z^{-1} + \cdots + b_n z^{-n}
\end{align*}
$$

(3)

(4)

(5)

(6)

where $y[k]$ and $u[k]$ are the plant output and the control input, respectively, at sampling instant $k$.

**Assumption 1** In the present paper, a sampled-data system is designed using the conditions:

- Control input is updated at intervals of $T_s$
- Plant output is sampled at intervals of $T_s$, where $l$ is an integer and $l \geq 2$

Hence, the designed sampled-data control system is a dual-rate system, where the sampling interval is $l$ times the update interval. Under Assumption 1, the present paper deals with a multi-input single-output single-rate system converted from a dual-rate system using techniques such as Lifting [6].

Multiplying both sides of (4) by

$$
C[z^{-1}] = \prod_{i=1}^{n} (1 + a_i z^{-1} + \cdots + a_{i-1} z^{-i})
$$

(7)

provides an equivalent non-minimal form of (4) [14],[15].

$$
\begin{align*}
A[z^{-1}] y[k] &= B[z^{-1}] u[k-1] \\
A[z^{-1}] &= A_1 C[z^{-1}] \\
&= 1 + a_1 z^{-1} + \cdots + a_n z^{-n} \\
B[z^{-1}] &= B_1 C[z^{-1}] \\
&= b_0 + b_1 z^{-1} + \cdots + b_n z^{-n}
\end{align*}
$$

(8)

(9)

(10)

In the present paper, (8) is rearranged as

$$
A[z^{-1}] y[k] = B[z^{-1}]^T u[k-l]
$$

$$
A[z^{-1}] = A[z^{-1}] = 1 + a_1 z^{-1} + \cdots + a_n z^{-n}
$$

$$
B[z^{-1}] u[k-1] = B[z^{-1}]^T u[k-l]
$$

$$
B[z^{-1}] = B_1 [z^{-1}] B_2 [z^{-1}] \cdots B_l [z^{-1}]^T
$$

$$
u[k] = [u[k] u[k+1] \cdots u[k+l-1]]^T
$$

(11)

(12)

(13)

(14)

(15)

(16)

and a dual-rate controller is designed using the $l$-inputs single-output single-rate model (11).

It is assumed that the following dual-rate control law applies and stable control of the plant is achieved by the control law:

$$
\begin{bmatrix}
y[k] \\
\vdots \\
y[k+l-1]
\end{bmatrix}
= \frac{K_1[z^{-1}] w[k] - X_1[z^{-1}] y[k]}{Y_1[z^{-1}]}
\quad (17)
$$

where scalar $w[k]$ is the reference input to be followed by the plant output $y[k]$, and $K_1[z^{-1}]$, $X_1[z^{-1}]$ and $Y_1[z^{-1}]$ are coefficient polynomials. The dual-rate controller (17) is designed as a multi-input single-output single-rate system.

(17) is rewritten as

$$
\begin{bmatrix}
y[k] \\
\vdots \\
y[k+l-1]
\end{bmatrix}
= \frac{K_1[z^{-1}] w[k] - X_1[z^{-1}] y[k]}{Y_1[z^{-1}]}
$$

(18)

where $Y_1[z^{-1}]$ is an $l$-dimensional polynomial diagonal matrix and is singular, $K_1[z^{-1}]$ and $X_1[z^{-1}]$ are $l$-dimensional polynomial vectors, and are given as:

$$
Y_1[z^{-1}] = \begin{bmatrix} Y_1[z^{-1}] & 0 \\
0 & Y_1[z^{-1}] \\
\vdots & \vdots \\
0 & Y_1[z^{-1}]
\end{bmatrix} 
$$

(19)

$$
K_1[z^{-1}] = \begin{bmatrix} K_1[z^{-1}] & \cdots & K_l[z^{-1}]
\end{bmatrix}^T 
$$

(20)

$$
X_1[z^{-1}] = \begin{bmatrix} X_1[z^{-1}] & \cdots & X_l[z^{-1}]
\end{bmatrix}^T 
$$

(21)

where $Y_1[z^{-1}]$ is non-singular.

In this case, the closed-loop system is calculated as:

$$
y[k] = \frac{z^{-1} Y_1[z^{-1}]^T K_1[z^{-1}] w[k]}{T[z^{-1}]}
$$

(22)

where

$$
T[z^{-1}] = Y_1[z^{-1}] A[z^{-1}]
$$

$$
+ z^{-1} Y_1[z^{-1}] X[z^{-1}]
$$

(23)

$$
Y_1[z^{-1}] = \sum_{i=1}^{l} Y_i[z^{-1}]
$$

(24)

$$
Y_1[z^{-1}] = \begin{bmatrix} Y_1[z^{-1}] & \cdots & Y_l[z^{-1}]
\end{bmatrix}^T 
$$

(25)

$$
Y_1[z^{-1}] = B_1[z^{-1}] \prod_{i=1}^{l} Y_i[z^{-1}].
$$

(26)

The derivation of (22) is shown in Appendix A.
From this assumption, a sampled plant output converges to a reference input and is given as a step-wise function, but the intersample response may oscillate. Hence, in the present paper, the intersample response is improved without deterioration of the closed-loop system (22). To this end, the control objective of the present paper is to make an intersample output converge to the reference input without changing the closed-loop system. In the next section, the dual-rate controller (18) is extended and the intersample response is improved.

3. Extension of the Dual-rate Controller

An extension of a dual-rate control law is proposed to improve the intersample behavior, and the advantages are shown. Design polynomial vectors $U_a[z_{i+1}]$ and $U_a[z_{i+1}]$ are newly introduced, and the dual-rate control law (18) is extended as:

$$Y_{i}[z_{i+1}]u[k] = K[z_{i+1}]w[k] - X_{i}[z_{i+1}]y[k] \quad (27)$$

$$X_{i}[z_{i+1}] = X_{i}[z_{i+1}] + U_a[z_{i+1}]A[z_{i+1}] \quad (28)$$

$$U_a[z_{i+1}] = \begin{bmatrix} u_{a,1}[z_{i+1}] & \cdots & u_{a,d}[z_{i+1}] \end{bmatrix}^T \quad (29)$$

$$U_{a_i}[z_{i+1}] = \begin{bmatrix} Y_{i,1}[z_{i+1}] & \cdots & Y_{i,d}[z_{i+1}] \end{bmatrix} \quad (30)$$

$$Y_{i_a}[z_{i+1}] = \begin{bmatrix} Y_{i,1}[z_{i+1}] - z_{i+1}Y_{a_a}[z_{i+1}]B[z_{i+1}] \quad (i = j) \end{bmatrix} \quad (31)$$

$$Y_{i_a}[z_{i+1}] = \begin{bmatrix} -z_{i+1}U_{a_a}[z_{i+1}]B[z_{i+1}] \quad (i \neq j) \end{bmatrix} \quad (32)$$

$$Y_{i}[z_{i+1}] = \begin{bmatrix} Y_{i,1}[z_{i+1}] - z_{i+1}U_{a_a}[z_{i+1}]B[z_{i+1}] \quad (i = j) \end{bmatrix} \quad (33)$$

The derivation of (37) is shown in Appendix D. It follows from (38) that the following theorems are obtained.

**Theorem 1** $U_a[z_{i+1}] = U_a[z_{i+1}], T_a[z_{i+1}] = 0$. The closed-loop system (37) obtained using the extended control law is equivalent to the closed-loop system (22). Therefore, using $U_a[z_{i+1}]$ and $U_a[z_{i+1}]$, the dual-rate controller (18) can be redesigned independent of the closed-loop system (22) under the condition $U_a[z_{i+1}] = U_a[z_{i+1}]$.

The primary advantage of Theorem 1 is that $T_a[z_{i+1}] = T_a[z_{i+1}]$ is achieved simply, although the constraint $U_a[z_{i+1}] = U_a[z_{i+1}]$ must be satisfied. In this case, the form of the extended controller differs from that of the original one, but the control performance is the same. In particular, the use of $U_a[z_{i+1}] = U_a[z_{i+1}] = 0$ gives the same controller as the original one. Therefore, Theorem 1 shows that the extension of the dual-rate controller is a general design method including the original controller.

**Theorem 2** $U_a[z_{i+1}] \neq U_a[z_{i+1}]$ and $\tilde{T}_a[z_{i+1}] = 0$ and $T_a[z_{i+1}] = T_a[z_{i+1}]$ can be achieved. Then, the closed-loop systems (22) and (37) are equivalent independent of the selection of $U_a[z_{i+1}]$ and $U_a[z_{i+1}]$.

Using $U_a[z_{i+1}]$ and $U_a[z_{i+1}]$ that satisfy $T_a[z_{i+1}] = 0$, the intersample response can be improved independent of the sampled response because a control law can be redesigned independent of a discrete-time closed-loop system. Using $U_a[z_{i+1}]$ and $U_a[z_{i+1}]$, the extended controller can improve the control performance. In the present paper, using Theorem 2, a design method for $U_a[z_{i+1}]$ and $U_a[z_{i+1}]$ is proposed.

$$U_a[z_{i+1}] = U_a[z_{i+1}]B[z_{i+1}]A[z_{i+1}] \quad (i \neq l) \quad (40)$$

$$U_a[z_{i+1}] = U_a[z_{i+1}]B[z_{i+1}]A[z_{i+1}] \quad (i \neq l) \quad (41)$$

$$U_a[1][z_{i+1}] = U_a[z_{i+1}]B[z_{i+1}]A[z_{i+1}] \quad (i \neq l) \quad (42)$$

$$U_a[z_{i+1}] = U_a[z_{i+1}]B[z_{i+1}]A[z_{i+1}] \quad (i \neq l) \quad (43)$$

where $U_a[z_{i+1}] (i = 1, \ldots, l)$ is a design polynomial. Using (40) ~ (43), $\tilde{T}_a[z_{i+1}] = 0$ is achieved independent of the selection of $U_a[z_{i+1}]$, and the sampled behavior is maintained. Hence, the intersample behavior can be improved because the control input between sampling instances is changed by $U_a[z_{i+1}]$. The closed-loop gain $C[z_{i+1}]$ from the reference input to the control input is given as:

$$C[z_{i+1}] = \begin{bmatrix} C_1[z_{i+1}] \\ C_2[z_{i+1}] \\ \vdots \\ C_l[z_{i+1}] \end{bmatrix} \quad (44)$$

where $M[z_{i+1}] = A[z_{i+1}]Y_{a}[z_{i+1}] + z_{i+1}X_{a}[z_{i+1}]B[z_{i+1}] \quad (45)$

In the present paper, a design method for $U_a[z_{i+1}]$ is proposed to suppress the intersample ripple in the steady state. To this end, the steady-state gain of $C[z_{i+1}]$ must satisfy

The ARMA model is given:

\[ C_1[1] = C_2[1] = \cdots = C_l[1] \tag{47} \]

where \( C_i[1] \) \((i = 1, \cdots, l)\) are defined as

\[
\begin{bmatrix}
C_1[1] \\
C_2[1] \\
\vdots \\
C_l[1]
\end{bmatrix} = \bar{M}[1]K[1] \tag{48}
\]

and \( \bar{M}[z^{-1}] \) denotes the cofactor matrix of \( M[z^{-1}] \). \( U_i[1] \) \((i = 1, \cdots, l)\) are obtained by solving equations: \( \bar{C}_1[1] = C_2[1], \bar{C}_2[1] = C_3[1], \cdots, \bar{C}_l[1] = \bar{C}_1[1]. \) Based on (47), (46) is satisfied and a control input does not fluctuate between sampling instants in the steady state, and intersample ripples do not emerge. In the present paper, since \( U_i[z^{-1}] \) has been designed to suppress intersample ripples in the steady state, intersample ripples do not emerge in the steady state even if \( U_i[z^{-1}] \) is a constant. However, because \( U_i[z^{-1}] \) can be designed as a polynomial, further improvement of the intersample response can be achieved by considering the transient response.

4. Numerical Example

Numerical examples demonstrate the effectiveness of the proposed method. In particular, the advantage of Theorem 2 is shown.

A controlled plant is given as the following transfer function.

\[ G(s) = \frac{1}{s^2 + 1.8s + 1} \tag{49} \]

The state-space representation of the transfer function model is given as:

\[
\begin{align*}
\dot{x}(t) &= \begin{bmatrix} -1.8 & -1 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) \\
y(t) &= \begin{bmatrix} 0 & 1 \end{bmatrix} x(t)
\end{align*} \tag{50, 51}
\]

If both the update interval of a control input and the sampling interval of a plant output are 2[s], the following discrete-time ARMA model is given:

\[
(1 - 0.74z^{-1} + 0.17z^{-2})y[k] = (0.28 + 0.15z^{-1})u[k - 1] \tag{52}
\]

Assume that the control input \( u[k] \) is updated every 1 step but the plant output \( y[k] \) is sampled every 2 steps. Therefore, \( i = 2 \), and a dual-rate controller is designed using the following two-input single-output single-rate system.

\[
(1 - 0.21z^{-1} + 0.0275z^{-2})y[k] = [0.36 + 0.025z^{-1} + 0.15z^{-2}]u[k - 2] \tag{53}
\]

A reference input to be followed by a plant output is set as a unit step signal.

First, a dual-rate controller, which stabilizes a closed-loop system, is obtained as

\[
\begin{bmatrix} 1 + 0.099z^{-1} \\
0 \end{bmatrix} u[k] = \begin{bmatrix} 0.93 \\
-0.78 - 0.73z^{-1} \end{bmatrix} y[k]. \tag{54}
\]

The use of the control law (54) makes a closed-loop system stable, where the characteristic polynomial is \( 1 - 0.6z^{-1} + 0.01z^{-2} \). The sampled response is illustrated by the circle in Fig. 1. The intersample response and the control input are illustrated by the solid lines in Fig. 1 and Fig. 2, respectively, and this result corresponds to \( U_i[z^{-1}] = U_i[z^{-1}] = 0 \) as described later herein. The plant output converges to the reference input at sampling instants, but the control input does not converge but oscillates, and the intersample output oscillates between sampled outputs because the steady-state gains from a reference input to a control input \((C[1] = \lim_{k \to \infty} \frac{\Delta y}{\Delta u})\) are \((2.2 - 0.071)\) and are not constant. The employed control law is equivalent to the proposed control law using \( U_i[z^{-1}] = U_i[z^{-1}] \).

Next, the intersample behavior is improved using the proposed design method. Using (40)–(43), \( U_a[z^{-1}] \) and \( U_b[z^{-1}] \) are given as follows.

\[
U_a[z^{-1}] = \begin{bmatrix} U_i[z^{-1}] & B_a[z^{-1}]Y_i[z^{-1}] \\
U_i[z^{-1}] & B_a[z^{-1}]Y_i[z^{-1}] \end{bmatrix} \tag{55}
\]

\[
U_b[z^{-1}] = \begin{bmatrix} U_i[z^{-1}] & B_a[z^{-1}]Y_i[z^{-1}] \\
U_i[z^{-1}] & B_a[z^{-1}]Y_i[z^{-1}] \end{bmatrix} \tag{56}
\]

Then, \( T_i[z^{-1}] = 0 \) can be achieved independent of the selection of \( U_i[z^{-1}] \) \((i = 1, 2)\), and the closed-loop system obtained using the extended control law is equivalent to that of (54). Therefore, proper design of \( U_i[z^{-1}] \) can improve the intersample response independent of the sampled response given by the conventional dual-rate controller (54). To make the steady-state gains \( C[1] = C_2[1], U_i[z^{-1}] \) is calculated using (44), (46), and (48), and the steady-state behavior of the intersample response is improved. To obtain a unique solution of \( C[1] = C_2[1] \) using (48), \( U_i[z^{-1}] \) is fixed as 0. The obtained \( U_i[z^{-1}] \) is then given as:

\[ U_i[z^{-1}] = -3.46 \tag{57} \]

Using (55), (56), and (57), the extended control law is given by:

\[
\begin{bmatrix} 1 + 0.44z^{-1} & -0.27z^{-1} \\
0.25z^{-2} & +0.33z^{-2} \\
+0.035z^{-3} & +0.12z^{-3} \\
+0.0013z^{-4} & +0.0085z^{-4} \end{bmatrix} u[k] = \begin{bmatrix} 0.93 \\
-0.78 - 0.73z^{-1} \end{bmatrix} y[k] \tag{58}
\]

Then, the steady-state gains are \( C[1] = \{1.1\} \), and (46) is satisfied. The simulation result obtained using the extended control law is illustrated by the dashed lines in Fig. 1, 2. It can be seen that the sampled response is the same as that of (54), and also that the intersample response can be improved independent of the sampled response and that it converges to the reference input without ripples. Consequently, the control inputs given by
the extended control law (58) calculated at sampling instants converge to $[1 \ 1]^T$ without oscillation. Consequently, the ripples between the sampled outputs could be eliminated. Furthermore, the proposed method is a general design method including the conventional dual-rate control because the simulation result using $U_1[z^{-1}] = 0$ is the same as that of (54).

**Fig. 1** Output results obtained by using dual-rate control law and extended control law.

**Fig. 2** Input results obtained by using dual-rate control law and extended dual-rate control law.

5. Conclusion

The present paper has proposed a new design method for a dual-rate control system, where the sampling interval of a plant output is an integer multiple of the update interval of a control input. A dual-rate control law is extended by introducing new design parameters. The extended control law has the advantage that the controller can be redesigned without changing a closed-loop system in discrete time, as has been proven in the present paper. Finally, simulation results that confirm the effectiveness of the proposed method have been shown.

The proposed method requires the assumption that the control law (17) is given. However, a control input can be adjusted independent of the sampled response. As a result, the intersample response can be improved independent of sampled response. In the present paper, the relation between the transient intersample response and the new design parameters are not made clear, but because the present paper has proposed a design method for parameters that satisfies the condition whereby the intersample ripples do not appear in the steady state, an intersample output could converge to a reference input as well as a sampled output without a change in the sampled response. In the future, an optimal design method for the design parameters $U_{i1}[z^{-1}]$ and $U_{i1}[z^{-1}]$ should be investigated to improve transient response. Furthermore, the proposed controller has been designed without taking model uncertainties into account. Therefore, further study is required in order to produce a system that is robust against uncertainties.

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**References**


**Appendix A** Derivation of Closed-loop System (22)

Since the inverse matrix of \( Y[z_i^{-1}] \) is given as:

\[
Y[z_i^{-1}]^{-1} = \frac{1}{Y_p[z_i^{-1}]} \cdot \text{diag} \left\{ Y[z_i^{-1}], \ldots, Y[L[z_i^{-1}]] \right\}
\]

the following equation is derived.

\[
Y_p[z_i^{-1}] A[z_i^{-1}] y[k] = z_i^{-1} Y_p[z_i^{-1}] \cdot (K[z_i^{-1}] + u[k] - X[z_i^{-1}] y[k]).
\]

It follows from (A.2) that the closed-loop system (22) is obtained.

**Appendix B** Proof of Lemma 1

**Proof 1** \( \forall i \in \{1, 2, \ldots, l\} \), \( i \)-th row in \( Y[z_i^{-1}] \) is multiplied by \( \prod_{k=1}^{l-1} U_{a_k}[z_i^{-1}] \), and \( \forall j \in \{1, 2, \ldots, l\} \), \( j \)-th column in \( Y[z_i^{-1}] \) is multiplied by \( \prod_{k=1}^{l-1} B_k[z_i^{-1}] \). Then, \( Y_{i,j}[z_i^{-1}] \) is obtained. The determinant of \( Y[z_i^{-1}] \) is given as:

\[
|Y_{i,j}[z_i^{-1}]| = \left( \prod_{k=1}^{l-1} U_{a_k}[z_i^{-1}] B_k[z_i^{-1}] \right)^{-1} |\tilde{Y}_{i,j}[z_i^{-1}]| (B.1)
\]

where \( i, j \) element in \( Y_{i,j}[z_i^{-1}] \)

\[
|Y_{i,j}[z_i^{-1}]| = \begin{cases} \prod_{k=1}^{l-1} U_{a_k}[z_i^{-1}] B_k[z_i^{-1}] & (i = j) \\ -z_i^{-1} \prod_{k=1}^{l-1} U_{a_k}[z_i^{-1}] B_k[z_i^{-1}] & (i \neq j) \end{cases}
\]

\(|Y_{i,j}[z_i^{-1}]|\) is obtained using \( M_i \) and \( N_j \) defined as follows:

\[
M_i = |\tilde{Y}_{i,j}[z_i^{-1}]| (B.3)
\]

\[
N_j = \begin{bmatrix} M_{i-1} & -B[z_i^{-1}] \\ -B[z_i^{-1}] & \ddots & -B[z_i^{-1}] \\ & \ddots & \ddots & -B[z_i^{-1}] \\ & & -B[z_i^{-1}] & -B[z_i^{-1}] \end{bmatrix} (B.4)
\]

\(M_i\) is calculated using the following equations:

\[
M_i = Y_i[z_i^{-1}] M_{i-1} + \tilde{Y}_{i-1,j}[z_i^{-1}] N_{j-1} (B.5)
\]

\[
N_i = Y_i[z_i^{-1}] N_{i-1} (B.6)
\]

where

\[
M_i = Y_i[z_i^{-1}] - B[z_i^{-1}] (B.7)
\]

\[
N_i = Y_i[z_i^{-1}] (B.8)
\]

\[
\tilde{Y}_{i,j}[z_i^{-1}] = Y_i[z_i^{-1}] \prod_{k=1}^{l-1} U_{a_k}[z_i^{-1}] B_k[z_i^{-1}] (B.9)
\]

\[
\tilde{Y}_{i,j}[z_i^{-1}] = z_i^{-1} \prod_{k=1}^{l} U_{a_k}[z_i^{-1}] B_k[z_i^{-1}] (B.10)
\]

The use of the properties of the determinant satisfies (35).

**Appendix C** Proof of Lemma 2

**Proof 2** \( \tilde{Y}_{i,j}[z_i^{-1}] \) is defined by

\[
\tilde{Y}_{i,j}[z_i^{-1}] = \begin{bmatrix} \tilde{Y}_{E,1}[z_i^{-1}] & \cdots & \tilde{Y}_{E,1}[z_i^{-1}] \\ \vdots & \ddots & \vdots \\ \tilde{Y}_{E,1}[z_i^{-1}] & \cdots & \tilde{Y}_{E,1}[z_i^{-1}] \end{bmatrix} (C.1)
\]

\(Y_{i,j}[z_i^{-1}]\) is obtained using (B.5), the diagonal elements are derived as

\[
\tilde{Y}_{E,1}[z_i^{-1}] = \prod_{k=1}^{l} Y_k[z_i^{-1}]
\]

\(\forall k \neq l\),

\[
\tilde{Y}_{E,1}[z_i^{-1}] = \sum_{k=1}^{l} U_{a_k}[z_i^{-1}] B_k[z_i^{-1}] \prod_{l=1}^{k-1} Y_k[z_i^{-1}].
\]

**non-diagonal elements** Calculation similar to that in (B.6) yields

\[
\tilde{Y}_{E,1}[z_i^{-1}] = z_i^{-1} U_{a_i}[z_i^{-1}] B_i[z_i^{-1}] \prod_{k=1}^{l} Y_k[z_i^{-1}](C.3)
\]

where \( i \neq j \).

Therefore, the elements on the right-hand side of (C.1) are obtained as (C.2) and (C.3).

**Appendix D** Derivation of the Closed-loop System (37)

Substitution of (27) into (11) yields the following:

\[
|Y_{i,j}[z_i^{-1}]| A[z_i^{-1}] y[k] = z_i^{-1} B[z_i^{-1}] \tilde{Y}_{i,j}[z_i^{-1}] \cdot (K[z_i^{-1}] + u[k] - X[z_i^{-1}] y[k]).
\]

The \( i \)-th element of \( B[z_i^{-1}] \tilde{Y}_{i,j}[z_i^{-1}] \) is calculated as follows:

\[
B[z_i^{-1}] \tilde{Y}_{i,j}[z_i^{-1}] = \sum_{k=1}^{l} B_k[z_i^{-1}] \tilde{Y}_{E,1}[z_i^{-1}] = \prod_{k=1}^{l} Y_k[z_i^{-1}]
\]

Consequently, the following relation is obtained:

\[
B[z_i^{-1}] \tilde{Y}_{i,j}[z_i^{-1}] = Y_{bl}[z_i^{-1}] t (D.3)
\]

Using (D.3), the closed-loop system (37) is calculated.

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