Inverse Optimal and Asymptotically Stable Adaptive Consensus Control of Multi-Agent Systems Based on $H_{\infty}$ Control Criterion

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Abstract: Design methods of asymptotically stable adaptive consensus control of multi-agent systems composed of the first-order and the second-order regression models are presented based on inverse optimal control criterion. The proposed control schemes are derived as solutions of certain $H_{\infty}$ control problems, where estimation errors of tuning parameters are regarded as external disturbances to the process. The resulting control systems are robust to uncertain system parameters and the desirable consensus tracking is achieved asymptotically via adaptation schemes and $L_2$-gain design parameters together with an introduction of a generating model of a leader.

Key Words: adaptive control, consensus control, multi-agent system, $H_{\infty}$ control.

1. Introduction

Distributed consensus tracking of multi-agent systems with limited communication networks, has been a basic and important issue in cooperative control problems of multi-agent systems, and various research results have been reported for various processes and under various conditions [1]–[4]. In those works, adaptive control or sliding mode control methodologies were also proposed in order to deal with uncertainties of agents, and stability of control systems was discussed via Lyapunov function analysis. However, so much attention does not have been paid on control performance such as optimal property or transient performance in those research works.

The purpose of the paper is to present design methods of asymptotically stable adaptive consensus control of multi-agent systems composed of the first-order and the second-order systems of multi-agent systems, and various research results have been reported for various processes and under various conditions [1]–[4]. In those works, adaptive control or sliding mode control methodologies were also proposed in order to deal with uncertainties of agents, and stability of control systems was discussed via Lyapunov function analysis. However, so much attention does not have been paid on control performance such as optimal property or transient performance in those research works.

The purpose of the paper is to present design methods of asymptotically stable adaptive consensus control of multi-agent systems composed of the first-order and the second-order regression models based on the notion of inverse optimality and $H_{\infty}$ control criterion [5],[6]. This is an extension of our previous work [7], where asymptotic consensus tracking was not assured generally. The proposed control schemes are derived as solutions of certain $H_{\infty}$ control problems, where estimation errors of tuning parameters are regarded as external disturbances to the process. The resulting control systems are robust to uncertain system parameters and the desirable consensus tracking is achieved asymptotically via adaptation schemes and $L_2$-gain design parameters together with an introduction of a generating model of a leader [8]. Several simulation studies also confirm the effectiveness of the proposed methodologies.

It should be noted that the cost functional in the present $H_{\infty}$ control problem, is not specified in advance. Instead, in the inverse optimal controller design, that is to be deduced at the final stage, as a possible choice of the meaningful cost functional from the total control problem.

2. Multi-Agent Systems and Information Network Graphs

In this section, mathematical preliminaries on information network graph of multi-agent systems are summarized [2],[3]. As a model of interaction among agents, a weighted undirected graph $G = (\mathcal{V}, \mathcal{E})$ is considered, where $\mathcal{V} = \{1, \ldots, N\}$ is a node set corresponding to a set of agents. $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is an edge set, and an edge $(i, j) \in \mathcal{E}$ indicates that the agent $i$ and $j$ can communicate with each other. Associated with the edge set $\mathcal{E}$, a weighted adjacency matrix $A = [a_{ij}] \in \mathbb{R}^{N \times N}$ is introduced, whose entry $a_{ij}$ is defined by

$$
\begin{cases}
    a_{ij} = a_{ji} > 0 & \text{if } (i, j) \in \mathcal{E}, \\
    a_{ij} = a_{ji} = 0 & \text{otherwise}.
\end{cases}
$$

A path is a sequence of edges in the forms $(i_1, i_2), (i_2, i_3), \ldots$ $(i_j \in \mathcal{V})$, and the undirected graph is called connected, if there is always an undirected path between every pair of distinct nodes. For the adjacency matrix $A = [a_{ij}]$, the Laplacian matrix $L = [l_{ij}] \in \mathbb{R}^{N \times N}$ is defined by

$$
l_{ij} = \sum_{j=1}^{N} a_{ij}, \quad l_{ij} = -a_{ij}, \quad (i \neq j).
$$

The Laplacian matrix is symmetric and positive-semidefinite. Furthermore, the Laplacian matrix has a simple 0 eigenvalue with the associated eigenvector $1 = [1 \cdots 1]^T$, and all other eigenvalues of $L$ are positive, if the corresponding undirected graph is connected.

In this manuscript, a consensus control problem of leader-follower type is considered, where $x_0$ is a leader which each agent $i \in \mathcal{V}$ (a follower) should follow. For the leader $x_0$ and the followers $i$, $a_{i0}$ is defined such as

$$
a_{i0} = \begin{cases} 
    > 0 & \text{leader’s information is available to follower } i, \\
    = 0 & \text{otherwise,}
\end{cases}
$$

and from $a_{i0}$ and $L$, the matrix $M = [m_{ij}] \in \mathbb{R}^{N \times N}$ is defined by

$$
M = L + \text{diag}(a_{i0} \ldots a_{N0}).
$$

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$M$ is symmetric and positive definite, if 1. at least one $a_0$ $(1 \leq i \leq N)$ is positive, and 2. the graph $G$ is connected [3]. Hereafter, it is assumed that those conditions 1 and 2 are satisfied.

3. Adaptive $H_\infty$ Consensus Control for First-Order Models

3.1 Problem Statement

A multi-agent system composed of the first-order regression models is considered.

$$\dot{x}_i(t) = \Omega x_i(t) \theta_i + B_i u_i(t), \quad (i = 1, \ldots, N),$$

where $x_i \in \mathbb{R}^n$ is a state, $u_i \in \mathbb{R}^m$ is an input, $\theta_i \in \mathbb{R}^l$ is an unknown parameter vector, and $\Omega_i \in \mathbb{R}^{n \times n}$ is a regressor matrix composed of $x_i$ and its structure is known. It is assumed that $\Omega_i$ is bounded for bounded $x_i$, $B_i \in \mathbb{R}^{n \times m}$ is an unknown matrix of the form

$$B_i = \text{diag}(b_{i1}, \ldots, b_{in}),$$

and the sign of $b_{ij}$ is known a priori. Hereafter, it is assumed that $b_{ij} > 0$ without loss of generality. The communication structure among agents and a leader is prescribed by the information network graph $G$ with the associated adjacency matrix $A$, the Laplacian matrix $L$, and the matrix $M$. The control objective is to achieve consensus tracking of the leader-follower type such as

$$x_i \to x_j, \quad (i, j = 1, \ldots, N),$$

$$x_i \to x_0, \quad (i = 1, \ldots, N),$$

where the leader $x_0$ is synthesized by (7) (a generating model of the leader)

$$\hat{x}_0(t) = f(x_0(t), t),$$

and the function $f(x_i(t) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ satisfies the following inequality for a certain constant $\rho > 0$ [9].

$$\|f(x_0, t) - f(x_i, t)\| \leq \rho \|x_0 - x_i\|,$$

$$\forall x_0, x_i \in \mathbb{R}^n.$$

Furthermore, another objective from the viewpoint of adaptive control, is to obtain an adaptive control structure whose stability is not seriously affected by the adaptation scheme, or whose performance is not significantly degraded by the estimation errors of the tuning parameters. For that purpose, in order to attenuate the estimation errors of the tuning parameters. For that purpose, in order to attenuate the estimation errors, the control scheme is to be deduced as a solution of certain $H_\infty$ control problem, where $L_2$-gains from the estimation errors of tuning parameters to the generalized output are prescribed by positive constants $\gamma_i$ (design parameters) such that

$$\int_0^\infty \|\text{generalized output}\|^2 dt \leq \sum_i \gamma_i \int_0^\infty \|\text{estimation errors of tuning parameters}\|^2 dt + \text{constant term},$$

where the constant term depends on the initial condition. The tuning parameters in the adaptation scheme are to be defined in the following subsections ($\hat{P}_i, \hat{\theta}_i, \hat{a}_i, \hat{b}_i$), but the generalized output ($\sqrt{\hat{q} + \hat{s}^T \hat{R} \hat{s}}$) is derived at the final stage as a possible choice of the meaningful cost functional for the corresponding Hamilton-Jacobi-Isaacs equation and its solution in the inverse optimal design procedure.

3.2 Control Law and Error Equation

Associated with the information network graph $G$, the following control law is adopted.

$$u_i(t) = \hat{P}_i(t) - \hat{Q}_i(t) \hat{\theta}_i(t)$$

$$+ v_i(t)$$

$$\equiv \hat{P}_i(t) u_0(t) + v_i(t),$$

where $a_{ij} (1 \leq i \leq N, 0 \leq j \leq N)$ is an entry of the adjacency matrix $A$ and $M$, and $\hat{a}_i (> 0)$ is a tuning parameter. $\hat{\cdot}$ is denoted as a current estimate of $\cdot$, and $P_i$ is defined by

$$P_i = \text{diag}(p_{i1}, \ldots, p_{in}), \quad p_{ij} = 1/b_{ij}.$$

Furthermore, $v_j$ is a stabilizing signal which is to be determined later based on $H_\infty$ control criterion. A tracking error between the leader $x_0$ and the follower $x_i$ is defined by

$$\hat{x}_i(t) \equiv x_i(t) - x_0(t),$$

and the substitution of (10) and (12) into (3) yields

$$\dot{\hat{x}}_i(t) = \Omega \hat{x}_i(t) \theta_i + B_i \hat{u}_i(t)$$

$$= \Omega \hat{x}_i(t) \theta_i - \hat{\theta}_i(t) + U_{i0}(t) B_i \hat{P}(t) - p_i$$

$$+ B_i v_i(t)$$

$$\hat{\theta}_i(t) + \sum_{j=1}^N \hat{\theta}_{ij} \hat{x}_j(t)$$

$$+ \{f(x_i, t) - f(x_0, t),\}$$

$$U_{i0} = \text{diag}(u_{i01}, \ldots, u_{i0n}),$$

$$u_{i0} = [u_{i01}, \ldots, u_{i0n}]^T,$$

$$p_i = [p_{i1}, \ldots, p_{in}]^T.$$
and $n_0 = 0$; otherwise. In the present case, the generating model of the leader $f(x_0(t), t)$ is employed in (10), instead of $f(x_0(t), t)$, since $x_0$ may not be known for all followers.

Remark 2. $f(x_0(t), t)$ can be also utilized in (10), if $a_0 > 0$. Stability analysis of that slightly modified version can be carried out similarly to the present one (10).

3.3 Adaptive $H_{\infty}$ Consensus Control for First-Order Models

A positive definite function $W_0$ of $\dot{x}$ and $(\dot{b} - b)$ is defined by

$$W_0(t) = \frac{1}{2}\dot{x}(t)^T(M \otimes I)\dot{x}(t) + \frac{1}{2}b(t) - b)^T\Gamma^{-1}_1 \left[b(t) - b\right], \quad \Gamma_1 = \Gamma_1^T > 0, \text{ diagonal},$$

$$b = [b_1, \ldots, b_m]^T, \quad b_i = [b_{i1}, \ldots, b_{im}]^T,$$

where $Pr(\cdot)$ are projection operations in which tuning parameters are constrained to bounded regions deduced from upper bounds and lower bounds of each element of $b$ (it means that when elements of $b$ are going out of those bounded regions, the corresponding elements of $b$ are set to 0) [10]. Then, the time derivative of $W_0$ along its trajectory is given as follows:

$$W_0(t) \leq \dot{x}(t)^T(M \otimes I)\Omega(t)(\dot{\theta} - \hat{\theta}(t)) + \dot{x}(t)^T(M \otimes I)U_0(t)B[\hat{p}(t) - p] + \dot{x}(t)^T(M \otimes I)[(M \otimes I)\hat{x}] + \dot{x}(t)^T(M \otimes I)\hat{B}(t)v(t),$$

where the following relation is utilized to deduce (34) (for detail, see Appendix)

$$\frac{d}{dt}\left[\frac{1}{2}b(t) - b\right)^T\Gamma^{-1}_1 \left[b(t) - b\right] \leq \dot{b}(t) - b)^T\Gamma_1\dot{x}(t),$$

and $[(M \otimes I)\hat{x}]$, $\alpha$ and $\hat{\alpha}$ are defined such as

$$[(M \otimes I)\hat{x}] = \text{block diag} \left(\sum_{j=1}^{N} m_j I\hat{x}_j, \ldots, \sum_{j=1}^{N} m_N I\hat{x}_j\right), \quad \alpha = [\alpha_1, \ldots, \hat{\alpha}_N]^T.$$

$\hat{\alpha}_i$ is an estimate of $\alpha_i$ and each $\hat{\alpha}_i$ is to be determined based on the local data in the neighborhood of the follower $i$. $\hat{B}(t)$ is a current estimate of $\dot{B}$, and its elements are constructed from the corresponding elements of $b(t)$ (a current estimate of $b$).

From the evaluation of $W_0$ (34), the next system is introduced virtually.

$$\dot{x} = f_0 + g_1 d_1 + g_1 d_2 + g_1 d_3 + g_1 v, \quad (39)$$

$$v = -\frac{1}{2}R^{-1}(L_{g_1} V_0)^T - \frac{1}{2}R^{-1}B^T(M \otimes I)\hat{x},$$

$$\dot{v} = -\frac{1}{4}\left(B^{-1}\Omega^T \hat{x} + B^{-1}U_0(\hat{B}^T + K_R) + K_R\right)^{-1}B^{-1}[[(M \otimes I)\hat{x}] [(M \otimes I)\hat{x}]^T \hat{x}^T + K_R \right) \dot{x} + \hat{x}.$$
\[
W_0 \leq \tilde{x}^T (M \otimes I) \Omega \dot{x} + U_0 d_2 + \{(M \otimes I) \tilde{x}\} d_1
\]

\[
- q - \frac{1}{4} \gamma_1 \tilde{x}^T (M \otimes I) \left( \frac{\Omega^T \tilde{x}}{\gamma_1} + \frac{U_0 d_2}{\gamma_2} \right)
\]

\[
+ \frac{(M \otimes I) \tilde{x}}{\gamma_1} \frac{\Omega (M \otimes I) \tilde{x}}{\gamma_1} + \frac{[M \otimes I] \tilde{x}}{\gamma_2} \frac{[M \otimes I] \tilde{x}}{\gamma_2}
\]

\[
= - q - v^T R_v
\]

\[
+ \left( v + \frac{1}{2} R_1 \tilde{B}^T (M \otimes I) \tilde{x} \right)^T R \cdot \left[ v + \frac{1}{2} R_1 \tilde{B}^T (M \otimes I) \tilde{x} \right]
\]

\[
+ \gamma_1^2 \|d_1\|^2 - \gamma_1^2 \|d_1\| - \tilde{\gamma}_1^2 \|d_1\| - \tilde{\gamma}_1^2 \|d_1\| - \tilde{\gamma}_1^2 \|d_1\|
\]

\[
+ \gamma_1^2 \|d_2\|^2 - \gamma_2^2 \|d_2\| - \tilde{\gamma}_2^2 \|d_2\| - \tilde{\gamma}_2^2 \|d_2\|
\]

\[
+ \gamma_2^2 \|d_3\|^2 - \gamma_2^2 \|d_3\| - \tilde{\gamma}_2^2 \|d_3\| - \tilde{\gamma}_2^2 \|d_3\|
\]

\[
\leq \frac{1}{2} \sum_{i=1}^3 \tilde{x}^T \Pi_i \tilde{x} + \sum_{i=1}^3 \tilde{\gamma}_i^2 \int_0^t \|d_i\|^2 \, d\tau + W_0(t)
\]

(49)

The first result of the paper is obtained.

**Theorem 1.** The partial adaptive control system (10), (48), (30) is uniformly bounded for arbitrary bounded design parameters \( \hat{\theta}, \hat{P}, \hat{\alpha} > 0 \), and \( v \) is a sub-optimal control input which minimizes the upper bound on the following cost functional \( J(t) \).

\[
J(t) = \sup \left\{ \int_0^t (q + v^T R_v) d\tau + W_0(t) \right\}
\]

(50)

**Proof.** From (49), it follows that

\[
\int_0^t (q + v^T R_v) d\tau + W_0(t) + \frac{1}{2} \sum_{i=1}^3 \tilde{\gamma}_i^2 \int_0^t \|d_i\|^2 d\tau
\]

\[
- \frac{1}{4} \gamma_1 \tilde{x}^T (M \otimes I) \left( \frac{\Omega^T \tilde{x}}{\gamma_1} + \frac{U_0 d_2}{\gamma_2} \right)
\]

\[
+ \frac{(M \otimes I) \tilde{x}}{\gamma_1} \frac{\Omega (M \otimes I) \tilde{x}}{\gamma_1} + \frac{[M \otimes I] \tilde{x}}{\gamma_2} \frac{[M \otimes I] \tilde{x}}{\gamma_2}
\]

\[
= - q - v^T R_v
\]

\[
+ \left( v + \frac{1}{2} R_1 \tilde{B}^T (M \otimes I) \tilde{x} \right)^T R \cdot \left[ v + \frac{1}{2} R_1 \tilde{B}^T (M \otimes I) \tilde{x} \right]
\]

\[
+ \gamma_1^2 \|d_1\|^2 - \gamma_1^2 \|d_1\| - \tilde{\gamma}_1^2 \|d_1\| - \tilde{\gamma}_1^2 \|d_1\|
\]

\[
+ \gamma_1^2 \|d_2\|^2 - \gamma_2^2 \|d_2\| - \tilde{\gamma}_2^2 \|d_2\| - \tilde{\gamma}_2^2 \|d_2\|
\]

\[
+ \gamma_2^2 \|d_3\|^2 - \gamma_2^2 \|d_3\| - \tilde{\gamma}_2^2 \|d_3\| - \tilde{\gamma}_2^2 \|d_3\|
\]

(51)

where the synthesis of \( v \) is considered to derive (53). From those two inequalities (52), (53), uniform boundedness of the control systems is assured for arbitrary bounded design parameters \( \hat{\theta}, \hat{P}, \hat{\alpha} > 0 \), and semi-optimality of \( v \) for the cost functional \( J(t) \) and the relation (51) are deduced.

**Remark 3.** Theorem 1 denotes the properties of the partial adaptive control system (10), (48), (30), where the tuning of \( \hat{\theta}, \hat{P}, \hat{\alpha} \) are not necessarily required. It indicates that the adaptive control structure whose stability is not strictly affected by the adaptation scheme, has been obtained in the proposed design procedure.

**Remark 4.** \( J(t) \) is a cost functional of the differential game, where the players \( d_1 \sim d_3 \) are to maximize \( J(t) \) and the player \( v \) is to minimize \( J(t) \). Its property was clarified in the inequality (52).

**Remark 5.** \( J(t) \) is a fictitious cost functional, since \( d_1 \sim d_3 \) are not actually external disturbances but estimations errors of tuning parameters, and since these do not generally converge to zero asymptotically. Nevertheless, \( v \), which is derived as a solution for that fictitious \( H_{\infty} \) control problem, attain the inequality (51) and stabilize the total systems, and it means that the \( L_2 \)-gains from the disturbances \( d_1 \sim d_3 \) to the generalized output \( \sqrt{q + v^T R_v} \) are prescribed by positive constants \( \gamma_1 \sim \gamma_2 \) for any finite \( t \). Hence, the \( H_{\infty} \) control problem has been solved from the viewpoint of inverse optimality, where the cost functional (or the generalized output) and the control structure are prescribed and constructed simultaneously.

Next, the tuning laws of \( \hat{\theta}, \hat{P}, \hat{\alpha} \) are determined as follows:

\[
\dot{\hat{\theta}}(t) = \Gamma_2 \Omega^T (M \otimes I) \tilde{x}(t),
\]

(54)

\[
\dot{\hat{P}}(t) = - \Gamma_3 U_0 (M \otimes I) \tilde{x}(t),
\]

(55)

\[
\dot{\hat{\alpha}}(t) = \Gamma_4 \sum_{i=1}^N m_i \tilde{x}_i(t),
\]

(56)

\( \Gamma_2 > 0, \Gamma_3 > 0, \Gamma_4 > 0 \),

where \( \Gamma_2 \) and \( \Gamma_3 \) are chosen as diagonal matrices. Although \( \hat{\alpha} \) is an estimate of the mutual \( \alpha \), the tuning of each \( \hat{\alpha} \) is decentralized based on the local data in the neighborhood of the follower \( i \).

A positive definite function \( W \) of \( \tilde{x}, (\dot{\tilde{x}} - \tilde{b}), (\dot{\tilde{x}} - \theta), (\dot{\tilde{x}} - \tilde{p}) \) is defined by

\[
W(t) = \frac{1}{2} \tilde{x}^T (M \otimes I) \tilde{x}(t)
\]

\[
+ \frac{1}{2} \|\dot{\tilde{x}} - \tilde{b}\|^2 \Gamma_1^{-1} \|\dot{\tilde{x}} - \tilde{b}\|
\]

\[
+ \frac{1}{2} \|\dot{\tilde{x}} - \theta\|^2 \Gamma_2^{-1} \|\dot{\tilde{x}} - \theta\|
\]

\[
+ \frac{1}{2} \|\dot{\tilde{x}} - \tilde{p}\|^2 \Gamma_3^{-1} \|\dot{\tilde{x}} - \tilde{p}\|
\]

\[
+ \frac{1}{2} \sum_{i=1}^N \|\hat{\alpha}_i(t) - \alpha\|^2 / \Gamma_4.
\]

(57)

Then, from the time derivative of \( W \) along its trajectory,

\[
\dot{W}(t) \leq - \frac{1}{2} \tilde{x}^T (M \otimes I) \dot{\tilde{x}}(t)
\]

\[
\leq 0,
\]

(58)

the following result is obtained.
Theorem 2. The total adaptive control system (10), (48), (30), (54), (55), (56) is uniformly bounded, and the consensus tracking is achieved asymptotically such that
\[
\lim_{t \to \infty} \tilde{x}(t) = 0.
\] (59)

Proof. Since all signals are assured to be bounded via Theorem 1, it follows that \( \dot{x} \) is also bounded, and thus Barbalat’s lemma can be applied to the properties \( \dot{x} \in L^2, \ddot{x} \in L^\infty \), and (59) is deduced. \( \square \)

Remark 6. Theorem 2 indicates that the asymptotic consensus tracking is achieved under the prescribed condition of the network graph. On the contrary, in our previous work [7], approximate tracking was assured under the same conditions of the graph, and asymptotic stability was achieved under the highly restricted conditions such that \( \dot{x}_0 \equiv 0 \) or the information of \( x_0 \) is available for all followers \( i \sim N \ (a_{0i} \sim a_{00} \neq 0) \).

Remark 7. Since \( B \) and \( \bar{B} \) are diagonal matrices, and \( \Omega \) and \( U_0 \) are block-diagonal ones, \( v \) in (48) can be implemented in a distributed fashion. Furthermore, tuning gain matrices \( \Gamma_1, \Gamma_2, \Gamma_3 \) are all diagonal in order to tune \( b, \bar{b}, \bar{b} \) in a distributed manner. Hence, the total adaptation schemes are constructed in a decentralized fashion.

4. Adaptive \( H_{\infty} \) Consensus Control for Second-Order Model

4.1 Problem Statement

Next, a multi-agent system composed of the second-order regression models is considered.
\[
\dot{x}_i(t) = \Omega_i(t)\dot{\theta}_i + B_iu_i(t), \quad (i = 1, \ldots, N),
\] (60)
where \( x_i, u_i, \theta_i \) are defined similarly to the previous case, and the form of \( B_i \) is the same as the former one. \( \Omega_i \) is a regressor matrix composed of \( x_i \) and \( \dot{x}_i \), and is bounded for bounded \( x_i \) and \( \dot{x}_i \). The communication structure among agents is prescribed by the information network graph \( \mathcal{G} \) with the associated adjacency matrix \( A \), the Laplacian matrix \( L \), and the matrix \( M \). The control objective is to achieve consensus tracking of the leader-follower type with velocity tracking such as
\[
x_i(t) \to x_j(t), \quad \dot{x}_i(t) \to \dot{x}_j(t), \quad (i, j = 1, \ldots, N),
\] (61)
\[
x_i(t) \to \dot{x}_0(t), \quad \dot{x}_i(t) \to \dot{x}_0(t), \quad (i = 1, \ldots, N),
\] (62)
where the leader \( x_0 \) is synthesized by (63) (a generating model of a leader)
\[
\ddot{x}_0(t) = f(x_0(t), \dot{x}_0(t), t),
\] (63)
and the function \( f(x, \dot{x}, t) : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \) satisfies the following inequality for certain constants \( \rho_1(>0) \) and \( \rho_2(>0) \) [9].
\[
\|f(x_0, \dot{x}_0, t) - f(x, \dot{x}, t)\| \leq \rho_1\|x_0 - x\| + \rho_2\|\dot{x}_0 - \dot{x}\|, \quad \forall x_0, x, \dot{x}_0, \dot{x} \in \mathbb{R}^n.
\] (64)

Similarly to the first order case, another control objective is to obtain an adaptive control structure whose performance is not significantly degraded by the estimation errors of the tuning parameters, and the cost functional or the generalized output of the \( H_{\infty} \) control problem, is not specified in advance, and is to be derived in the final stage of the inverse optimal controller design.

4.2 Control Law and Error Equation

Associated with the information network graph, the following control law is adopted.
\[
u_i(t) = \dot{\hat{\theta}}_i(t) - \Omega_i(t)\dot{\theta}_i(t) - \sum_{j=0}^{N-1} a_{ij}(x_i(t) - x_j(t)) - \alpha \sum_{j=0}^{N-1} a_{ij}(\dot{x}_i(t) - \dot{x}_j(t)) + f(x_i(t), \dot{x}_i(t), t)
\]
\[
- \tilde{k}_i(t) \sum_{j=1}^{N} m_{ij} I \tilde{x}_j(t) + v_i(t)
\]
\[
= \ddot{\hat{\theta}}_i(t) - \tilde{k}_i(t) \sum_{j=1}^{N} m_{ij} I \tilde{x}_j(t) + v_i(t), \quad (65)
\]
where the definitions of \( a_{ij} \) (1 \( \leq i \leq N \), 0 \( \leq j \leq N \)), \( \rho_\alpha \), \( \nu \), are the same as the previous case. \( \alpha \) is a positive constant, \( \tilde{k}_i(t) \) is a positive tuning parameter, and \( \tilde{x}_i(t) \) is the \( i \)-th element of the vector signal \( \tilde{s}(t) \) which is to be determined later. The consensus tracking error \( \tilde{x}_i \) is denoted by (12), and the substitution of (65) and (12) into (66) yields
\[
\ddot{\tilde{x}}_i(t) = \Omega_i(t)\theta_i + B_iu_i(t) - \tilde{x}_0(t)
\]
\[
= \Omega_i(t)(\dot{\theta}_i - \tilde{\theta}_i(t)) + U_0(t)B_i[\tilde{\theta}_i(t) - p_i] + Bu_i(t)
\]
\[
+ \left\{ -(l_{ii} + a_{ii})\tilde{x}_i(t) - \sum_{j=1}^{N} l_{ij}\tilde{x}_j(t) \right\}
\]
\[
+ \left\{ -(l_{ii} + a_{ii})\tilde{x}_i(t) - \sum_{j=1}^{N} l_{ij}\tilde{x}_j(t) \right\}
\]
\[
\to \tilde{k}_i(t) \sum_{j=1}^{N} m_{ij} I \tilde{x}_j(t) + [f(x_i(t), \dot{x}_i(t), t) - f(x_0(t), \dot{x}_0(t), t)],
\] (66)
where the definitions of \( U_0, u_0, p \) are defined similarly to the former case. Then, the total representation of the multi-agent system is given as follows:
\[
\ddot{x}(t) = \Omega(t)\theta - \tilde{\theta}(t) + U_0(t)B[p - \tilde{p}(t)]
\]
\[
- (M \otimes I) \hat{x}(t) - \alpha (M \otimes I) \hat{x}(t)
\]
\[
- (\tilde{k}(t) \otimes I) (M \otimes I) \hat{x}(t)
\]
\[
+ \Delta F(x, \dot{x}, x_0, \dot{x}_0, t) + \eta(t),
\] (67)
where the definitions of \( \tilde{x}, \Omega, \theta, U_0, B, p, \nu, \otimes, \otimes \) are the same as the first-order case. On the contrary, \( \tilde{k} \) and \( \Delta F(x, \dot{x}, x_0, \dot{x}_0, t) \) are defined such as
\[
\tilde{k} = \text{diag} (\tilde{k}_1, \ldots, \tilde{k}_N),
\] (68)
\[
\Delta F(x, \dot{x}, x_0, \dot{x}_0, t) = [(f(x_j, \dot{x}_j, t) - f(x_0, \dot{x}_0, t))^T, \ldots, (f(x_N, \dot{x}_N, t) - f(x_0, \dot{x}_0, t))^T]^T.
\] (69)
4.3 Adaptive $H_\infty$ Consensus Control for Second-Order Models

For the matrix $M$ and positive constants $\alpha, \gamma$, the matrices $P$ and $Q$ are defined by

$$ P = \begin{bmatrix} \frac{1}{2}M^T & \frac{1}{2}M \\ \frac{1}{2}M & \frac{1}{2}M \end{bmatrix}, \quad Q = \begin{bmatrix} \frac{1}{2}M^2 & \frac{1}{2}M^2 \\ \frac{1}{2}M^2 & \frac{1}{2}M^2 \end{bmatrix}. \quad (70) $$

$P$ and $Q$ are both positive definite, if $\gamma$ satisfies the next condition [3].

$$ 0 < \gamma < \min \left\{ \sqrt{\lambda_{\text{min}}(M)}, \frac{4\alpha \lambda_{\text{min}}(M)}{4 + \alpha^2 \lambda_{\text{min}}(M)} \right\}. \quad (71) $$

Hereafter, it is assumed that $\gamma$ satisfies (71), and by utilizing $\gamma$, $\tilde{s}(t)$ is defined such that

$$ \tilde{s}(t) \equiv \tilde{s}(t) + \gamma \tilde{\tilde{s}}(t). \quad (72) $$

Adopting the positive definite $P$, a positive definite function $W_0$ of $\tilde{z}$, $(\tilde{b} - b)$ is defined by

$$ W_0(t) = \tilde{z}(t)^T P (M \otimes I) \tilde{z}(t) + \frac{1}{2} \tilde{b}(t) - b)^T \Gamma_1^{-1} \tilde{b}(t) - b), \quad (73) $$

$$ \tilde{z} = [\tilde{s}, \tilde{\tilde{s}}]^T. \quad (74) $$

where $\Gamma_1 = \Gamma_1^T > 0$ (diagonal), and $b$ is defined similarly to the first-order case. The tuning law of $\tilde{b}$ is chosen such as

$$ \dot{\tilde{b}}(t) = \text{Pr} [\Gamma_1 V(t)^T (M \otimes I) \tilde{z}(t)], \quad (75) $$

where the definitions of $V$, $\text{Pr}(\cdot)$ are the same as the previous ones. Then, the time derivative of $W_0$ along its trajectory is evaluated as follows:

$$ \dot{W}_0(t) \leq \tilde{s}(t)^T (M \otimes I) \Omega(t) (\tilde{s} - \tilde{b}(t)) + \tilde{\tilde{s}}(t)^T (M \otimes I) \tilde{b}(t) (\tilde{b} - b) - p) + \tilde{s}(t)^T (M \otimes I) \Omega(t) \tilde{b}(t) (\tilde{b} - b) \quad (76) $$

where

$$ [(M \otimes I) \tilde{s}] = \text{block diag} \left( \sum_{j=1}^{N} m_1^j I_3, \ldots, \sum_{j=1}^{N} m_N^j I_3 \right). \quad (77) $$

$$ k = \max \left\{ \frac{\lambda_{\text{max}}(Q)}{\lambda_{\text{min}}(Q)}, \frac{\lambda_2}{\lambda_{\text{min}}(Q)} \right\}, \quad (78) $$

$$ \tilde{k} = [\tilde{k}_1, \ldots, \tilde{k}_N]^T. \quad (79) $$

The next result is obtained.

**Theorem 3.** The partial adaptive control system (65), (89), (75) is uniformly bounded for arbitrary bounded design parameters $\theta, \tilde{b}, \tilde{k}(0)$, and $v$ is a sub-optimal control input which minimizes the upper bound on the following cost functional $J$.

$$ J(t) = \sup_{d_l, d_2, \tilde{d}_l, \tilde{\tilde{d}}_l} \left[ \int_0^t \left( q + v^T R v \right) dt + W_0(t) \right] - \sum_{l=1}^3 \lambda_l^2 \int_0^t \left\| d_l \right\|^2 dt. \quad (90) $$

Also the next inequality holds.

$$ \int_0^t \left( q + v^T R v \right) dt + W_0(t) \leq \sum_{l=1}^3 \lambda_l^2 \int_0^t \left\| d_l \right\|^2 dt + W_0(0). \quad (91) $$

Next, the tuning laws of $\theta, \tilde{b}, \tilde{k}$ are determined as follows:

$$ \dot{\theta}(t) = \Gamma_3 \Omega(t)^T (M \otimes I) \tilde{s}(t), \quad (92) $$

$$ \dot{\tilde{b}}(t) = -\Gamma_3 \tilde{b}(t) (M \otimes I) \tilde{s}(t), \quad (93) $$

$$ \dot{\tilde{k}}(t) = \Gamma_4 \left[ \sum_{j=1}^{N} m_1^j I_3 \tilde{s}(t) \right], \quad (94) $$

$$ (\Gamma_2 = \Gamma_1^T > 0, \Gamma_3 = \Gamma_2^T > 0, \Gamma_4 > 0). \quad (95) $$
where \( \Gamma_2 \) and \( \Gamma_3 \) are chosen as diagonal matrices. Although \( \hat{k}_i \) is an estimate of the mutual \( k_i \), the tuning of each \( \hat{k}_i \) is decentralized based on the local data in the neighborhood of the follower \( i \).

Then, similarly to the previous case, for positive definite function \( W(t) \) of \( z(\hat{b} - b), \hat{b} - \theta, (\hat{p} - p), (\hat{k}_i - k) \), defined such that

\[
W(t) = z(t)^T \left( P \otimes I \right) z(t) + \frac{1}{2} \left[ \hat{b}(t) - b \right]^T \Gamma_1^{-1} \left[ \hat{b}(t) - b \right] + \frac{1}{2} \left[ \hat{b}(t) - \theta \right]^T \Gamma_2^{-1} \left[ \hat{b}(t) - \theta \right] + \frac{1}{2} \left[ \hat{p}(t) - p \right]^T \Gamma_3^{-1} \left[ \hat{p}(t) - p \right] + \frac{1}{2} \sum_{i=1}^{N} \left[ \hat{k}_i(t) - k_i \right]^2 / \Gamma_4_i,
\]

the time derivative of \( W \) is evaluated as follows:

\[
\dot{W}(t) \leq -\frac{1}{2} z(t)^T (M \otimes I) \dot{\hat{b}}(t) K_R \dot{R}(t) (M \otimes I) z(t) \leq 0,
\]

and the next theorem is obtained.

**Theorem 4.** The total adaptive control system (65), (89), (75), (92), (93), (94) is uniformly bounded, and the consensus tracking is achieved asymptotically such that

\[
\lim_{t \to \infty} \bar{x}(t) = \lim_{t \to \infty} \hat{x}(t) = 0.
\]

**Remark 8.** Theorem 3 denotes the properties of the partial adaptive control system (65), (89), (75), where the tunings of \( \hat{b}_0, \hat{P}, \hat{k}_i \) are not necessarily required. Theorem 4 indicates that the asymptotic consensus tracking is achieved under the prescribed condition of the network graph. This is an extension of our previous work \([7]\), where approximate tracking was assured under the same conditions of the graph, and asymptotic stability was achieved under the highly restricted conditions such that \( \bar{x}_0 \equiv 0 \) or the information of \( x_0 \) is available for all followers \( 1 \sim N \) (that is, \( a_{10} \sim a_{N0} \neq 0 \)).

### 4.4 Adaptive \( H_\infty \) Consensus Control for Second-Order System with Velocity Tracking

As a specified version of the control scheme for the second-order case, \( \gamma = 0 \) can be also adopted. where \( \bar{x}(t) \) is equal to \( \hat{x}(t) \). Then, although \( P \) remains positive definite, \( Q \) becomes positive semidefinite

\[
P = \begin{bmatrix}
\frac{1}{2}M^2 & 0 \\
0 & \frac{1}{2}M
\end{bmatrix}, \quad Q = \begin{bmatrix}
0 & 0 \\
0 & \alpha M^2
\end{bmatrix},
\]

and for the leader \( x_0 \) synthesized by

\[
\bar{x}_0(t) = f(x_0(t), t),
\]

\[
\|f(x_0(t), t) - f(\bar{x}_0(t), t)\| \leq \rho \|x_0 - \bar{x}_0\|,
\]

the consensus tracking of velocity is achieved.

\[
\bar{x}_i \to \bar{x}_j, \quad (i, j = 1, \ldots, N),
\]

\[
\bar{x}_i \to \bar{x}_0, \quad (i = 1, \ldots, N).
\]

By replacing \( \bar{x}(t) \) by \( \hat{x}(t) \), and by utilizing newly defined \( P \) and \( Q \) (98), the following two theorems are obtained.

**Theorem 5.** In the partial adaptive control system (65), (89), (75) together with \( \gamma = 0 \) (that is, \( \bar{x} \) is replaced by \( \hat{x} \)), the velocity tracking error \( \hat{x} \) is uniformly bounded for arbitrary bounded design parameters \( \hat{b}_0, \hat{P}, \hat{k}_i \), and \( v \) is a sub-optimal control input which minimizes the upper bound on the cost functional \( J(50) \). Also the inequality (51) holds.

**Theorem 6.** In the total adaptive control system (65), (89), (75), (92), (93), (94) together with \( \gamma = 0 \) (\( \bar{x} \) is replaced by \( \hat{x} \)), the velocity tracking error \( \hat{x} \) and the tuning parameters \( \hat{b}_0, \hat{P}, \hat{B}_i, \hat{k}_i \) are uniformly bounded, and the consensus tracking of velocity is achieved asymptotically such that

\[
\lim_{t \to \infty} \hat{x}(t) = 0.
\]

### 5. Numerical Example

In order to verify the effectiveness of the proposed methodology, numerical experimental studies for the second-order regression models are performed.

A multi-agent system composed of the second-order regression models is considered such as

\[
\bar{x}_i(t) = \theta_i x_i(t) + u_i(t), \quad (i = 1, 2, 3),
\]

\[
x_i(0) = 0, \quad x_0(t) = 0, \quad x_0(0) = 0,
\]

where \( x_i \in \mathbb{R}, u_i \in \mathbb{R}, \) and \( \theta_i \in \mathbb{R} \) is an unknown parameter. Associated with the information network (Fig. 1), the adjacency matrix \( A = [a_{ij}] \) and \( a_{00} \) are chosen such that

\[
A = \begin{bmatrix}
0 & 1 & 0 \\
1 & 1 & 0 \\
0 & 1 & 0
\end{bmatrix}, \quad a_{10} = 1, \quad a_{20} = a_{30} = 0.
\]

The control objective is to achieve consensus tracking

\[
x_i \to x_j, \quad \bar{x}_i \to \bar{x}_j,
\]

\[
x_i \to x_0, \quad \bar{x}_i \to \bar{x}_0, \quad (i, j = 1, 2, 3),
\]

where the virtual leader \( x_0 \) is determined such as

\[
x_0(t) = \sin t,
\]

or equivalently

\[
\bar{x}_0 = f(x_0(t), \bar{x}_0(t), t) = -x_0(t),
\]

\[
(x_0(0) = 0, \quad \bar{x}_0(0) = 1).
\]

The design parameters are chosen as follows:

\[
\Gamma = 10I, \quad K_R = 20I, \quad \alpha = 1, \quad \gamma = 0.4, \quad \gamma_i = 0.01.
\]

As system parameters, both time-invariant (TI) and time-varying (TV) (or piecewise time-invariant (Fig. 2)) cases are considered such that

\[
\theta_1 = 1, \quad \theta_2 = 2, \quad \theta_3 = 3, \quad (TI),
\]

\[
\theta_1 = f_S(t), \quad \theta_2 = 2 f_S(t), \quad \theta_3 = 3 f_S(t), \quad (TV),
\]

\[
f_S(t) = \begin{cases}
1 & 0 \leq t < 2.5, \\
5 & 2.5 \leq t < 7.5, \\
7.5 & 7.5 \leq t < 10, \\
0 & 2.5 \leq t \leq 5, \quad 7.5 \leq t \leq 10, \quad \ldots.
\end{cases}
\]

The simulation results of the proposed design scheme (Theorem 3 and Theorem 4) are shown in Fig. 3 (time-invariant case) and Fig. 4 (time-varying case). For comparison, the adaptive control systems which do not contain the stabilizing signal \( v(t) \), are also shown for both cases; Fig. 5 (time-invariant case) and Fig. 6 (time-varying case).

From those results, it is seen that the proposed \( H_\infty \) adaptive control strategies which contain \( v(t) \), achieve better tracking...
Fig. 1 Network Graph.

Fig. 2 $f_\theta(t)$: TV (piecewise time-invariant) function.

Fig. 3 Simulation result for time-invariant models.

Fig. 4 Simulation result for time-varying models.

Fig. 5 Simulation result for time-invariant models without $\nu$.

Fig. 6 Simulation result for time-varying models without $\nu$.

and convergence properties together with robustness to abrupt changes of the system parameters. Those are owing to the disturbance attenuation features to estimation errors of tuning parameters in the proposed $H_\infty$ control scheme.

**Remark 9.** Compared with the numerical example in the previous work [7], there are no apparent differences in the tracking and convergence properties between those two control schemes ([7] and the present strategy). However, the present methodology provided clear assurance of asymptotic convergence of consensus tracking.

### 6. Concluding Remarks

Design methods of asymptotically stable adaptive $H_\infty$ consensus control of multi-agent systems composed of the first-order and the second-order regression models have been presented as an extension of our previous work [7]. In the inverse optimal controller design, the cost functional or the generalized output and the control scheme are prescribed and constructed simultaneously, and then, the proposed control schemes are derived as solutions of certain $H_\infty$ control problems, where estimation errors of tuning parameters are regarded as external disturbances to the process. The resulting control systems are shown to be robust to uncertain system parameters, and the desirable consensus tracking is achieved asymptotically via adaptation schemes and $L_2$-gain design parameters together with an introduction of a generating model of a leader. Thus, it can be said that the $H_\infty$ control problem has been solved among the
proposed controller design. Effectiveness of the proposed design schemes were also confirmed by the simulation studies, and those results and Theorem 1, 3, and 5 (Remark 3) indicated that the adaptive control structure whose stability is not affected by the adaptation laws, or whose performance is not seriously degraded by the estimation errors of the tuning parameters, has been obtained in the manuscript.

The present paper is an extended version of [8], and includes detailed proofs of theorems and provides several numerical examples.

References


Appendix A Derivation of the Inequality (35) and Treatment of Continuity of $\dot{\hat{b}}$

For the derivation of the inequality (35), it is sufficient to consider the following three cases.

**Case 1.** When an element of $\hat{b}$ is becoming greater than its upper bound, it follows that the corresponding element of $(\hat{b} - b)$ is positive and the corresponding argument of $Pr$ is positive. For such case, the corresponding element of $\dot{\hat{b}}$ is set to 0, and the inequality (35) holds ($0 \leq$ positive term).

**Case 2.** When an element of $\hat{b}$ is becoming less than its lower bound, it follows that the corresponding element of $(\hat{b} - b)$ is negative and the corresponding argument of $Pr$ is negative. For such case, the corresponding element of $\dot{\hat{b}}$ is set to 0, and the inequality (35) holds ($0 \leq$ positive term).

**Case 3.** Otherwise, when the corresponding element of $\hat{b}$ is not set to zero (or the corresponding element of $\hat{b} \in [its lower bound, its upper bound]$), (35) holds with equality condition.

For careful treatment of continuity of $\dot{\hat{b}}$, the projection-type adaptive laws where continuity of the time derivatives of tuning parameters are assured, can be also employed in the proposed methodology. However, the detailed discussion to such case is omitted, since it is not the main issue of the present topic.

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