Distributed Source Identification by Two-Hop Consensus Dynamics with Uniform Time-Varying Communication Time-Delays

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Abstract: This paper considers the average consensus problem by the 2-hop communication with time-varying time-delays. Each agent updates its state by a discrete-time consensus dynamics with fixed and undirected communication networks to achieve the average consensus. We also consider an application of the proposed consensus dynamics to an inverse problem of diffusion phenomena in order to find a source of a diffusion process from time-series data measured by agents. The inverse problem of diffusion phenomena can be formulated as a linear least squares problem by the difference approximation. We show that agents can estimate the initial distribution of concentration by solving the least squares problem with the proposed consensus dynamics.

Key Words: multi-agent systems, consensus problem, inverse problem, diffusion equation.

1. Introduction

Recently, the consensus problem of multi-agent systems has attracted much attention for large-scale distributed computation [1]–[3]. In such large-scale networks, communication time-delays cannot be ignored and there are a number of analyses on the effect of time-delays such as time-varying time-delays in switching communication graphs [4], random communication delays in sampled-data multi-agent systems [5], average consensus with constant but heterogeneous time-delays [6], and discrete-time average consensus dynamics with bounded time-delays under fixed and directed networks [7]. The authors of [8] considered the 2-hop consensus dynamics with constant time-delays. The advantage of the multi-hop communication is that agents can convey information by following several communication links even if they cannot make a direct link. Hence the multi-hop communication can accelerate convergence speed without adding extra communication links. The authors in [9] and [10] considered 2-hop consensus dynamics with communication time-delays which are different for different hop levels.

This paper considers the average consensus problem by the 2-hop communication with time-varying time-delays. Each agent updates its state by a discrete-time consensus dynamics with the information of the 2-hop neighbors. We show a sufficient condition to achieve the average consensus for fixed and undirected communication graphs by a gradient-like optimization algorithm [11]. Most of the existing consensus control based on the multi-hop communication has been considered for constant time-delays [8]–[10]. On the other hand, the proposed method considers the case when communication time-delays are time-varying.

In this paper, we also consider an application of the proposed consensus dynamics to a source identification problem of diffusion phenomena [12]. Demetriou considered estimation of concentration of gas dispersion over a 2-dimensional (2-D) field by a modified Luenberger-type observer [13]. Burns and Rautenberg proposed an infinite-dimensional optimal filter for distributed parameter systems to compute optimal placements and trajectories of stationary and mobile sensors [14]. Zou et al. considered a source localization problem with a group of mobile robots by particle swarm optimization [15]. For a distributed source identification problem, Cao et al. proposed a distributed sequential convex combination method with the advection-diffusion model [16]. Murray-Bruce and Dragotti considered a consensus-based distributed source identification problem by the quantized gossip algorithm [17].

In this paper, we propose a distributed source identification problem of a diffusion process by solving an inverse problem of the diffusion equation. For example, estimating a hot zone contaminated by chemical or radioactive substances from the information of concentration observed by sensor agents can be formulated as an inverse problem of the diffusion equation [18],[19]. We consider a multi-agent system where each agent has a discretized gradient of concentration as its state. The inverse problem of the diffusion equation can be formulated as a linear least squares problem by the difference approximation. Based on the method of [20], we extend the preliminary result for the 1-D diffusion process [21] to the source identification problem on the 2-D planar region. We show that the multi-agent system can estimate the initial distribution of the 2-D diffusion phenomena by the proposed 2-hop consensus dynamics with communication time-delays. In contrast to the existing consensus-based source identification methods by [16],[17], and [21], the proposed method explicitly considers the effects of the multi-hop communication and communication time-delays which are unavoidable for large-scale sensor networks.

The rest of this paper is organized as follows: Section 2 presents the 2-hop average consensus dynamics with time-varying time-delays. Section 3 summarizes the difference approximation of the 2-D diffusion equation. Section 4 considers a distributed inverse problem of the 2-D diffusion phenomena by the proposed consensus dynamics. In Section 5, we present...
2. Two-Hop Consensus Dynamics with Time-Delays

In this section, we consider the average consensus dynamics with time-varying communication time-delays.

An undirected graph $G = (V,E)$ consists of a finite and nonempty node set $V = \{1, 2, \ldots, n\}$ and an edge set $E \subseteq V \times V$. Each node $i$ in graph $G$ represents each agent $i$ and each (undirected) edge $(i,j) \in E$ indicates that agents $i$ and $j$ exchange their data with each other. In this paper, we make the following assumption for local information exchanges:

**Assumption 1** The undirected graph $G$ is static and connected.

We consider a multi-agent system where each agent $i \in V$ has the state $\xi_i(k) \in \mathbb{R}$ at discrete time $k \in \mathbb{N}$. Suppose that $N_i = \{j \in V \mid (j,i) \in E, j \neq i\}$ is the neighbor set of agent $i$. In large-scale sensor networks, communication time-delays cannot be ignored. Let $\theta(k)$ be the maximal allowable time-delay for each communication link at time $k$. We consider the following discrete-time consensus dynamics with communication time-delays:

$$\xi_i(k+1) = \xi_i(k) + \sum_{j=1}^{n} p_{ij}(\xi_j(k-T(k)) - \xi_i(k-T(k))) + \sum_{j=1}^{n} p_{ji}(\xi_j(k) - \xi_i(k)), \quad (1)$$

where $\xi_j(k) = \sum_{\ell=1}^{\infty} p_{ij}(\xi_i(\ell-T(k)) - \xi_j(k-T(k)))$, $T(k) = \theta(k-1) + \theta(k)$ is the time-delay at time $k$, and $p_{ij}$ is the weight such that $p_{ij} > 0$ if $\ell \in N_i$ and $p_{ij} = 0$ otherwise.

Note that the auxiliary variable $\xi_j(k)$ shows the sum of the differences of the states with the 1-hop neighbor agents. On the other hand, the term $\sum_{j=1}^{n} p_{ij}(\xi_j(k) - \xi_i(k))$ in (1) shows the sum of the differences of the auxiliary variables between agent $i$ and its neighbors. It shows that the term $\sum_{j=1}^{n} p_{ij}(\xi_j(k) - \xi_i(k))$ depends on the states of the 2-hop neighbors which are connected to agent $i$ by following two communication links. Therefore the proposed consensus dynamics (1) is the dynamics with the 2-hop communication and the time-varying time-delay $T(k)$.

**Remark 1** In the proposed consensus dynamics (2), the time delay at time $k$ is defined as $\theta(k-1) + \theta(k)$. This assumption might be conservative for the analysis of the average consensus. For the time-invariant time-delays, the authors of [9],[10] have considered multi-hop consensus dynamics with different time-delays for different hop levels. The extension of the proposed two-hop consensus dynamics to those with hop-level time-varying time-delays will be our subsequent work.

**Remark 2** The proposed dynamics (1) can be extended to the more general case with $m$-hop communication ($m \geq 2$) and the analysis for the 2-hop case is also valid for the $m$-hop case. Therefore, for simplicity of argument, we focus on the 2-hop communication dynamics (1) in this paper.

For simplicity of notation, we here introduce the parameter $\tau(k)$ such that $\tau(k) = k - T(k)$ to represent the time-delay at time $k$. In this paper, we make the following assumption on the time-delay [11].

**Assumption 2** There exists a positive integer $\eta$ such that, for each $k \in \mathbb{N}$, there holds $\max(0, k - \eta + 1) \leq \tau(k) \leq k$.

Assumption 2 implies that the communication time-delays are bounded. Then the dynamics (1) can be rewritten as follows:

$$\xi_i(k+1) = \xi_i(k) + \xi_i(k) + \sum_{j=1}^{n} p_{ij}(\xi_j(k) - \xi_i(k)), \quad (2)$$

where $\xi_j(k) = \sum_{\ell=1}^{\infty} p_{ij}(\xi_i(\ell-T(k)) - \xi_j(k-T(k)))$. We consider the vectors $\xi_i(k) \in \mathbb{R}^n$ and $\xi_j(k) \in \mathbb{R}^n$ which are obtained by stacking the elements $\xi_i(k)$ and $\xi_j(k)$ of all $n$ agents, that is, $\xi(k) = [\xi_1(k) \xi_2(k) \cdots \xi_n(k)]^T$ and $\xi(k) = [\xi_1(k) \xi_2(k) \cdots \xi_n(k)]^T$. From the definition of $\xi_i(k)$, we have $\xi(k) = -L\xi(k)$. Thus we have the vector form of the dynamics (2) as follows:

$$\xi(k+1) = \xi(k) + \xi(k) + L\xi(k)$$

$$= \xi(k) - L\xi(k) + L(-L\xi(k))$$

$$= \xi(k) - (L + L^2)\xi(k). \quad (3)$$

In this paper, we define the average consensus as follows:

**Definition 1** A group of agents is said to achieve the average consensus if (4) holds for any initial state.

$$\lim_{k \to \infty} \xi_i(k) = \frac{1}{n} \sum_{i=1}^{n} \xi_i(0), \quad \forall i \in V. \quad (4)$$

The following lemma shows that the sum of the states is constant at all times.

**Lemma 1** Under Assumption 1, there holds

$$\sum_{i=1}^{n} \xi_i(k) = \sum_{i=1}^{n} \xi_i(0), \quad \forall k \in \mathbb{N}. \quad (5)$$

**Proof** From Assumption 1, the sum of the differences of the auxiliary variables is given by

$$\sum_{i=1}^{n} \sum_{j=1}^{n} p_{ij}(\xi_j(\tau(k)) - \xi_i(\tau(k)))$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} p_{ij}(\xi_j(\tau(k)) - \xi_i(\tau(k)))$$

$$= \sum_{j=1}^{n} \sum_{i=1}^{n} p_{ij}(\xi_j(\tau(k)) - \xi_j(\tau(k)))$$

$$= - \sum_{j=1}^{n} \sum_{i=1}^{n} p_{ij}(\xi_j(\tau(k)) - \xi_j(\tau(k))). \quad (6)$$

The first equality of (6) comes from Assumption 1 ($p_{ij} = p_{ji}$) and the second equality is obtained by exchanging subscripts $i$ and $j$. From (6), we have $\sum_{i=1}^{n} \xi_i(k) = 0$. Similarly, we also have $\sum_{i=1}^{n} \sum_{j=1}^{n} p_{ij}(\xi_j(k) - \xi_j(k)) = 0$. Thus we have $\sum_{i=1}^{n} \xi_i(k+1) = \sum_{i=1}^{n} \xi_i(k)$ for all $k \in \mathbb{N}$. \hfill $\square$

Next, we consider a sufficient condition to achieve the average consensus by (1). Let $F : \mathbb{R}^n \to \mathbb{R}$ be a cost function such that

$$F(\xi) = \frac{1}{2} \xi^\top (L + L^2)\xi, \quad (7)$$
where $\xi \in \mathbb{R}^n$ and $L = \{e_{ij}\} \in \mathbb{R}^{n \times n}$ is the graph Laplacian of the graph $G$, that is, $e_{ij} = -p_{ij}$ for $i \neq j$ and $e_{ii} = \sum_{j \neq i} p_{ij}$.

Since $F$ is continuously differentiable and $L + L^2$ is symmetric, we have

$$\nabla F(\xi) = (L + L^2)\xi.$$  \hfill (8)

Now we define a gradient of the cost function $s_i$ as follows:

$$s_i(k) = -\nabla_i F(\xi(\tau(k))),$$  \hfill (9)

where $\nabla_i F(\xi)$ is the $i$-th element of $\nabla F(\xi)$. From (3) and (9), the consensus dynamics (2) is represented by

$$\xi(k + 1) = \xi(k) + s_i(k).$$  \hfill (10)

Bertsekas and Tsitsiklis have shown that the discrete-time asynchronous difference equation of the form (10) results in $\lim_{k \to \infty} F(\xi(k)) = 0$ (Proposition 5.1 in [11]). Note that, in the proposed method, $\nabla F(\xi(k)) = (L + L^2)\xi(k)$ is 0 implies the average consensus since the graph $G$ is undirected and connected. Based on this observation, we have the following theorem for the average consensus.

**Theorem 1** If (11) holds with Assumptions 1 and 2, the average consensus is achieved by the consensus dynamics (1):

$$0 < p_{ij} < \frac{\sqrt{S + 1 + \eta \eta} - \sqrt{S + 1 + \eta \eta}}{4d_{\max} \sqrt{1 + \eta \eta}}, \quad \forall i \in V, j \in N_i,$$  \hfill (11)

where $d_{\max} = \max_{\xi \in V} |\xi|_1$ and $|\cdot|$ is the cardinality of a set.

**Proof** The matrix $L + L^2$ is positive semidefinite because $L$ is real symmetric and all its eigenvalues are non-negative from the Gershgorin disc theorem [22]. Thus $F(\xi) \geq 0$ for each $\xi \in \mathbb{R}^n$.

From (8), for all $\xi, \zeta \in \mathbb{R}^n$, we have

$$\nabla F(\xi) - \nabla F(\zeta) = (L + L^2)(\xi - \zeta).$$

Thus we have

$$\|\nabla F(\xi) - \nabla F(\zeta)\| \leq (\|L\|_2 + \|L^2\|_2)\|\xi - \zeta\|_2,$$

where $\|L\|_2$ is the induced 2-norm of $L$. Since $L$ is a symmetric matrix, we have $\|L\|_2 = \rho(L)$, where $\rho(L)$ is the spectral radius of $L$. From the Gershgorin disc theorem, we obtain

$$\rho(L) \leq 2 \left( \max_{i \in V} \sum_{j \neq i} p_{ij} \right) \leq 2p_{\max}d_{\max},$$

where $p_{\max} = \max_{i \in V} p_{ij}$. This implies that

$$\|\nabla F(\xi) - \nabla F(\zeta)\| \leq K_1\|\xi - \zeta\|_2, \quad \forall \xi, \zeta \in \mathbb{R}^n,$$

where $K_1 = 2p_{\max}d_{\max}(1 + 2p_{\max}d_{\max})$.

Here, we consider the following function $\sigma: \mathbb{R} \rightarrow \mathbb{R}$:

$$\sigma(x) = 4d_{\max}^2(1 + \eta \eta)\sqrt{x}^2 + 2d_{\max}(1 + \eta \eta)\eta - 1.$$

If $0 < x < \frac{\sqrt{S + 1 + \eta \eta} - \sqrt{S + 1 + \eta \eta}}{4d_{\max} \sqrt{1 + \eta \eta}}$, we obtain $\sigma(x) < 0$. Thus, (11) yields that

$$4d_{\max}^2(1 + \eta \eta)p_{\max}^2 + 2d_{\max}(1 + \eta \eta)p_{\max} < 1.$$  \hfill (12)

By dividing the both sides of (12) by $4d_{\max}^2(1 + \eta \eta)p_{\max}^2 + 2d_{\max}(1 + \eta \eta)p_{\max} > 0$, we have

$$1 < \frac{1}{2p_{\max}d_{\max}(1 + 2p_{\max}d_{\max})(1 + \eta \eta)} = K_1(1 + \eta \eta).$$  \hfill (13)

Therefore, from Proposition 5.1 in [11], we obtain

$$\lim_{k \to \infty} F(\xi(k)) = \lim_{k \to \infty} (I + L)\xi(k) = 0,$$  \hfill (14)

where $I \in \mathbb{R}^{n \times n}$ is the identity matrix. Since $L$ is positive semidefinite, $\lambda_i(I + L) = 1 + \lambda_i(L) > 0$ for all $i \in V$, where $\lambda_i(\cdot)$ is an eigenvalue of a square matrix. Therefore $I + L$ is invertible, and hence, we have $\lim_{k \to \infty} \xi(k) = 0$.

From Assumption 1, this yields that $\lim_{k \to \infty} \xi(k) = c_{\text{ln}},$ where $c \in \mathbb{R}$ and $1_k = [1 \ 1 \cdots 1] \in \mathbb{R}^n$ [11]. Moreover, from Lemma 1, the sum of the states is constant at all times, that is, $\sum_{i=1}^n \xi_i(k) = \sum_{i=1}^n \xi_i(0), \forall k \in \mathbb{N}$. It follows that $\lim_{k \to \infty} \xi(k) = \frac{1}{n} \sum_{i=1}^n \xi_i(0)1_n$. \hfill \Box

### 3. Two-Dimensional Diffusion Phenomena

In this section, we review the difference approximation of the 2-D diffusion equation [18],[19]. We consider a diffusion process on a planar region $\Pi = \{(x,y) \in \mathbb{R}^2 \mid a_s \leq x \leq b_s, \; a_t \leq y \leq b_t\}$ as shown in Fig. 1, where $a_s, b_s, a_t,$ and $b_t$ are positive constants such that $a_s < b_s$ and $a_t < b_t$. Let $U(x,y,t) \in \mathbb{R}$ be the concentration on the position $(x,y)$ at time $t$. The initial concentration is given by $U(x,y,0)$ for $(x,y) \in \Pi$. The two-dimensional (2-D) diffusion equation is given by

$$\frac{\partial U(x,y,t)}{\partial t} = D \left( \frac{\partial^2 U(x,y,t)}{\partial x^2} + \frac{\partial^2 U(x,y,t)}{\partial y^2} \right),$$

$$\forall (x,y) \in \Pi \setminus \partial \Pi, \quad t > 0,$$

$$U(x,y,t) = 0, \quad \forall (x,y) \in \partial \Pi, \; t > 0,$$

where $\partial \Pi$ is the boundary of the planar region $\Pi$ and $D > 0$ is a given diffusion coefficient [23]. We discretize the 2-D planar region and the time by the following grid points:

$$x_p = a_s + p\Delta x, \quad p = 0, 1, 2, \ldots, h_x,$n

$$y_q = a_t + q\Delta y, \quad q = 0, 1, 2, \ldots, h_y,$n

$$t_r = r\Delta t, \quad r = 0, 1, 2, \ldots,$n

where $\Delta x = (b_x - a_x)/h_x, \Delta y = (b_y - a_y)/h_y, \Delta t$ is a grid-point interval of the time.

By the Forward in Time, Central in Space (FTCS) method [23], we have the following discretized diffusion equation:

$$U_{p,q,r+1} - U_{p,q,r} \over \Delta t = D \left( U_{p+1,q,r} - 2U_{p,q,r} + U_{p-1,q,r} \over \Delta x^2 + U_{p,q+1,r} - 2U_{p,q,r} + U_{p,q-1,r} \over \Delta y^2 \right),$$

$$\forall p \in I_p, \; q \in I_q, \; r \in \mathbb{N},$$  \hfill (15)

$$U_{0,q,r} = U_{p,0,r} = U_{p,h_r,r} = 0,$$  \hfill (16)
where $U_{p,q,r} = U(x_p,y_q,t_r) \in \mathbb{R}$, $I_p = \{1,2,\ldots,h_x - 1\}$, and $I_q = \{1,2,\ldots,h_y - 1\}$. Then we have

$$U_{p,q,r+1} = \alpha(U_{p+1,q,r} + U_{p-1,q,r}) + \beta(U_{p,q+1,r} + U_{p,q-1,r}) + \gamma U_{p,q,r},$$  \hfill (17)

where $\alpha = D\Delta t/\Delta x^2$, $\beta = D\Delta t/\Delta y^2$, and $\gamma = 1 - 2\alpha - 2\beta$. Equation (17) is numerically stable [23] if

$$0 < \Delta t < \frac{\Delta x^2\Delta y^2}{2D(\Delta x^2 + \Delta y^2)}. \hfill (18)$$

We here define a matrix $\Omega = [\omega_{ij}] \in \mathbb{R}^{h \times h}$ as follows:

$$\omega_{ij} = \begin{cases} \alpha & \text{if } (i,j) = (p+1,q), (p-1,q), \\ \beta & \text{if } (i,j) = (p,q+1), (p,q-1), \\ \gamma & \text{if } (i,j) = (p,q), \end{cases}$$

where $p \in I_p$, $q \in I_q$, and $h = (h_x - 1)(h_y - 1)$. Then the discretized diffusion equation (17) can be expressed in a matrix form as follows:

$$u_{r+1} = \Omega u_r, \hfill (19)$$

where $u_r = [U_{1,1}, U_{1,2}, \ldots, U_{h_x-1,h_y-1}]^T \in \mathbb{R}^h$ is the vector of concentrations of all the discretized points on the planar region $\Pi$ at time $t_r$.

4. Consensus-Based Distributed Inverse Problem

We consider $n$ static sensor agents which are placed on some of the discretized grid points of the planar region $\Pi$ as shown in Fig. 2 ($2 \leq n \leq h$). The position of agent $i$ ($i \in V$) is represented by $(x_i, y_i)$. Each agent $i$ measures the following discretized gradients of concentration:

$$\frac{\partial U(x,y,t_r)}{\partial x} \bigg|_{x=x_i} = \frac{U_{i+1,j,r} - U_{i,j,r}}{\Delta x},$$  \hfill (20)

$$\frac{\partial U(x,y,t_r)}{\partial y} \bigg|_{y=y_i} = \frac{U_{i,j+1,r} - U_{i,j,r}}{\Delta y}. \hfill (21)$$

We assume that both the diffusion coefficient and the grid intervals satisfy (18) for numerical stability.

The inverse problem considered in this paper is to estimate the vector of the initial distribution of concentration $u_0 = [U_{1,1,0}, U_{1,2,0}, \ldots, U_{h_x-1,h_y-1,0}]^T \in \mathbb{R}^h$ from the time-series data (20) and (21). We assume that agents use the time-series data in the measurement interval $\Delta t \leq t \leq h_t \Delta t$.

We define a matrix $M_i = [m_{ij}] \in \mathbb{R}^{2h \times h}$ such that

$$m_{ij} = \begin{cases} 1 & \text{if } (i',j') = (1,\mu_i), (2,\nu_i + 1), \\ -1 & \text{if } (i',j') = (1,\nu_i), (2,\nu_i), \\ 0 & \text{otherwise,} \end{cases}$$

where $\mu_i = \xi_i(h_x - 1) + \zeta_i$ and $\nu_i = (\xi_i - 1)(h_y - 1) + \zeta_i$. The matrix $M_i$ extracts the data measured by agent $i$ from the entire measurements by all the agents. We also define a matrix $g_i$ which represents the diffusion process at the position of agent $i$ in the time interval $\Delta t \leq t \leq h_t \Delta t$:

$$g_i = \begin{bmatrix} M_i \Omega \\ M_i \Omega^2 \\ \vdots \\ M_i \Omega^{h_t} \end{bmatrix} \in \mathbb{R}^{2h \times h}.$$ 

Thus, from (19), we have $g_i u_0 = \varphi_i$, where

$$\varphi_i = \begin{bmatrix} U_{i+1,1,1} - U_{i,1,1} \\ U_{i,2,1} - U_{i,1,1} \\ U_{i,1,2} - U_{i,1,1} \\ \vdots \\ U_{i+1,h_y,1} - U_{i,h_y,1} \\ U_{i,h_x+1,1} - U_{i,h_y,1} \end{bmatrix} \in \mathbb{R}^{2h},$$

and $U_{h_x+1,1,r} = U_{p,h_y+1,r} = 0$ ($p \in I_p, q \in I_q, r \in I_t = \{1,2,\ldots,h_t\}$). The vector $\varphi_i$ represents the discretized differences of measurements related to agent $i$. We here define $A_i$ and $b_i$ as follows:

$$A_i = \Psi g_i \in \mathbb{R}^{h \times h},$$  \hfill (22)

$$b_i = \Psi \varphi_i \in \mathbb{R}^{2h}, \hfill (23)$$

where $\Psi$ is a scaling matrix with the grid intervals $\Delta x$ and $\Delta y$ of the discretized planar region $\Pi$:

$$\Psi = \text{diag} \left\{ \begin{array}{cc} \frac{1}{\Delta x} & 0 \\ \frac{1}{\Delta y} & 0 \\ \frac{1}{\Delta x} & \frac{1}{\Delta y} \\ \vdots \\ \frac{1}{\Delta x} & \frac{1}{\Delta y} \end{array} \right\} \in \mathbb{R}^{2h \times 2h}.$$ 

The matrix $A_i$ shows the discretized diffusion process at the position of agent $i$ scaled by the grid intervals $\Delta x$ and $\Delta y$. The vector $b_i$ is the actual measurements (20) and (21) of agent $i$ in the interval $\Delta t \leq t \leq h_t \Delta t$.

Then we obtain the following linear equation:
\[ Au_0 = b, \]

where
\[ A = [A_1^T A_2^T \cdots A_n^T]^T \in \mathbb{R}^{2nh \times nh}, \]
\[ b = [b_1^T b_2^T \cdots b_n^T]^T \in \mathbb{R}^{2nh}. \]

The matrix \( A \) shows the discretized and scaled diffusion process at the positions of all \( n \) agents and the vector \( b \) is the actual measurements of all the agents in the interval \( \Delta t \leq t \leq h_1 \Delta t \).

The inverse problem considered in this paper is to identify the initial diffusion \( u_0 \) on the planar region \( \Pi \) from the measurements of the gradients of concentration (20) and (21). Therefore the inverse problem of the diffusion equation (17) is equivalent to finding \( \hat{u}_0 \) such that
\[ \hat{u}_0 = \min_{u \in \mathbb{R}^{\mathbb{R}^{\Pi}}} \|Au_0 - b\|_2. \]

The solution of the inverse problem \( \hat{u}_0 \) satisfies the normal equation \((A^T A)\hat{u}_0 = A^T b\). Thus, under the assumption that \( A^T A \) is invertible, \( \hat{u}_0 \) is given as follows [20]:
\[ \hat{u}_0 = C^{-1} d, \]
where
\[ C := \frac{1}{n} A^T A = \frac{1}{n} \sum_{i=1}^{n} A_i^T A_i = \frac{1}{n} \sum_{i=1}^{n} C_i, \]
\[ d := \frac{1}{n} A^T b = \frac{1}{n} \sum_{i=1}^{n} A_i^T b_i, \]
\[ C_i := A_i^T A_i \in \mathbb{R}^{h \times h}, \]
\[ d_i := A_i^T b_i \in \mathbb{R}^h. \]

Note that the discussion in Section 2 is still valid for the higher dimensional case \( \xi_i \in \mathbb{R}^m \) \((m \geq 2)\) by applying the proposed method for each element \( \xi_i \) \((i = 1, 2, \ldots, m)\). We consider a multi-agent system where each agent has a state \( \xi_i(k) = [\xi_{i1}(k) \; \xi_{i2}(k) \cdots \xi_{ih(k)}(k)]^T \in \mathbb{R}^{h(k) + 1} \) and the following discrete-time consensus dynamics (\( i \in I \in \{1, 2, \ldots, h(k) + 1\})
\[ \xi_i(k+1) = \xi_i(k) + \sum_{j=1}^{n} p_{ij}(\xi_i(k) - \xi_j(k)), \]
where \( \xi_i(k) = [\xi_{i1}(k) \; \xi_{i2}(k) \cdots \xi_{ih(k)}(k)]^T \in \mathbb{R}^{h(k) + 1} \).

From (33), we see that if there are eigenvalues such that \( \lambda_i \approx 0 \), even small perturbations can be largely blown up by the term \( 1/h \). To avoid this, we truncate the terms of the eigenvalues such that \( |\lambda_i| < \epsilon_{th} \), where \( \epsilon_{th} \) is a sufficiently small positive constant. Then we have an approximated solution \( \tilde{u}_0 \) as follows:
\[ \tilde{u}_0 = \frac{1}{n} \sum_{i=1}^{n} \xi_i(k). \]
The bound of time-delays is given by $\xi_i$ to an inverse problem of diffusion phenomena in order to find a source of a diffusion process from time-series data measured by agents. We showed that multi-agent systems can estimate the initial distribution by solving the least squares problem with the proposed consensus dynamics.

In this paper, we assumed that local communications among agents are bidirectional and time-delays are common for all communication links. The analysis for the more general communication mechanism will be the subject of our future work.

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**References**


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