State Estimation under Lebesgue Sampling and an Approach to Event-Triggered Control

Takahiro KAWAGUCHI*, Masaki INOUE*, and Shuichi ADACHI*

Abstract: In a standard setup of conventional state estimation problems, the output signal of a dynamical system is sampled at every regular time interval. This paper addresses an estimation problem under event-triggered sampling: an irregular sampling rule in which the output signal is sampled only when a specified event occurs. In particular, this paper focuses on the Lebesgue sampling, which is a type of event-triggered sampling such that the output signal is sampled only when it crosses specific thresholds. In the proposed estimation method, not only information in the sampled points but also that in the inter-sample intervals are utilized to improve the estimation accuracy efficiently. The particle filter is applied to the estimation since the inter-sample information makes the distribution of the estimates non-Gaussian. Furthermore, the proposed estimator is combined with a linear-quadratic-regulator (LQR) type state-feedback controller. The derived control law can be an effective approach to an event-triggered control problem in which the Lebesgue-sampled output is utilized. The effectiveness of both of estimation and control methods is examined through numerical examples.

Key Words: state estimation, Lebesgue sampling, event-triggered control, particle filter.

1. Introduction

In state estimation problems, the state of a dynamical system is estimated by using the output signal, which includes partial state information and, inevitably, measurement noise. A well-known state estimator is the Kalman filter (e.g., [1]). In the state estimation problems, a type of sampling is not explicitly considered well. One of the most commonly used sampling rules is to sample the signal at every regular time interval. However, in practical applications, the regular sampling rule is not always available, because there can be limitations of measurement instruments and communication bandwidth to transmit the data. One of the typical limitations comes from that a system and an estimator are connected through a network as shown in Fig. 1. In that case, the amount of data transmitted from the system to the estimator is limited and depends on network load. Due to the limitations, many practical systems often apply event-triggered sampling, which irregularly samples the output signal only when a specified event occurs. Estimation and control problems based on the sampling rule have been attracted much attention in recent years, see e.g., [2]–[4], and references therein.

This paper addresses the state estimation under the Lebesgue sampling [5], which is a type of event-triggered sampling. In the Lebesgue sampling, a signal is sampled only when it crosses specific thresholds as shown in Fig. 2(a). In contrast, the standard sampling, in which a signal is sampled at every regular time step as shown in Fig. 2(b), is called the Riemann sampling. The Lebesgue sampling tends to collect informative data, while the Riemann sampling does not. For example, in the last half period in Fig. 2(b), there is little information of the signal since its value is almost zero and does not vary.

It should be noted that the data sampled by the Lebesgue sampling has inter-sample information in addition to information on the sampled points. A time interval between sequential two samples informs us that the signal does not cross the...
thresholds. In this paper, the inter-sample information is taken into state estimators to improve the estimation accuracy.

A general description of event-based sampling and a design method of state estimator are proposed in [6]. The estimator in [6] utilizes the inter-sample information and estimates the state as an approximated Gaussian distribution. However, due to the approximation, the estimation accuracy cannot be sufficiently improved.

This paper proposes a new state estimation method that takes the inter-sample information into account in a different way from [6]. In the proposed method, the particle filter is applied to estimate the non-Gaussian distribution of the state without approximating it to Gaussian. Furthermore, the proposed estimator is combined with a linear-quadratic-regulator (LQR) type state-feedback controller. The derived control law can be an effective approach to an event-triggered control problem in which the Lebesgue sampled output is utilized. The effectiveness of the proposed estimation and control methods is examined through numerical examples.

2. Problem Formulation

In this section, the state estimation problem is formulated. Let the scalar time-series \( y_k \) be generated by the following discrete-time state-space model \( \Sigma \):

\[
x_{k+1} = Ax_k + b_1 u_k + b_2 y_k,
\]

\[
y_k = c^T x_k + w_k,
\]

where \( x_k \in \mathbb{R}^n \) is the state-vector, and \( u_k \in \mathbb{R} \) is the input. \( A \in \mathbb{R}^{n \times n}, b_1 \in \mathbb{R}^{n \times 1}, b_2 \in \mathbb{R}^{n \times 1}, \) and \( c \in \mathbb{R}^{1 \times n} \) are the constant matrices, which characterize the dynamics of the system. System noise \( v_k \in \mathbb{R} \) and measurement noise \( w_k \in \mathbb{R} \) are the zero-mean white Gaussian noises with variance of \( \sigma_v^2 \) and \( \sigma_w^2 \), respectively. As illustrated in Fig. 3, the output of the system \( y_k \) is sampled by the Lebesgue sampler \( \mathcal{L} \), which performs the Lebesgue sampling as follows.

The mechanism of the Lebesgue sampling in this paper is a realistic implementation of the Lebesgue sampling for discrete-time signals, which is called the approximate Lebesgue sampling in [5]. In the sampling mechanism, \( y_k \) for every \( k \) is classified into a sampled set or not, which is shown in Fig. 4. Let \( \eta_m, m \in \{1, 2, \ldots, M\} \) denote the thresholds satisfying \( \eta_1 < \eta_2 < \cdots < \eta_M \). It is assumed that \( \eta_1 \) and \( \eta_M \) are sufficiently small and large, respectively, so that there is no \( y_k \) satisfying \( y_k < \eta_1 \) or \( \eta_M < y_k \).

Given \( y_k \), find \( m_k \in \mathbb{N} \) which satisfies

\[
\eta_{m_k} \leq y_k < \eta_{m_k+1}.
\]

Then, \( y_k \) is sampled only when \( m_k \) is different from \( m_{k-1} \).

**Example 1** An example of the thresholds is given as

\[
\eta_m = \left( m - M - \frac{1}{2} \right) d,
\]

where \( d \) denotes the interval of thresholds and \( M \) is a sufficiently large even number.

Then, this paper proposes a state estimation method. In the method, the state \( x_k \) is estimated by using the Lebesgue-sampled output

**Remark 1** The system \( \Sigma \) in Fig. 3 is a discrete-time system and it generates the output \( y_k \) at regular intervals. The system \( \Sigma \) can be derived from a continuous-time dynamical system and a measurement instrument. In the case, the sampling period of \( \Sigma \) may be set as the minimum feasible period of the measurement instrument.

3. State Estimation under Lebesgue Sampling

In this section, a state estimation method under the Lebesgue sampling is proposed.

The state estimator based on the Bayesian approach such as the Kalman filter, requires a priori estimate and the likelihood function. They are utilized to calculate a posteriori estimate. In order to estimate \( x_k \) in (1) based on the Lebesgue-sampled output \( z_k \), the Bayesian approach is utilized. The likelihood function of \( x_k \) given \( z_k \) is equal to \( p(z_k|x_k) \), which is the conditional probability of \( z_k \) given \( x_k \). Deriving \( p(z_k|x_k) \) for \( k \in S \) and \( k \notin S \), we can estimate the states with utilizing the inter-sample and sample information.

3.1 Likelihood Formulation

The conditional probability density of \( y_k \) given \( x_k \) for some \( k \in S \) is described as

\[
p(y_k|x_k) = \frac{1}{\sqrt{2\pi\sigma_w}} \exp \left( -\frac{(y_k - c^T x_k)^2}{2\sigma_w^2} \right).
\]

Note that \( z_k = \phi \), for which \( y_k \) is not sampled for \( k \notin S \), implies \( \eta_{m_k} \leq y_k < \eta_{m_k+1} \). We have the information that the inequality holds even in sequential two sampled points. This is called the
inter-sample information in this paper. The information is integrated into the estimator through the conditional probability \( p(\phi|\mathbf{x}_k) \), which is the probability of that the output \( y_k \) is not sampled for \( k \notin S \) given \( \mathbf{x}_k \). The probability \( p(\phi|\mathbf{x}_k) \) is described as

\[
p(\phi|\mathbf{x}_k) = \int_{\eta_{mk}}^{\eta_{mk+1}} \frac{1}{\sqrt{2\pi\sigma_w^2}} \exp \left( -\frac{(y_k - c^T\mathbf{x})^2}{2\sigma_w^2} \right) dy_k
\]

where

\[
f(m, x) = \text{erf} \left( \frac{\eta_m - c^T x}{\sqrt{2\sigma_w^2}} \right)
\]

and

\[
\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2) dt
\]

The function of the probability in (8) for various \( \sigma_w^2 \) is depicted in Fig. 5. It is clear that smaller \( \sigma_w^2 \) leads larger non-Gaussianity of the probability.

Although \( m_k \) is required to evaluate (8), it is not obtained directly if \( k \notin S \) since the Lebesgue sampler does not sample the data at the time \( k \). In order to evaluate (8) even at \( k \notin S \), we need to use a past index on the thresholds. Let \( j \in S \) denote the closest index of \( k \notin S \) and \( j < k \). Then, \( m_k = m_j \) holds.

### 3.2 State Estimation

The two probabilities \( p(y_k|\mathbf{x}_k) \) and \( p(\phi|\mathbf{x}_k) \) are derived as (7) and (8), respectively. They are utilized for the state estimation. However, the standard Kalman filter is not directly applicable to the estimation since \( p(\phi|\mathbf{x}_k) \) in (8) does not obey the Gaussian distribution. Therefore, this paper applies the particle filter [7] and proposes a state estimation method using (7) and (8). The algorithm of the proposed method is described as follows.

**STEP 0 Initialization**

Set \( k = 1 \) and choose an appropriate a priori distribution \( p(\mathbf{x}_0) \) and the number of the particles \( L \). Draw the samples \( \mathbf{x}_{i,0}^{(1)}, i \in \{1, \ldots, L\} \) from \( p(\mathbf{x}_0) \). Then, obtain a set \( X_{1/0} = \{ \mathbf{x}_{1/0}^{(1)}, \ldots, \mathbf{x}_{1/0}^{(L)} \} \).

**STEP 1 Update**

**STEP 1.1 Calculation of the likelihood**

If the output signal \( y_k \) is sampled at \( k \), calculate the likelihood as

\[
\alpha_k^{(i)} = p(y_k|\mathbf{x}_{k|k-1}^{(i)}), \; i \in \{1, \ldots, L\},
\]

otherwise

\[
\alpha_k^{(i)} = p(\phi|\mathbf{x}_{k|k-1}^{(i)}), \; i \in \{1, \ldots, L\}.
\]

**STEP 1.2 Resampling**

Generate samples \( \mathbf{x}_{k|k}^{(i)}, j \in \{1, \ldots, L\} \) by resampling from \( X_{k|k-1} \) with probability \( \alpha_k^{(i)}/\sum_{i=1}^{L} \alpha_k^{(i)} \). Then, obtain the set \( X_{k|k} = \{ \mathbf{x}_{k|k}^{(1)}, \ldots, \mathbf{x}_{k|k}^{(L)} \} \).

**STEP 1.3 Updating estimates**

Calculate the filtered estimate by

\[
\hat{x}_k = \frac{1}{L} \sum_{i=1}^{L} \mathbf{x}_{k|k}^{(i)}.
\]

**STEP 2 Prediction**

Draw the samples of a system noise \( \mathbf{v}_k^{(i)}, i \in \{1, \ldots, L\} \) from

\[
p(\mathbf{v}_i) = \frac{1}{\sqrt{2\pi\sigma_v^2}} \exp \left( -\frac{\mathbf{v}_i^2}{2\sigma_v^2} \right),
\]

and calculate

\[
\mathbf{x}_{k+1|k}^{(i)} = \mathbf{A}\mathbf{x}_{k|k}^{(i)} + \mathbf{b}_u \mathbf{u}_k^{(i)} + \mathbf{b}_v \mathbf{v}_k^{(i)}, \; i \in \{1, \ldots, L\}.
\]

Then, obtain the set

\[
X_{k+1|k} = \{ \mathbf{x}_{k+1|k}^{(1)}, \ldots, \mathbf{x}_{k+1|k}^{(L)} \},
\]

which is used in the next time step \( k + 1 \).

Go to STEP 1 replacing \( k \) by \( k + 1 \).

STEP 1 and STEP 2 in the procedure are repeated, and the state estimate \( \hat{x}_k \) is given by (13).

**Remark 2** For comparison, we consider a simplistic event-based estimation method: the resampling process in STEP 1.2 in the proposed method is skipped when the output signal \( y_k \) is not sampled, and (13) is replaced by

\[
\hat{x}_k = \frac{1}{L} \sum_{i=1}^{L} \mathbf{x}_{k|k-1}^{(i)}.
\]

in STEP 1.3. We emphasize that the proposed method utilizes the inter-sample information in STEP 1.2 with the likelihood (12), while the simplistic estimation does not.

**Remark 3** It should be noted that the algorithm stated above is directly applicable to any nonlinear dynamical system. In such an application, the number of particle becomes more important for accurate estimation [8],[9].
4. An Approach to Event-Triggered Control

In this section, the proposed estimator is applied to a control problem under the Lebesgue sampling, and an approach to the event-triggered control is further proposed.

We consider a control system depicted in Fig. 6. A dynamical system $\Sigma$ and a state estimator are connected over a network, and the output of the system $\Sigma$ is sampled by the Lebesgue sampler. An actuator is connected to the estimator and the system $\Sigma$ through no network, so the control input to $\Sigma$ can also be changed even at the time when the estimator does not receive any signal. This assumption is different from that in the optimal control problem [10].

The control system $\Sigma$ is described by the state-space model in (1) and (2). The problem is finding the input $u_k$ which minimizes the cost function

$$J_1 = E\left[ x_k^T S_f x_k + \sum_{i=1}^{N-1} (x_i^T Q x_i + r u_i^2) \right], \quad (17)$$

where $N$ is the length of the evaluated period, and $S_f$, $Q$, and $r$ are the weights, which are designed by users.

Suppose that the output of $\Sigma$ is sampled for all time. Then, the control problem above is reduced to the well-known linear quadratic Gaussian (LQG) control. The optimal control law is obtained by designing the state estimator and the state feedback law, separately, which is known as the separation principle [11].

The optimal input $u_k$ is given by

$$u_k = -f^T_k \hat{x}_k, \quad (18)$$

where $\hat{x}_k$ is the estimated state by the Kalman filter, and feedback gain $f^T_k$ is obtained by minimizing

$$J_2 = \hat{x}_k^T S_f \hat{x}_k + \sum_{i=1}^{N-1} (\hat{x}_i^T Q \hat{x}_i + r u_i^2), \quad (19)$$

where

$$x_{k+1} = Ax_k + B u_k, \quad (20)$$

and

$$u'_k = -f^T_k \hat{x}_k. \quad (21)$$

Let us consider the control problem in which the output signal of the system $\Sigma$ is sampled by the Lebesgue sampler as analogy with the LQG control shown in Fig. 7. The input is given by (18), where $\hat{x}_k$ is the estimates obtained by the proposed estimator. The entire control system is shown in Fig. 8. The proposed control law is expected to improve the performance defined by (17). This is because the estimator utilizes the intersample information and gives the accurate estimation compared with the simplistic estimator.

5. Numerical Examples

5.1 State Estimation under the Lebesgue Sampling

A numerical experiment is performed to illustrate the effectiveness of the proposed state estimation method, which is given in Section 3.

Consider a scalar dynamical system described by

$$x_{k+1} = ax_k + bv_k, \quad (22)$$

$$y_k = cx_k + w_k, \quad (23)$$

where $a = 0.9048$, $b = 0.25$, and $c = 0.3807$. The variances of $v_k$ and $w_k$ are $\sigma^2_v = 1$ and $\sigma^2_w = 0.1^2$, respectively. The thresholds of the Lebesgue sampling are defined as (4), where $d = 0.23$. An example of the output signal is shown in Fig. 9.

Experiments of the state estimation are performed in 1000 trials with different realization of $v_k$ and $w_k$. In each experiment, the proposed method is applied to (22) and (23). The number of particles is $L = 200$. The method is compared with the simplistic method, defined in Subsection 3.2.

The estimated states by the standard Kalman filter in which the output signal $y_k$ is fully sampled, the simplistic estima-
Fig. 10 The results of the state estimation. In the figures, the dashed line is the time series of the true state, and the solid line is the estimated state. The time when the output signal $y_k$ is sampled is depicted by vertical dotted lines. In (a), 50 sampled points are utilized while 17 sampled points are utilized in (b) and (c).

The mean and the standard deviation of the root mean square error (RMSE) in 1000 trials are calculated and summarized in Table 1. It is clear that the RMSE of the proposed method is smaller than the simplistic one, and it is verified that the proposed method is effective under the Lebesgue sampling.

Remark 4 The computation time linearly depends on the number of particles. It is known that the required number of the particles for accurate estimation monotonically increases with the increase in the order of the system. The proposed method cannot be directly applied to large-scale systems.

5.2 Control under the Lebesgue Sampling

A numerical experiment is performed to illustrate the effectiveness of the proposed control method, which is given in Section 4.

A magnetic levitation system which is depicted in Fig. 11, is considered. It floats an iron ball of mass $M$ by controlling the suction force of the electromagnet. The dynamics of the system around the equilibrium point are described by a state-space model (1) and (2), where

$$A = \exp(A'T)$$
$$b_v = b_v = \int_0^T \exp(A'\tau)b'd\tau$$
$$c = [1 \ 0]^T$$
$$A' = \begin{bmatrix} 0 & 1 \\ K_x/M & 0 \end{bmatrix}$$
$$b' = \begin{bmatrix} 0 \\ -K_x/M \end{bmatrix}.$$  

Note that the system is unstable since eigenvalues of $A'$ are $\pm \sqrt{K_x/M}$. In (24)–(28), $T$ is the sampling interval, and $K_x$ and $K_r$ are constant numbers related to the electromagnetic suction force depending on the equilibrium point. In this numerical example, $T = 10^{-3}$, $K_x = 177$, $K_r = 5.187$ and $M = 0.358$, and $v_k$ and $w_k$ are zero-mean Gaussian white noise of variance $0.1/10^{-6}$, respectively. The thresholds of the Lebesgue sampling are defined as (4) and $d = 10^{-2}$. The feedback gain $f_k^*$ is decided by minimizing (19), where $S_f = I$, $Q = I$, and $r = 0.1$.

The results of the control by the LQG control in which the output signal $y_k$ is fully sampled, the simplistic control method, and the proposed control method are illustrated in Figs. 12 (a), 12 (b), and 12 (c), respectively. Note that the simplistic control method applies the simplistic estimation method and the feedback gain $f_k^*$. From Fig. 12 (b), we see that the fluctuation of the states is large, that is, the control performance is deteriorated comparing with the full sampling case shown in the increase.
Fig. 12 The time series of the states under the control.

Fig. 12: The result by the proposed method is shown in Fig. 12 (c), and it is seen that the control performance is particularly improved than the simplistic case shown in Fig. 12 (b).

The cost functions in (19) are calculated and summarized in Table 2:

<table>
<thead>
<tr>
<th>Method</th>
<th>Cost Function</th>
</tr>
</thead>
<tbody>
<tr>
<td>Proposed method</td>
<td>223.5</td>
</tr>
<tr>
<td>Simplistic method</td>
<td>962.4</td>
</tr>
<tr>
<td>Kalman filter with full sampling</td>
<td>78.2</td>
</tr>
</tbody>
</table>

Table 2. The value of the cost function by the proposed method is smaller than that by the simplistic method, and the effectiveness of the proposed method is examined.

6. Conclusion and Future Work

This paper proposed a state estimation method and an event-triggered control method under the Lebesgue sampling. In the state estimation method, not only sample-point information of the output but also inter-sample information is utilized to particularly improve the estimation accuracy. The estimation method was combined with a state feedback controller to derive a novel approach to an event-triggered control problem under the Lebesgue sampling. The effectiveness of the proposed methods was illustrated in numerical simulations.

In future, we will find a class of dynamical systems for which the proposed methods are particularly effective. Analysis of stability and convergence of the proposed estimation and control methods will be studied. We will also address a design problem of the thresholds for the Lebesgue sampling.

References

Takahiro Kawaguchi (Student Member)

He received his B.E. and M.E. degrees from Keio University, Yokohama, Japan, in 2011 and 2013, respectively. From 2013 to 2015, he was employed by Toshiba Research and Development Center. Since 2014, he has been in a doctoral course of Keio University. His research interests include the system identification and the state estimation theory and its application to real systems.

Masaki Inoue (Member)

He received the M.E. and Ph.D. degrees in mechanical engineering from Osaka University, Japan in 2009 and 2012, respectively. From 2010 to 2012, he was a Research Fellow of the Japan Society for the Promotion of Science. From 2012 to 2014, he was a Project Researcher of the FIRST (Funding Program for World-Leading Innovative R&D on Science and Technology) Aihara Innovative Mathematical Modelling Project, and also a Doctoral Researcher of the Graduate School of Information Science and Engineering, Tokyo Institute of Technology, Japan. Currently, he is an Assistant Professor in the Department of Applied Physics and Physico-informatics, Faculty of Science and Technology, Keio University, Japan. His research interests include stability theory of dynamical systems. He is a member of IEEE and ISCIE.

Shuichi Adachi (Member, Fellow)

He received the B.E., M.E., and Ph.D. degrees in Electrical Engineering all from Keio University, Yokohama, Japan, in 1981, 1983, and 1986, respectively. From 1986 to 1990, he was employed by Toshiba Research and Development Center. In 1990, he joined Department of Electrical and Electronic Engineering, Utsunomiya University as an associate professor, and in 2002 he became a professor. From 2003 to 2004, he was a visiting researcher at Engineering Department of Cambridge University. Since 2006, he has been a professor in the Department of Applied Physics and Physico-Informatics, Keio University. His current research interests include system identification theory and its application to real systems, for example, automobile, aerospace, acoustic systems, and so on. He is a member of the IEEE, the JSME, IEE of Japan, and so on.