A Boundary-Value-Free Reconstruction Method for Magnetic Resonance Electrical Properties Tomography Based on the Neumann-Type Integral Formula over a Circular Region

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Abstract: The electrical properties (EPs) of biological tissue, consisting of conductivity and permittivity, provide useful information for the diagnosis of malignant tissues and the evaluation of heat absorption rates. Recently, magnetic resonance electrical properties tomography (MREPT), by which EPs are reconstructed from internal magnetic field data measured by using magnetic resonance imaging (MRI), has been actively studied. We previously proposed an explicit pointwise reconstruction method for MREPT based on a complex partial differential equation (PDE), known as the D-bar equation, of the electric field and its explicit solution given by an integral formula. In this method, as in some other conventional methods, EP values on the boundary of the region of interest must be given as a Dirichlet boundary condition of the PDE. However, it is difficult to know these values precisely in practical situations. Therefore, in this paper, we propose a novel method for reconstructing EPs in a circular region without any knowledge of boundary EP values. Starting from the integral solution to solve the D-bar equation in a circular region with the Neumann boundary condition, we show that the contour integral term of the integral formula is eliminated by using Faraday’s law and solve the PDE based only on magnetic field data measured by using MRI. Numerical simulations show that the proposed method yields a good reconstruction results without any knowledge of boundary EP values.

Key Words: electrical properties (EPs), magnetic resonance imaging (MRI), magnetic resonance electrical properties tomography (MREPT), Dirichlet-type integral formula, Neumann-type integral formula.

1. Introduction

The electrical properties (EPs) of biological tissues, consisting of conductivity and permittivity, have been studied over a wide range of frequencies [1]–[3]. Several studies have suggested that some malignant tissues have significantly different EP values from those of normal tissues [4]. Therefore, EP maps are expected to be useful for the diagnosis of malignant tissues. In addition, conductivity maps can be used to calculate the specific absorption rate (SAR) distribution [5], which is the heating criterion of tissues in recent high field MRI systems.

Among many proposed approaches to EP map reconstruction, magnetic resonance electrical properties tomography (MREPT) has attracted attention because of its advantages of non-invasiveness and high internal spatial resolution [6],[7]. MREPT reconstructs EP distributions from magnetic field data distorted by the loaded object using the so-called B1 mapping technique [8]–[10]. The interaction between electromagnetic field transceived by MR radio frequency (RF) coils and EPs are described by the following time-harmonic Maxwell equations:

\[ \nabla \times E = -i \omega \mu_0 H, \]  
\[ \nabla \times H = i \omega (\epsilon_\sigma E - i \sigma / \omega) E + i \omega \sigma E, \]  

where \( \mu_0 \) is the magnetic permeability of free space, \( \epsilon_\sigma \) is the permittivity in free space, and \( \omega / 2\pi \) is the Larmor frequency.

The complex permittivity, \( \kappa \), is defined by \( \epsilon_\sigma \epsilon_0 - i \sigma / \omega \). We can assume that the magnetic permeability is identical to \( \mu_0 \) through the entire body, because the magnetic susceptibility of the body is sufficiently small [11]. Conversely, \( \kappa \) is the distribution parameter we aim to reconstruct. By eliminating the electric fields from Eqs. (1) and (2), it holds that

\[ \Delta H + \frac{\nabla \kappa}{\kappa} \times (\nabla \times H) + \omega^2 \mu_0 \kappa H = 0. \]  

This is a non-linear PDE of \( \kappa \). Because the positively rotating component \( H_\perp \equiv (H_x + iH_y)/2 \) of the RF magnetic field (where \( H_x \) and \( H_y \) are the \( x \)- and \( y \)-components of \( H \), respectively) is a measurable quantity in a typical MRI system, the MREPT problem is to reconstruct \( \kappa \) from measured \( H^\perp \) data.

A primitive approach to solve Eq. (3) is to reduce it into an algebraic equation by neglecting the spatial gradient of \( \kappa \). Haacke et al. [12] and later Wen [13] introduced a formula \( \kappa = - (\Delta H^\perp) / (i \omega^2 \mu_0 H^\perp) \) by assuming that \( \nabla \kappa = 0 \); stabilized versions of this approach were also proposed [14],[15]. However, the assumption that \( \nabla \kappa = 0 \) results in severe artifacts in the tissue transition regions where EPs vary [16],[17].

Various type of methods have been proposed to tackle the transition region artifacts. The first approach is to solve the PDE using an iterative optimization calculation [18]–[21]. In these iterative methods, it is important to give the optimal initial value, without which a local minimum solution can be obtained. To give an appropriate initial value, direct methods with low computational costs are necessary. The second approach is to derive a linear system of equations. Some methods that regard the gradients of \( \kappa \) as independent variables and solve the simultaneous equation for \( \kappa \) and \( \nabla \kappa \) from multiple sets of mag-

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netic field data are proposed. These methods work by switching the excitation mode of the coil [22] or by using a multichannel coil [23]–[28]. Hafalar et al. [29] proposed a cr-MREPT method by which a convection-reaction equation of inverse of $\kappa$ is introduced and solved by an finite element method (FEM). However, the Laplacian of $H^*$ is needed in this method, thus amplifying the noise. The third approach is to derive an explicit pointwise formula of $\kappa$. Nachman [30] proposed a method in which he took the inner product of Eq. (3) and $\nabla \times \mathbf{H}$ and introduced the formula $\kappa = -\langle \Delta \mathbf{H} \cdot (\nabla \times \mathbf{H}) \rangle / (\omega^2 \mu_0 \mathbf{H} \cdot (\nabla \times \mathbf{H}))$. However, this equation breaks down when $\mathbf{H}$ and $\nabla \times \mathbf{H}$ are orthogonal. Palamodov [31] introduced a complex linear PDE of inverse of $\kappa$. According to this method, the Riemann–Hilbert problem is solved to reconstruct EPs analytically.

We previously proposed an explicit pointwise reconstruction formula for MREPT [32], which is classified as the third approach. In this method, a complex linear PDE, known as the D-bar equation, of the electric field $\mathbf{E}$ is derived and solved in an explicit pointwise manner by the generalized Cauchy integral formula with the Dirichlet boundary condition. Then, from Ampere’s law, $\kappa$ is reconstructed by the ratio of $\partial H^*$ to $E_z$. This method does not need to calculate the Laplacian of $H^*$; therefore, it is more robust against the noise.

However, the problem with this formula is that the EP values on the boundary of the region of interest (ROI) must be given as a Dirichlet boundary condition, which is not the case in practical situations. In this paper, we propose a novel method for reconstructing EPs in a circular region that does not need EP values on the boundary. Starting from an integral formula that solves the D-bar equation in a circular region with the Neumann boundary condition, that is, a normal derivative of the electric field, we show that the contour integral term of the integral formula is eliminated using Faraday’s law and the PDE is solved from only magnetic field data measured by using MRI.

Therefore, our proposed method is more practical than previous ones although the ROI is restricted to a circular region. In Section 2, the D-bar equation of $E_z$, and its solution, as previously proposed in our former paper, is revisited. We propose an integral formula to reconstruct EPs in a circular region without the boundary EP values in Section 3. Section 4 is devoted to numerical simulations. Finally, Section 5 concludes this paper.

2. Reconstruction Method Based on the Dirichlet-Type Integral Formula [32]

2.1 Complex Representation of the Governing Equations

In this section, we revisit the complex representation of the governing equations that we previously introduced. By using complex differential operators $\partial \equiv \frac{1}{2}(\partial_x - i\partial_y)$, $\bar{\partial} \equiv \frac{1}{2}(\partial_x + i\partial_y)$ and taking the ($x$-component) + $i$ ($y$-component) of Eqs. (1) and (2), the following equations hold:

$$\bar{\partial}E_z - \partial E^+ = \omega \mu_0 H^*,$$

$$\bar{\partial}H_z - \partial H^* = -\omega \kappa E^*, \quad \text{(5)}$$

where $E^+ \equiv (E_x + iE_y)/2$. On the other hand, by taking the $z$-component of Eq. (1) and using $\nabla \cdot \mathbf{H} = 0$, we obtain the following:

$$4\partial H^* + \bar{\partial}H_z = -\omega \kappa E_z. \quad \text{(6)}$$

Now, we make some assumptions to reduce the problem in the 2D plane. First, we assume that $H_z \approx 0$. This is valid when a birdcage coil is used [14] and assumed in almost all conventional methods. By this assumption, Eq. (6) is rewritten as

$$\kappa = \frac{-4\partial H^*}{\omega E_z}, \quad \text{(7)}$$

where $E_z \neq 0$. This can be regarded as the definition of $\kappa$: that is, $\kappa$ is the ratio of $-4\partial H^*$ to $\omega E_z$. Hence, if we can express $E_z$ in terms of $H^*$, the reconstruction formula of $\kappa$ is obtained. Note that if $E_z = 0$, $\partial H^*$ is also zero and we cannot define $\kappa$ using Eq. (7). In fact, such a point is observed near the center of the loaded object when the birdcage coil is used in a quadrature excitation mode [22],[29]. The methods for displacing the zeros of $E_z$ have already been proposed and we provide a specific example in Section 4.2.

Second, we assume that $\partial H^* \approx 0$ to obtain a simple formula by making a 2D-plane approximation. The same assumption is made in Theorem 1 in [31]. From Eq. (5) with the first and second assumptions, we have $E^+ = 0$. Substituting this into Eq. (4) yields

$$\bar{\partial}E_z = \omega \mu_0 H^*. \quad \text{(8)}$$

This type of PDE is known as the D-bar equation, which can be explicitly solved with appropriate boundary conditions. The conventional approaches in [29],[31],[32] assume that $\kappa$ on the boundary of the ROI is given a priori, resulting in the Dirichlet boundary condition $E_z = -\omega \partial H^*/\omega \kappa$. In Section 2.2, our previous method assuming the Dirichlet boundary condition is summarized. In Section 2.3, a novel method assuming the Neumann boundary condition is proposed.

2.2 Explicit Pointwise Solution to the D-Bar Equation

Now, we set the regular domain $D \subset \mathbb{C}$ as our 2D ROI, and denote its simple closed contour by $C$. In general, if a complex-valued function $f$ is differentiable in $D$ and continuous on $C$, then the generalized Cauchy integral formula holds [33]:

$$f(\zeta_0) = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{\zeta - \zeta_0} d\zeta - \frac{1}{\pi} \int_D \frac{\bar{\partial} f}{\zeta - \zeta_0} d\zeta d\eta, \quad \text{(9)}$$

where $\zeta_0 \in D$ and $\zeta = \xi + i\eta$. This is the representation formula of $f$ at an arbitrary point $\zeta_0$ in $D$ in terms of the contour integral of the Dirichlet boundary value $f$ on $C$ and the 2D area integral of $\bar{\partial} f$ in $D$.

In the case of $f = E_z$ in Eq. (9), the substitution of $\bar{\partial}E_z$ in Eq. (8) yields

$$E_z(\zeta_0) = \frac{1}{2\pi i} \oint_C \frac{E_z(\zeta)}{\zeta - \zeta_0} d\zeta - \frac{1}{\pi} \int_D \frac{\omega \mu_0 H^*}{\zeta - \zeta_0} d\zeta d\eta, \quad \text{(10)}$$

which is rewritten by using Eq. (7) as

$$E_z(\zeta_0) = \frac{1}{2\pi i} \oint_C \frac{-\omega \partial H^*}{\zeta - \zeta_0} d\zeta - \frac{1}{\pi} \int_D \frac{\omega \mu_0 H^*}{\zeta - \zeta_0} d\zeta d\eta. \quad \text{(11)}$$

We refer to Eq. (11) as Dirichlet-type integral formula hereafter. By substituting Eq. (11) into Eq. (7), $\kappa$ is reconstructed as follows:

$$\kappa(\zeta_0) = \frac{-4\omega \partial H^*}{2\pi i} \oint_C \frac{-\omega \partial H^*}{\zeta - \zeta_0} d\zeta - \frac{1}{\pi} \int_D \frac{\omega \mu_0 H^*}{\zeta - \zeta_0} d\zeta d\eta. \quad \text{(12)}$$
This is an explicit pointwise reconstruction formula of $\kappa$ in $D$ given by $H^*$ in $D$ and $\partial H^*$ and $\kappa$ on $C$. Eq. (12) does not include the Laplacian of $H^*$ and is therefore more robust against noise. However, the problem is that $\kappa$ on $C$ should be known a priori, which is not the case in practical situations. We overcome this problem by solving the Neumann boundary value problem for Eq. (8) in the next section.

3. Reconstruction Method Based on the Neumann-Type Integral Formula

In this section, we derive an explicit pointwise reconstruction formula of $\kappa$ in which its boundary value is unnecessary. Our proposed method is restricted to the case where the ROI is a circular area. Nevertheless, this method improves the applicability to practical situations to which EP values can vary on the boundary of the ROI, and we can consider local inner regions in which the EP values are totally unknown. Extending our proposed method to the general-shaped ROIs is future work.

When $D$ is a unit circle and $C = \partial D$, for $f$ differentiable in $D$ and continuous on $C$, a Neumann-type integral representation formula of $f$ at a point in $D$ is given as [33]

$$f(\zeta_0) = -\frac{1}{2\pi i} \oint_C \log \frac{1 - \zeta_0/\zeta}{\zeta} (\partial_\zeta f - \frac{\zeta}{a} \partial_a f) d\zeta,$$

where $\partial_a$ represents the outward normal derivative and $c = f(0)$. Note that the boundary values required in Eq. (13) are $\partial_\zeta f$ and $\partial_a f$ on $C$, while that in Eq. (9) is $f$ on $C$. It is straightforward to extend Eq. (13) to the case where $D$ is a disc with a radius of $a$, which is given by

$$f(\zeta_0) = -\frac{1}{2\pi i} \oint_C \frac{a \log(1 - \zeta_0/\zeta)}{\zeta} (\partial_\zeta f - \frac{\zeta}{a} \partial_a f) d\zeta,$$

where $c = f(0)$. For the detailed derivation of Eq. (14), see Appendix.

Letting $f = E_z$ in Eq. (14) and using Eq. (8) leads to

$$E_z(\zeta_0) = -\frac{1}{2\pi i} \oint_C \frac{a \log(1 - \zeta_0/\zeta)}{\zeta} (\partial_\zeta E_z - \frac{\zeta}{a} \partial_a E_z) d\zeta,$$

We show in the following that the boundary integral term in Eq. (15) vanishes. First, on the boundary $C$, it holds that

$$\partial_a E_z = \frac{\partial E_z}{\partial n} - \frac{\partial E_z}{\partial n} = \frac{\partial E_z}{\partial n},$$

Next, by taking the (x-component) = $i\gamma$(y-component) of Eq. (1) and using the assumption $E^* = 0$, it holds that

$$\partial E_z = -\omega_0 H^*.$$ (17)

Substituting Eqs. (8) and (17) into Eq. (16) yields

$$\partial_a E_z = -\frac{\zeta}{a} \partial_\zeta E_z + \frac{\zeta}{a} \partial_a E_z = \frac{\zeta}{a} E_z + \frac{\zeta}{a} E_z.$$ (16)

By this formula, $\kappa$ is reconstructed using only the measured quantity $H^*$ and its first derivative without any knowledge of $\kappa$, whereas Eq. (12) needs $\kappa$ on the boundary. According to the definition of $\kappa$, conductivity and permittivity maps are computed by simply taking the real and imaginary parts of the reconstructed $\kappa$.

4. Numerical Simulations

4.1 Models and Conditions

To verify our proposed method based on Eq. (25), we conducted a series of simulations. We compare the proposed method with our previous method based on Eq. (12). We examined three cases: reconstruction of $\kappa$ by Eq. (12) with the...
true boundary EP values, reconstruction of \( x \) by Eq. (12) with the average EP values on the boundary for the most “desirable” case, and reconstruction using Eq. (25) without any boundary EP values.

As shown in Fig. 1, a 16-leg high-pass shielded birdcage coil with a diameter of 240 mm and height of 270 mm was constructed and a cylindrical phantom with a diameter of 144 mm and height of 270 mm was loaded into the birdcage coil. The optimal capacitance value of the birdcage coil was determined according to the method proposed by Gurler et al. [34]. The magnetic field at 123.2 MHz (corresponding to 2.89 T MRI system) was excited in a quadrature excitation mode in which the coil was driven by two ports 90° apart from each other and with a 90° phase difference. \( H^* \) was obtained on a 180 mm \( \times \) 180 mm square region centered at the origin in the central slice \((z = 0)\), which we call the \( \xi-\eta \) plane hereafter, with a matrix size of 128 \( \times \) 128 (a resolution of about 1.4 mm \( \times \) 1.4 mm).

Two models (model 1 in Fig. 2 (a) and model 2 in Fig. 2 (b)) were constructed as loading objects. Both models were composed of normal tissue regions (region 2) and malignant tissue regions (region 1). Table 1 shows the EP values in each region. EP values are constant in region 1, whereas they linearly increase along the \( x \)-axis in region 2. The conductivity and the relative permittivity in region 2 are given as follows:

\[
\sigma(x) = 0.55 + \frac{0.05}{0.072^2} x \quad (-0.072 \leq x \leq 0.072),
\]
\[
\epsilon_r(x) = 75 + \frac{5}{0.072} x \quad (-0.072 \leq x \leq 0.072).
\]

Note that both models have homogeneous EP values along the \( z \)-axis. The computation of \( H^* \) was performed using the FEM software, COMSOL Multiphysics 5.2a (COMSOL Inc.) and MATLAB R2013a (The MathWorks Inc.) was used for all reconstruction processes.

To calculate \( \partial H^* \), we used the Savitzky–Golay filter for 2D image data [35]. \( H^* \) in the region \( A_{ij} \), neighboring a pixel at \((i, j)\), is approximated as the following second-order polynomial:

\[
H^*_{ij} = c_0 + c_{10} \xi_{ij} + c_{01} \eta_{ij} + c_{20} \xi_{ij}^2 + c_{11} \xi_{ij} \eta_{ij} + c_{02} \eta_{ij}^2.
\]

We take \( A_{ij} \) as the 5 px \( \times \) 5 px square region centered at \((i, j)\). By the 25 equations for each pixel in \( A_{ij} \), 6 coefficients are determined in the least square sense. Then, spatial derivatives of \( H^* \) were calculated as \( \partial_\xi H^*_{ij} = c_{10} + 2c_{20} \xi_{ij} + c_{11} \eta_{ij} \) and \( \partial_\eta H^*_{ij} = c_{01} + 2c_{02} \eta_{ij} + c_{11} \xi_{ij} \) and \( \partial^2 H^*_{ij} = (\partial_\xi H^*_{ij} - i \partial_\eta H^*_{ij})/2 \). The ROI was set as the circular region centered at the origin with a diameter of 135 mm for both models.

The integral terms were calculated with the following linear approximation in polar coordinates:

\[
\int_C f(\zeta) d\zeta \approx \sum_{i=0}^{L-1} \sum_{j=0}^{K-1} f(a_i, b_j) i a_i e^{i \psi} \Delta \theta,
\]
\[
\int_D f(\zeta) d\zeta d\eta \approx \sum_{i=0}^{L-1} \sum_{j=0}^{K-1} f(p_i, q_j) p_i q_j \Delta \theta \Delta \phi,
\]

where \( \Delta \theta = 2\pi/L, \Delta \phi = \phi/20, \theta_i = l \Delta \phi, \) and \( \phi_k = k \Delta \phi \). We take \( \Delta \phi \) and \( \Delta \theta \) as 1.4 mm and 2\pi/192 rad respectively. \( H^* \) and \( \partial H^* \) values at the nodes \((p_k, q_l)\) were calculated by bi-linear interpolation from Cartesian gridded data.

To test the stability of the proposed method, Gaussian noise was added to both the magnitude and phase of \( H^* \) in model 2. In this case, the Gaussian filter with a standard deviation of 1.4 mm (size of 5 px) was first applied to simulated noisy \( H^* \) data.

### 4.2 Result

By the simulation using model 1, we obtained the magnitude and phase of \( H^* \) shown in Fig. 3. As the model was loaded inside the birdcage coil, \( H^* \) was distorted, rather than being homogeneous when nothing was loaded. Figure 4 (a) shows the...
magnitude of $\partial H^+$ calculated from $H^+$ obtained by simulation. It was observed that $\partial H^+$ falls to zero at $z^* = 0 + 0i$ (center of the phantom). Figure 4 (b) is the magnitude of $E_z$ calculated from $\partial H^+$ obtained by simulation. It was observed that $\partial H^+$ falls to zero at $z^* = 0 + 0i$ (center of the phantom). Figure 4 (b) is the magnitude of $E_z$ estimated by the Neumann-type integral formula. As mentioned in Section 3, the constant $c$ in Eq. (20) is uniquely determined such that $E_z$ gets zero at $z^*$. Finally, EPs are reconstructed as the ratio of $\partial H^+$ and $E_z$ according to Eq. (7).

Figure 5 shows the reconstruction results of model 1. The Dirichlet-type integral formula reconstructs EPs correctly outside of the region where $|\partial H^+|$ and $|E_z|$ are low when the true boundary condition is given. As mentioned in Section 2.1, this spot-like artifact occurs in regions where $|\partial H^+|$ and $|E_z|$ become zero and most state-of-the-art methods using single channel birdcage coils have the same problem [19],[20],[29],[31],[32]. The method of displacing this spot-like artifact by inserting a dielectric pad was proposed in [29] and is applicable to both our previous and proposed methods. In Fig. 5 (b), some errors are also observed in region 1. This is due to the assumption that $\partial H^+ \approx 0$. In [32], we have already proposed an iterative method to correct the error in this assumption. We confirmed that a single iteration significantly suppressed the error in Fig. 5 (b).

If the boundary conditions are given as the average values, the spot-like artifact spreads around the center and the EP values are distorted over all of region 2. Conversely, our proposed method obtained good results for both conductivity and relative permittivity, although no information of boundary EP values are given. The conductivity result has the same quality as the result of the Dirichlet-type integral formula with the true boundary condition. The relative permittivity is also reconstructed better than the Dirichlet-type integral formula with the average boundary condition. Note that the range of Fig. 5 is restricted for visibility, although the values are out of range near the center. Figure 6 shows plots of the reconstructed conductivity and relative permittivity at the line $\eta = 0$ and more clearly shows the validity of our proposed method. The result at the low-electric field point is better than that of the Dirichlet-type integral formula, and the effect of artifacts is suppressed near this region.

We calculate the relative error of the reconstructed conductivity and relative permittivity by the $L^2$-norm as follows:

$$E(\chi) \equiv 100 \left( \frac{\sum_k \sum_l \left( \chi_t(\rho_k, \theta_l) - \chi(\rho_k, \theta_l) \right)^2 \rho_k \Delta \rho \Delta \theta}{\sum_k \sum_l \chi_t(\rho_k, \theta_l)^2 \rho_k \Delta \rho \Delta \theta} \right)^{1/2},$$

(31)

where $\chi$ is $\sigma$ or $\epsilon_r$, and $\chi_t$ is the true value of $\sigma$ or $\epsilon_r$, respectively. Note that we have excluded the point at which $E_z = 0$ when calculating Eq. (31) because we could not determine the conductivity and permittivity values there. For the result of Eq. (12) under the true boundary condition, $E(\sigma)$ and $E(\epsilon_r)$ are
Fig. 7 Reconstruction results for model 2 when 1% Gaussian noise was added. Upper row is for conductivities (S/m) and lower row is for relative permittivities. (a) True, (b) reconstructed by the Dirichlet-type integral formula with the true boundary condition, (c) reconstructed by the Dirichlet-type integral formula with the average boundary condition, (d) reconstructed by the Neumann-type integral formula without any knowledge of the boundary EP values.

Fig. 8 $L^2$ error ratio of the reconstructed (a) conductivity and (b) relative permittivity of model 2 when 0%–2% Gaussian noise is added. The relative error of the proposed method is lower than that of the Dirichlet-type integral formula with average boundary values.

5.0% and 3.8%, respectively. When the average boundary values are given, the errors increase to 10.3% and 12.8%, respectively, whereas the errors of our proposed method are 6.6% and 8.3%.

The error in the proposed method outside of the low-electric field region is mainly due to the assumption that $H^- \approx 0$. We confirmed that the result of the proposed method improves when we obtained $H^-$ in addition to $H^+$ from the simulation and used both to give the normal derivative of $E_z$ based on Eq. (18). The method to obtain both $H^+$ and $H^-$ by multichannel surface coils in addition to the quadrature birdcage coil is referred to in [14]. Obtaining $H^-$ by this method and using it to correct the Neumann boundary condition will improve the result of our proposed method.

Figure 7 shows the reconstruction results of model 2 when 1% Gaussian noise is added. Even with noisy data, three malignant tissue regions can be distinguished when the true boundary condition is given. If the boundary condition is given as an average value, the spot-like artifact spreads as in the case of model 1. The proposed method still reconstructs EPs as well as the Dirichlet-type integral formula with the true boundary condition.

We also calculate the ratio of relative errors for model 2 when 0%–2% Gaussian noise is added. This noise ratio is reasonable for regular MRI measurements [29]. As shown in Fig. 8, the relative error of our proposed method is lower than that in the case where the average boundary condition is given through the entire case where 0%–2% Gaussian noise was added. The proposed method has a higher relative error ratio than the case where the true boundary condition is given. However, it is difficult to give the true EP values in the practical case. In this paper, we compared the proposed method with the case where boundary EP values different from the true ones were given as an estimate and verified that our proposed method yielded better results. It can be said that the proposed method, which does not require boundary EP values and thus does not depend upon the precision of the estimation of these values, is important for practical applications.

5. Conclusion

In this paper, we proposed a novel method for MREPT which does not require the EP values on the boundary of the ROI. In this method, starting from the integral formula that solves the D-bar equation in a circular region with the Neumann boundary condition, the contour integral term of the integral formula is eliminated using Faraday’s law and the PDE is solved from only magnetic field data measured by using MRI. Numerical simulations showed that our proposed method keeps a good reconstruction result without any knowledge of EP values on the boundary of the ROI, thereby improving the applicability of our method to practical situations.

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References


Appendix  Derivation of the Neumann-Type Integral Formula in a Circular Region

In this appendix, we derive the Neumann-type integral formula in a circle \( D \) with a radius of a centered at the origin according to [33], in which the formula for a unit disc is given.

First, we treat an analytic function \( f \) on \( D \) which is given. According to [33], in which the formula for a unit disc is given.

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\[ \partial_a f = \partial f = \frac{\partial f}{\partial \zeta} \partial \zeta + \frac{\partial f}{\partial \bar{\zeta}} \partial \bar{\zeta} = e^{\theta} \partial f + e^{-\theta} \partial \bar{f} = \zeta \partial f + \bar{\zeta} \partial \bar{f} = \zeta \partial f, \quad (A.1) \]
due to the analyticity of \( f \). Substituting Eq. (A.1) into Cauchy’s integral formula for analytic functions yields
\[ \frac{\zeta_0}{a} \bar{\partial} f(\zeta_0) = \frac{1}{2\pi i} \oint_C \frac{\partial f}{\zeta - \zeta_0} d\zeta. \quad (A.2) \]

Especially when \( \zeta_0 = 0 \), it holds that
\[ 0 = \frac{1}{2\pi i} \oint_C \frac{\partial f}{\zeta} d\zeta. \quad (A.3) \]

Subtracting Eq. (A.3) from Eq. (A.2), we obtain
\[ \frac{\zeta_0}{a} \bar{\partial} f(\zeta_0) = \frac{1}{2\pi i} \oint_C \frac{\zeta_0}{\zeta(\zeta - \zeta_0)} \partial_a f d\zeta. \quad (A.4) \]

By dividing \( \zeta_0/a \) and integrating both sides with respect to \( \zeta_0 \), we have
\[ f(\zeta_0) = -\frac{1}{2\pi i} \oint_C \frac{a \log (1 - \zeta_0/\zeta)}{\zeta} \partial_a f d\zeta + c, \quad (A.5) \]
where \( c = f(0) \).

Next, we extend Eq. (A.5) to general functions that are not necessarily analytic. According to [33], we define two integral operators \( T \) and \( \Pi \) as follows:
\[ Tf(\zeta) = -\frac{1}{\pi} \int_D \frac{f(\zeta)}{\zeta - \zeta'} d\zeta d\eta, \quad (A.6) \]
\[ \Pi f(\zeta) = -\frac{1}{\pi} \int_D \frac{f(\zeta)}{\zeta' - \zeta^2} d\zeta d\eta. \quad (A.7) \]

These operators have the properties \( \partial T f = \Pi f \) and \( \partial \bar{T} f = f \) as proved in [33]. Now, let us define the function \( \phi = f - T \bar{f} \).
Then, by the properties of the operators, they hold that \( \bar{\partial} \phi = \bar{\partial} f - \bar{\partial} \bar{f} = 0 \) and \( \bar{\partial}_a \phi = \bar{\partial}_a f - (\zeta/a) \Pi \bar{f} - (\bar{\zeta}/a) \bar{\partial} f \).
Since \( \phi \) is analytic, we can substitute \( \phi \) into Eq. (A.5) to obtain
\[ f(\zeta_0) - T \bar{f} = \frac{1}{2\pi i} \oint_C \frac{a \log (1 - \zeta_0/\zeta)}{\zeta} (\bar{\partial}_a f - \frac{\zeta_0}{a} \Pi \bar{f} - \frac{\bar{\zeta}}{a} \bar{\partial} f) d\zeta + f(0) - T \bar{f}(0). \quad (A.8) \]

However, the second contour integral term vanishes as follows:
\[ \frac{1}{2\pi i} \oint_C \log (1 - \zeta_0/\zeta) \left( -\frac{\zeta_0}{a} \Pi \bar{f} \right) d\zeta = \frac{1}{2\pi i} \oint_C \log (1 - \zeta_0/\zeta^2) \left( \frac{1}{\zeta - \zeta'} \bar{f} \right) d\zeta d\eta = \frac{1}{\pi} \oint_D \left( \frac{1}{2\pi i} \oint_C \log (1 - \zeta_0/\zeta^2) \frac{d\zeta}{(\zeta - \zeta')^2} \right) \bar{f} d\zeta d\eta = 0. \quad (A.9) \]
The last equation holds because the integrand has no singularity in \( D \) and thus is anti-analytic there. Finally, we have
\[ f(\zeta_0) = -\frac{1}{2\pi i} \oint_C \frac{a \log (1 - \zeta_0/\zeta)}{\zeta} (\bar{\partial}_a f - \frac{\zeta_0}{a} \bar{\partial} f) d\zeta - \frac{1}{\pi} \int_D \frac{\zeta_0}{\zeta - \zeta_0} \bar{f} d\zeta d\eta + c, \quad (A.10) \]
where \( c = f(0) \). This is the Neumann-type integral formula in a circular region given by Eq. (14) in Section 3.