Simultaneous Design of Discrete-Time Observer-Based Robust Scaled-$H_\infty$ Controllers and Scaling Matrices

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Abstract: This paper addresses the simultaneous design of observer-based robust scaled-$H_\infty$ controllers and scaling matrices for discrete-time linear time-invariant polytopic systems. Based on the so-called dilated linear matrix inequality approach, we propose a design method for our problem with a structural constraint for the decision matrix which is introduced in “dilation” procedure. The structural constraint inevitably introduces conservatism; however, numerical examples illustrate the effectiveness of the simultaneous design of controller matrices, i.e. observer gain, state-feedback gain, the state transition matrix of the observer, and the input or the output matrix of the observer, as well as the scaling matrix.

Key Words: discrete-time system, observer-based output feedback controller, scaled-$H_\infty$ control, polytopic system, D-scaling.

1. Introduction

It is well known that $H_\infty$ controllers are very effective for controlling systems which contain bounded uncertainties, and many applications of $H_\infty$ controllers have been reported. For example, attitude control of spacecraft [1], compact disc player [2], flight controller for helicopter [3], etc. This is because $H_\infty$ controllers can have a certain amount of robustness against the uncertainties given in the frequency domain. To this end, mixed-sensitivity synthesis, e.g. [4], is widely applied. However, it is not so easy to design appropriate $H_\infty$ controllers with several design requirements, e.g. disturbance suppression, precise tracking, prevention of high frequency component input command, etc., in the framework of mixed-sensitivity synthesis. To accommodate this difficulty, multiple uncertainty blocks are introduced to represent those design requirements. However, if $H_\infty$ controllers are designed without consideration of the multiple uncertainty blocks; that is, if $H_\infty$ controllers are designed under the supposition that the multiple uncertainty blocks are considered as a single unstructured uncertainty block, conservative controllers are then obtained. As a remedy for tackling the conservatism due to the multiple uncertainty blocks, scaled-$H_\infty$ controller design and $\mu$-synthesis have been proposed, and the effectiveness of these methods have also been well illustrated. (For example, see [5],[6], etc.) However, numerical complexity is increased as an exchange for conservatism reduction. That is, scaled-$H_\infty$ controller design and $\mu$-synthesis are both formulated in terms of bilinear matrix inequalities (BMIs) and an iterative algorithm, so-called $D-K$ iteration, is applied to obtain locally optimal controllers. This is a sharp contrast to the simple formulation of $H_\infty$ controller design, which is formulated in terms of linear matrix inequalities (LMIs) [7]–[10].

If the plant model is given as a single continuous-time linear time-invariant (LTI) model with some mild constraints, a design method in [11], in which continuous-time observer-based scaled-$H_\infty$ controllers and so-called “D-scaling” matrices [5] are simultaneously optimized, can be applied. The design method has a single line search parameter which is necessarily introduced when “dilation” procedure [12]–[16] is applied to keep the equivalence between the original LMI condition and the dilated LMI condition. Though, this line search correspondingly increases the numerical complexity. In contrast to continuous-time case, dilation procedure for discrete-time systems

1 The obtained LMIs are originally called as “extended LMIs” [13].
required to design the input matrix of the observer, as demonstrated in [17]; however, in our method, we design the input or the output matrix of the observer without using any iterative algorithms.

We use the following notations in this paper: \( \mathbf{I} \) and \( \mathbf{0} \) respectively denote an identity matrix and a zero matrix of appropriate dimensions, \( \mathbb{R}^n, \mathbb{R}^{n \times m}, \mathbb{S}^n \), \( C \) and \( \mathbb{C}^{n \times m} \) respectively denote the sets of \( n \)-dimensional real vectors, \( n \times m \)-dimensional real matrices, \( n \times n \)-dimensional symmetric real matrices, complex numbers and \( n \times m \)-dimensional complex matrices, \( X^T \) for matrix \( X \) denotes the transpose of matrix \( X \), \( x \) denotes an abbreviated off-diagonal block in a symmetric matrix, \( \text{He}(X) \) denotes \( X + X^T \), and \( \text{diag}(X_1, \ldots, X_k) \) denotes a block-diagonal matrix composed of \( X_1, \ldots, X_k \). For the state vectors of discrete-time systems, superscript "\( * \)" denotes the state vector at the next step; that is, \( x^* \) denotes \( x(k+1) \).

The remainder is composed as follows: Section 2 defines plant system, the controller to be designed, our addressed problem, and then several related lemmas are shown; Section 3 gives our proposed method and then shows numerical examples to illustrate the effectiveness of our method; and finally concluding remarks are given in Section 4. In the appendix, our proposition for the system which has the dual form of the system addressed in the main body is given; however, the effectiveness will be illustrated in Section 3.

2. Preliminaries

2.1 Plant System

Let us consider the following discrete-time polytopic system:

\[
G(\xi) : \begin{bmatrix} x^T \\ y \end{bmatrix} = \begin{bmatrix} A(\xi) & B_1(\xi) & B_2(\xi) \\ C_1(\xi) & D_{11}(\xi) & D_{12}(\xi) \end{bmatrix} \begin{bmatrix} x \\ w \\ u \end{bmatrix},
\]

where \( x \in \mathbb{R}^n, z \in \mathbb{R}^n, y \in \mathbb{R}^n, w \in \mathbb{R}^n \) and \( u \in \mathbb{R}^m \) respectively denote the state with \( \mathbf{0} \) as its initial value, the performance output, the measurement output, the external input and the control input. All matrices in (1) are supposed to be time-invariant and to have compatible dimensions; however, matrices \( A(\xi), C_1(\xi) \), etc. are parametrically affine and given as:

\[
\begin{bmatrix} A(\xi) & B_1(\xi) & B_2(\xi) \\ C_1(\xi) & D_{11}(\xi) & D_{12}(\xi) \end{bmatrix} = \sum_{i=1}^{N} \xi_i \begin{bmatrix} A_i & B_{1i} & B_{2i} \\ C_i & D_{11i} & D_{12i} \end{bmatrix},
\]

with \( \xi = [\xi_1, \ldots, \xi_N]^T \) and compatibly dimensioned constant matrices \( A_i, C_i, \) etc.

The vector \( \xi \) is supposed to belong to the unit simplex:

\[
\Xi = \left\{ \xi \in \mathbb{R}^N : \xi_i \geq 0, \sum_{i=1}^{N} \xi_i = 1 \right\}.
\]

The following assumption is made for the matrix \( C_2 \) in (1), similarly to [13],[18].

**Assumption 1** The matrix \( C_2 \) is row full rank and is given as

\[
C_2 = \begin{bmatrix} \tilde{C}_2 & \mathbf{0} \end{bmatrix},
\]

where \( \tilde{C}_2 \in \mathbb{R}^{n \times k} \) is non-singular.

This assumption is easily satisfied by the following procedures.

1. Eliminate the redundant measurements which reflect the state variables, and
2. apply state transformation \( x' = Tx \) with appropriately defined nonsingular matrix \( T \).

Therefore, in practice, Assumption 1 does not introduce any restrictions at all.

The external input and the performance output are supposed to be structured and to satisfy the following assumption.

**Assumption 2**

\[
w = [w_1^T, \ldots, w_m^T]^T, w_i \in \mathbb{R}^{n_i}, \]

\[
z = [z_1^T, \ldots, z_n^T]^T, z_i \in \mathbb{R}^{n_i},
\]

\[
n_p := \sum_{i=1}^{m} n_{p_i} = n_w = n_z.
\]

This assumption can be easily satisfied by adding fictitious zero signals to \( w \) and/or \( z \) appropriately. This remedy has no effect on \( H_n \) norm, because adding zero signals to \( w \) and/or \( z \) corresponds to adding zero columns and/or rows to the original transfer matrix function.

The uncertainty block is defined as follows:

\[
\Delta = \text{diag}(\Delta_1, \ldots, \Delta_m), \Delta_i \in \mathbb{C}^{n_i \times n_i}.
\]

The set of constant scaling matrices, so-called "D-scaling" matrices, is correspondingly defined:

\[
L^\Delta := \{ S \in \mathbb{S}^n : S > 0, \Delta S = S \Delta \}.
\]

2.2 Controller

We now define an observer-based output feedback controller as follows:

\[
C : \begin{cases} \dot{x}^* = A_0 \hat{x} + B_0 u - L (y - C_2 \hat{x} - D_{22} u), \\ u = K \hat{x}, \end{cases}
\]

where \( \hat{x} \) denotes the estimated state of \( G(\xi) \) with its initial value being \( 0, A_0 \in \mathbb{R}^{n \times n}, B_0 \in \mathbb{R}^{n \times m}, L \in \mathbb{R}^{n \times n} \) and \( K \in \mathbb{R}^{n \times m} \), which are all to be designed, respectively denote the state transition matrix of the observer, the input matrix of the observer, the observer gain and the state-feedback gain.

In contrast to the case in which the plant is supposed to be given as a single LTI system, e.g. [11], the state transition matrix and the input matrix are both set to be designed, since there is no guarantee that the nominal state transition matrix and the nominal input matrix of \( G(\xi) \) give the best performance.

The observer-based output feedback controller \( C \) is equivalently expressed as \( \tilde{C} \):

\[
\tilde{C} : \begin{cases} \dot{x}^* = (A_0 + B_0 K) \hat{x} + L C_2 \hat{x} - L (y - D_{22} u), \\ u = K \hat{x}. \end{cases}
\]

This transformation indicates that we have no need to design \( A_0 \) and \( B_0 \) independently, and only have to design \( A_0 + B_0 K \) as a single matrix. This property plays a key role for our proposition.

2.3 Closed-Loop System

The closed-loop system comprising \( G(\xi) \) and \( C \) is given below:
\[ G_{c}\xi(\xi) : \begin{bmatrix} x_{d} \\
 z \end{bmatrix} = \begin{bmatrix} A_{c}(\xi) & B_{c}(\xi) \\
 C_{c}(\xi) & D_{c}(\xi) \end{bmatrix} \begin{bmatrix} x_{d} \\
 w \end{bmatrix}, \]

where \( x_{d} = [x^{T} - \hat{x}^{T} \hat{y}^{T}]^{T} \) denotes the state of \( G_{c}\xi(\xi) \), and the matrices \( A_{c}(\xi) \), etc. are calculated as follows:

\[
\begin{bmatrix}
A_{c}(\xi)B_{c}(\xi) \\
C_{c}(\xi)D_{c}(\xi)
\end{bmatrix} = \begin{bmatrix}
A(\xi) + LC_{2} \hat{A}(\xi) & -A_{0} + (B_{2}(\xi) - B_{0}) K' B_{1}(\xi) \\
-\hat{C}_{1}(\xi) - C_{0} \hat{A}(\xi) + D_{12}(\xi)K' & -D_{11}(\hat{\xi})
\end{bmatrix}.
\]

The state-space matrices are parametrically affine with respect to \( \xi \).

### 2.4 Problem Definition

We are ready to define our problem.

**Problem 1** For given \( \gamma > 0 \), find an observer-based robust controller \( C \) defined in (7) and a scaling matrix \( S \in \mathbb{L}^{n_{p}} \) such that the closed-loop system \( G_{c}\xi(\xi) \) is stabilized and satisfies (10) for all possible \( \xi \in \Xi \).

\[
\|S^{-1/2}G_{c}(s;\xi)S^{1/2}\|_{\infty} < \gamma. \tag{10}
\]

where \( G_{c}(s;\xi) \) denotes the transfer matrix function of \( G(\xi) \).

This is robust scaled-\( H_{\infty} \) controller design with the controller being set as an observer-based output feedback controller.

In this paper, the parameters \( \hat{\xi} \)'s are supposed to be time-invariant. If the variation bounds are bounded and \textit{a priori} known, then the extension to the case is possible. However, the performance index (10) should be correspondingly revised to accommodate the variations of parameters. For simplicity, this paper thus focuses on the time-invariant parameter case only.

### 2.5 Related Lemmas

First, we show a well-known result which characterizes the stability and (10) for \( G_{c}\xi(\xi) \).

**Lemma 1** (cf. [19]) The closed-loop system \( G_{c}\xi(\xi) \) is stable and satisfies (10) if and only if the following condition holds for given positive \( \gamma \):

- \( \exists P(\xi) \in \mathbb{S}^{2n}, \ S \in \mathbb{L}^{n_{p}} \) s.t.

\[
\begin{bmatrix}
P(\xi) & 0 \\
0 & S^{-1}
\end{bmatrix} \begin{bmatrix}
P(\xi)A_{c}(\xi) & P(\xi)B_{c}(\xi) \\
0 & S^{-1}
\end{bmatrix} > 0, \ \forall \xi \in \Xi. \tag{11}
\]

This is a version of bounded real lemma for discrete-time systems using parameter-dependent Lyapunov matrix \( P(\xi) \) with constant scaling matrix \( S \) being incorporated.

We now show three conditions equivalent to the condition in Lemma 1.

**Lemma 2** For given \( \gamma > 0 \), the following three conditions are equivalent to the condition in Lemma 1.

i) \( \exists X(\xi) \in \mathbb{S}^{2n}, \ S \in \mathbb{L}^{n_{p}} \) s.t.

\[
\begin{bmatrix}
X(\xi) & 0 \\
0 & S^{-1}
\end{bmatrix} \begin{bmatrix}
P(\xi)A_{c}(\xi) & P(\xi)B_{c}(\xi) \\
0 & S^{-1}
\end{bmatrix} > 0, \ \forall \xi \in \Xi. \tag{12}
\]

ii) \( \exists P(\xi) \in \mathbb{S}^{2n}, \ Q(\xi) \in \mathbb{R}^{2n \times 2n}, \ S \in \mathbb{L}^{n_{p}} \) s.t.

\[
\begin{bmatrix}
P(\xi) & 0 \\
0 & S^{-1}
\end{bmatrix} \begin{bmatrix}
P(\xi)A_{c}(\xi) & P(\xi)B_{c}(\xi) \\
0 & S^{-1}
\end{bmatrix} > 0, \ \forall \xi \in \Xi. \tag{13}
\]

iii) \( \exists X(\xi) \in \mathbb{S}^{2n}, \ Q(\xi) \in \mathbb{R}^{2n \times 2n}, \ S \in \mathbb{L}^{n_{p}} \) s.t.

\[
\begin{bmatrix}
-X(\xi) & 0 \\
0 & X(\xi)
\end{bmatrix} > 0, \ \forall \xi \in \Xi. \tag{14}
\]

The proof for robust \( H_{\infty} \) performance without scaling matrices has been presented in literature, i.e. [13],[16]; however, for completeness, a brief proof is given below.

**Proof** The equivalence between the condition in Lemma 1 and the condition i) can be easily confirmed after some algebraic manipulations with setting \( X(\xi) = P(\xi)^{-1} \).

The equivalence between the condition in Lemma 1 and the condition ii) is next addressed. By using Elimination Lemma [20], the condition ii) is equivalent to the following:

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
\Psi(\xi)_{1,3} & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix} > 0, \ \forall \xi \in \Xi. \tag{15}
\]

Here, \( \Psi(\xi)_{1,3} \) denotes the sum of the 1st and 3rd terms of the left-hand side of (13). The first inequality in (15) is given as follows:

\[
\begin{bmatrix}
P(\xi) & 0 \\
0 & S^{-1}
\end{bmatrix} \begin{bmatrix}
P(\xi)A_{c}(\xi) & P(\xi)B_{c}(\xi) \\
0 & S^{-1}
\end{bmatrix} > 0, \ \forall \xi \in \Xi. \tag{16}
\]

This inequality can be also derived by multiplying \( \begin{bmatrix} 1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} \) and its transpose from left and right, respectively, to (11).
Furthermore, the second inequality in (15) can be equivalently transformed to (11) after some algebraic manipulations and Schur complement. Therefore, the equivalence between the condition in Lemma 1 and the condition ii) is proved.

The equivalence between the conditions i) and iii) can be easily proved by using Elimination Lemma, similarly to the above argument.

□

In the next section, we show our proposition for Problem 1 using the condition iii) in Lemma 2. The condition ii) will be used in the appendix.

3. Main Results

3.1 Proposed Method

Our proposed method for Problem 1 is given below.

**Theorem 1** For given $\gamma > 0$, if there exist a parametrically affine matrix $X(\xi) \in \mathbb{S}^p$, and constant matrices $Y \in \mathbb{R}^{n_1}$, $L \in \mathbb{R}^{n \times n}$, $K \in \mathbb{R}^{p \times n}$, $Q$ structured as in (17) and $S \in \mathbb{L}^{n \times n}$ such that (16) holds for all vertices of $\Xi$, then the observer-based robust controller $\hat{C}$ with $A_0 + \lambda K = YQ^-1, L = LQ^-1C^-1$ and $K = KQ^-1$ stabilizes the closed-loop system $G_d(\xi)$ and satisfies (10) for all possible $\xi \in \Xi$ with $S = S$.

\[
\begin{bmatrix}
-X(\xi) & 0 & 0 \\
0 & X(\xi) & 0 \\
0 & 0 & \gamma^2 S
\end{bmatrix} + \text{He} \begin{bmatrix}
Q & \gamma^2 \mathcal{L}(\xi) & \mathcal{L}(\xi) \\
\gamma^2 \mathcal{L}(\xi) & \mathcal{L}(\xi) & 0 \\
\mathcal{L}(\xi) & 0 & 0
\end{bmatrix} 
= 0,
\]

(16)

where $\mathcal{L}(\xi)$ and $\mathcal{L}(\xi)$ are defined as follows:

\[
\mathcal{L}(\xi) = \begin{bmatrix}
A_0(\xi)Q_1 + [L \ 0] A_0(\xi)Q_2 + B_2(\xi)K - Y & Y \\
-L & 0
\end{bmatrix},
\]

\[
\mathcal{L}(\xi) = \begin{bmatrix}
C_0(\xi)Q_1 & C_0(\xi)Q_2 + D_1(\xi)K
\end{bmatrix}.
\]

The matrix $Q$ is structured as follows:

\[
Q = \begin{bmatrix}
Q_{11} & 0 \\
Q_{21} & Q_{22}
\end{bmatrix},
\]

(17)

where $Q_{11} \in \mathbb{R}^{n \times n}$, $Q_{21} \in \mathbb{R}^{(n-n_1) \times n}$, $Q_{22} \in \mathbb{R}^{(n-n_1) \times (n-n_1)}$, and $Q_2 \in \mathbb{R}^{n \times p}$.

**Proof** First, note that the inequality (16) is a quadratically parameter-dependent; however, due to the positivity of $S$, if the inequality (16) holds for all vertices of $\Xi$ then the inequality holds for all $\xi \in \Xi$, and vice versa. This is so-called “multi-convexity” [21]. The remaining proof is straightforward by noting the change-of-variables, $L = \mathcal{L}\xi Q_{11}$, $K = KQ_2$, and $Y = (A_0 + \lambda K)Q_2$. The non-singularity of matrix $Q$ is confirmed from the (1,1) and (2,2) blocks of (16); that is, $\text{He} \{ Q \} > X(\xi)$ holds from $\text{He} \{ Q \} - X(\xi) > 0$ in the (1,1) block and $X(\xi) > 0$ holds from $X(\xi) - [B_1(\xi)^T \ 0] S [B_1(\xi)^T \ 0] > 0$ in the (2,2) block.

□

**Remark 1** In order to obtain trustworthy solutions, it is recommended to set a single decision variable to be unity. For example, set the (1,1) block in $S$ as an identity matrix. This revision does not change the feasibility of the condition because the multiplication of any positive scalar has no effect on the feasibility of (16) and this multiplication can set the (1,1) block in $S$ as an identity matrix.

3.2 Numerical Examples

We consider a discrete-time linear time-invariant system whose state-space matrices are given below:

\[
\begin{bmatrix}
A_0 & B_1 & B_2 \\
C_1, D_{11} & D_{12} \\
C_2, D_{21} & D_{22}
\end{bmatrix} = \begin{bmatrix}
0.8 & -1 & 0 & 1 & 1 \\
1 & 0.2 & 1 & 1 & 0 \\
0 & 1 & 1 & 0.2 & 1 \\
0 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 1
\end{bmatrix}.
\]

(18)

Using this system, we set two polytopic systems $G_1(\xi)$ and $G_2(\xi)$ with $N = 2$.

We first define $G(\xi)$ in (1) with the following state-space matrices:

\[
\begin{bmatrix}
A_1 & B_1 & B_2, C_1, D_{11} & D_{12} \\
A_2, D_{11} & D_{12} \\
C_2, D_{21} & D_{22}
\end{bmatrix} \begin{bmatrix}
0.9 \times A_0, B_1, B_2, C_1, D_{11}, D_{12} \\
1.1 \times A_0, B_1, B_2, C_1, D_{11}, D_{12}
\end{bmatrix}.
\]

(19)

We refer to this system as $G_1(\xi)$.

We next define $G(\xi)$ in (A.1) with the following state-space matrices:

\[
\begin{bmatrix}
A_1 & B_1 & B_2, C_1, D_{11} & D_{12} \\
A_2, D_{11} & D_{12} \\
C_2, D_{21} & D_{22}
\end{bmatrix} \begin{bmatrix}
0.9 \times A_0, B_1, B_2, C_1, D_{11}, D_{12} \\
1.1 \times A_0, B_1, B_2, C_1, D_{11}, D_{12}
\end{bmatrix}.
\]

(20)

We refer to this system as $G_2(\xi)$.

Note that the both vertex models are unstable; that is, the spectral radii of $0.9 \times A_0$ and $1.1 \times A_0$ are over the unity.

For these systems, three scenarios are considered as the uncertainty model.

**Scalar block** $\Delta = \delta_1 I_2$, $\delta_1 \in \mathbb{C}$

**Diagonal block** $\Delta = \text{diag}(\delta_1, \delta_2)$, $\delta_1, \delta_2 \in \mathbb{C}$

**Full block** $\Delta = \delta_1 I_2$, $\delta_1 \in \mathbb{C}^{2 \times 2}$

We design observer-based robust scaled-$H_{\infty}$ controllers for $G_1(\xi)$ and $G_2(\xi)$ respectively by using Theorem 1 and Theorem 2. (The latter is given in the appendix.)

We first design observer-based robust scaled-$H_{\infty}$ controllers $\hat{C}$ for $G_1(\xi)$ by using Theorem 1. The results are summarized in Table 1, which is at the top of the next page. The results of a posteriori analysis using the condition iii) in Lemma 2 with constant $Q$ are also given to confirm the conservatism due to the structural constraint for $Q$ in (17). In the table, for reference, $A_0 + B_2, K$ with the designed $K$ is also given.

For comparison, under the scenario that the uncertainty is supposed to be full block, we design LTI full-order controllers for $G_1(\xi)$ with 2000 gridded fixed $\xi$ by using Matlab® command dhinf1mi. The results are shown in Fig. 1. Note that
The results are only valid for each model with specific \( \xi \), not for polytopic system \( G_1(\xi) \). Thus, the best achievable \( \gamma \) with robust controllers is more than or equal to 25.

We next design observer-based robust scaled-\( H_\infty \) controllers \( \hat{C} \) for \( G_2(\xi) \) by using Theorem 2. The results are summarized in Table 2. The results of a posteriori analysis using the condition ii) in Lemma 2 with constant \( \gamma \) are also given to confirm the conservatism due to the structural constraint for \( Q \) in (A. 8).

For comparison, under the scenario that the uncertainty is supposed to be full block, we design LTI full-order controllers for \( G_2(\xi) \) with 2000 gridded fixed \( \xi \) by using Matlab® command \texttt{hinfsyn}. The results are shown in Fig. 2. Note again that the results are only valid for each model with specific \( \xi \), not for polytopic system \( G_2(\xi) \). Thus, the best achievable \( \gamma \) with robust controllers is more than or equal to 11.674.

Note that the effects of the structural constraints in (17) and (A. 8) appear the gap between the optimal \( \gamma \)'s in design and a posteriori analysis in Tables 1 and 2. Similarly, the combined effect of the structural constraints in (17) and (A. 8) as well as the structured controllers, i.e. observer-based output controllers, appears the gap between the optimal designed \( \gamma \)'s in those tables under full block uncertainty scenario and the worse performances in Figs. 1 and 2.

With these in mind, the comparisons between Table 1 and Fig. 1, and between Table 2 and Fig. 2 indicate the following:

**Conservatism due to structural constraints in \( Q \):** In the scalar block and the full block cases in Table 1, \( \gamma \)'s in synthesis are almost the same as the \( \gamma \)'s in a posteriori analysis, which means that the structural constraint for \( Q \) in (17) hardly introduces conservatism for these cases.

On the contrary, in the diagonal block case in Table 1 and all the cases in Table 2, the structural constraints as in (17) and (A. 8) introduce non-negligible conservatism.

**Effectiveness of simultaneous design:** In Tables 1 and 2, \( \gamma \)'s in synthesis for the scalar block and the diagonal block cases are smaller than those for the full block case, which means that the simultaneous design of controllers and scal-

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### Table 1 Design results for \( G_1(\xi) \) by using Theorem 1, a posteriori analysis using condition iii) in Lemma 2 with constant \( Q \) and \( A_N + B_K \).

<table>
<thead>
<tr>
<th>Block type</th>
<th>( \min \gamma )</th>
<th>( K' )</th>
<th>( L )</th>
<th>( A_N + B_K )</th>
<th>( S )</th>
<th>( \min \gamma )</th>
<th>( A_N + B_K )</th>
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<td>0.840</td>
<td>-0.781</td>
<td>0.616</td>
<td>-0.8</td>
<td>0</td>
</tr>
</tbody>
</table>

---

### Table 2 Design results for \( G_2(\xi) \) by using Theorem 2, a posteriori analysis using condition ii) in Lemma 2 with constant \( Q \) and \( A_N + LC_\omega \).

<table>
<thead>
<tr>
<th>Block type</th>
<th>( \min \gamma )</th>
<th>( K' )</th>
<th>( L )</th>
<th>( A_N + LC_\omega )</th>
<th>( S^{-1} )</th>
<th>( \min \gamma )</th>
<th>( A_N + LC_\omega )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Scalar</td>
<td>17.587</td>
<td>-0.344</td>
<td>-0.716</td>
<td>0.010</td>
<td>0.384</td>
<td>[1 0.648 0.648 1.040]</td>
<td>16.381</td>
</tr>
<tr>
<td></td>
<td></td>
<td>-0.187</td>
<td>-0.958</td>
<td>0.120</td>
<td>0.274</td>
<td>0.021</td>
<td>0.577</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1.012</td>
<td>1.357</td>
<td>0.538</td>
<td>0.172</td>
<td>1.021</td>
<td>0.2</td>
</tr>
<tr>
<td>Diagonal</td>
<td>17.587</td>
<td>-0.230</td>
<td>-0.718</td>
<td>-0.025</td>
<td>-0.1035</td>
<td>[1 0 0 0.506]</td>
<td>16.631</td>
</tr>
<tr>
<td></td>
<td></td>
<td>-0.182</td>
<td>-0.941</td>
<td>0.166</td>
<td>0.746</td>
<td>1.021</td>
<td>0.2</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1.061</td>
<td>1.378</td>
<td>0.569</td>
<td>0.218</td>
<td>-0.8</td>
<td>0</td>
</tr>
<tr>
<td>Full</td>
<td>18.375</td>
<td>-0.250</td>
<td>-0.939</td>
<td>-0.189</td>
<td>-0.1048</td>
<td>[1 0 0 1]</td>
<td>17.613</td>
</tr>
<tr>
<td></td>
<td></td>
<td>-0.165</td>
<td>-0.750</td>
<td>0.309</td>
<td>0.238</td>
<td>1.021</td>
<td>0.2</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1.082</td>
<td>1.319</td>
<td>0.514</td>
<td>0.215</td>
<td>-0.8</td>
<td>0</td>
</tr>
</tbody>
</table>
ing matrices are effective to reduce the conservatism if the uncertainty block is structured.

**Overall conservatism:** The combined conservatism due to the structural constraints for $Q$ and the structured controllers, i.e. observer-based controllers, is negligible in the design for $G_1(\xi)$, because for the full block case in Table 1 $\gamma$ in synthesis is the same as the worst performance in Fig. 1. However, the combined conservatism is apparent in the design for $G_2(\xi)$, $\gamma$ for the full block case in Table 2 is much larger than the worst performance in Fig. 2.

Note also that the combined matrices comprising the state transition matrices and the input or the output matrices, i.e. $A_0 + B_0K$ for $G_1(\xi)$ and $A_0 + LC_0$ for $G_2(\xi)$, are different from those using the nominal matrices, i.e. $A_n$ and $B_{2n}$ or $C_{2n}$, and the designed $K$ or $L$. This indicates that the design of the transition matrix and the input or the output matrices reduces conservatism.

In summary, the examples show the effectiveness with respect to the simultaneous design of controllers and scaling matrices, as well as the conservatism due to the structural constraints for the introduced matrices in dilation procedure as well as the controller structure, i.e. observer-based controllers.

### 4. Conclusions

This paper addresses the design problem of discrete-time observer-based robust scaled-$H_{\infty}$ controllers for discrete-time linear time-invariant polytopic systems. Conventional design methods for scaled-$H_{\infty}$ controllers require iterative algorithms, i.e. $D - K$ iteration, to obtain suboptimal solutions. However, numerical inaccuracies sometimes prevent the iteration from proceeding smoothly. Furthermore, observer-based output feedback controllers are attractive because the roles of the controllers’ parts are very clear; that is, the observer estimates the plant state, and the control input is generated by using the estimated state. This property enables us to easily evaluate the validity of designed controllers.

We therefore address the design problem in which the observer-based controllers as well as the scaling matrices are simultaneously optimized, and we successfully propose a design method for the problem by using dilated LMI approach. One of the unique features of our proposition is that many controller matrices, i.e. observer gain, state-feedback gain, the state transition matrix of the observer, and the input or the output matrix of the observer, are set as decision matrices. Our method has a structural constraint for the matrix which is introduced in the dilation procedure; that is, some conservatism is inevitably introduced. However, numerical examples demonstrate that the conservatism can be negligible for some cases, and they also illustrate the effectiveness of the simultaneous design of scaling matrices and the controller matrices.

### Acknowledgments

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### References


Appendix  Proposition for Dual System

In this appendix, the following discrete-time polytopic system is considered:

\[
G^d(\xi) : \begin{bmatrix} x^T \\ z \\ y \\ w \end{bmatrix} = \begin{bmatrix} A(\xi) & B_1(\xi) & B_2 \\ C_1(\xi) & D_{11}(\xi) & 0 \\ C_2(\xi) & D_{21}(\xi) & D_{22} \end{bmatrix} \begin{bmatrix} x \\ w \\ u \end{bmatrix},
\]

(A.1)

where \(x \in \mathbb{R}^n, z \in \mathbb{R}^m, y \in \mathbb{R}^p, w \in \mathbb{R}^q\) and \(u \in \mathbb{R}^r\) respectively denote the state with 0 as its initial value, the performance output, the measurement output, the external input and the control input. All matrices in (A.1) are supposed to be time-invariant and to have compatible dimensions; however, the matrices \(A(\xi), \xi\) etc. are parametrically affine and given as follows:

\[
\begin{bmatrix} A(\xi) & B_1(\xi) \\ C_1(\xi) & D_{11}(\xi) \\ C_2(\xi) & D_{21}(\xi) \end{bmatrix} = \sum_{i=1}^{N} \xi_i \begin{bmatrix} A_i & B_{1i} \\ C_{1i} & D_{11i} \\ C_{2i} & D_{21i} \end{bmatrix},
\]

(A.2)

with \(\xi = [\xi_1, \xi_2, \ldots, \xi_N]^T \in \Xi\), which is the same as in (3), and compatibly dimensioned constant matrices \(A_i\), etc.

The following assumption is made for the matrix \(B_2\) in (A.1).

Assumption 3  The matrix \(B_2\) is column full rank and is given as

\[
B_2 = \begin{bmatrix} B_2 \\ 0 \end{bmatrix},
\]

(A.3)

where \(B_2 \in \mathbb{R}^{m \times n}\) is non-singular.

For system (A.1), Assumption 2 is supposed. Correspondingly, the uncertainty block given in (5) is supposed and the scaling matrix set \(\mathbb{L}_p\) given in (6) is set.

We now define an observer-based output feedback controller.

\[
C^d : \begin{cases} \dot{x}^\dagger = A_0 \dot{x}^\dagger + B_2 u - L(y - C_0 \dot{x} - D_{22} u), \\ u = K \dot{x}. \end{cases}
\]

(A.4)

where \(A_0 \in \mathbb{R}^{n_0 \times n_0}, C_0 \in \mathbb{R}^{n_0 \times n}, L \in \mathbb{R}^{n \times n_0}, \) and \(K \in \mathbb{R}^{n \times n_0}\) which are all to be designed, respectively denote the state transition matrix of the observer, the output matrix of the observer, the observer gain and the state-feedback gain.

The observer-based output feedback controller \(C^d\) is equivalently expressed as \(\hat{C}^d\).

\[
\hat{C}^d : \begin{cases} \dot{x}^\dagger = (A_0 + LC_0) \dot{x} + B_2 K \dot{x} - L(y - D_{22} u), \\ u = K \dot{x}. \end{cases}
\]

(A.5)

This transformation indicates that we have no need to design \(A_0\) and \(C_0\) independently and only have to design \(A_0 + LC_0\) as a single matrix.

The closed-loop system comprising \(G^d(\xi)\) in (A.1) and \(C^d\) in (A.4) is given below:

\[
G^d_{\hat{C}}(\xi) : \begin{bmatrix} x_{\hat{C}}^T \\ z \\ y \end{bmatrix} = \begin{bmatrix} A_{\hat{C}}(\xi) & B_1^\dagger(\xi) \\ C_{\hat{C}}(\xi) & D_{11}^\dagger(\xi) \\ C_{\hat{C}}(\xi) & D_{21}^\dagger(\xi) \end{bmatrix} \begin{bmatrix} x_{\hat{C}} \\ w \end{bmatrix},
\]

(A.6)

where \(x_{\hat{C}} = [x^T - \hat{x}^T \ x^T]^T\) denotes the state of \(G^d_{\hat{C}}(\xi)\), and the matrices \(A_{\hat{C}}(\xi), B_1^\dagger(\xi), \xi\) etc. are calculated as follows:

\[
\begin{aligned}
A_{\hat{C}}(\xi) &= A(\xi) - A_0 + \frac{L(C_0(\xi) - C_0)}{C_0(\xi) - C_0} B_1(\xi) + LD_{21}(\xi)
\end{aligned}
\]

\[
\begin{aligned}
B_1^\dagger(\xi) &= B_2 K + A(\xi) + B_1 K
\end{aligned}
\]

\[
\begin{aligned}
C_{\hat{C}}(\xi) &= C_2(\xi) + C_1(\xi) + D_{21}(\xi)
\end{aligned}
\]

Using the condition ii) in Lemma 2, the following method is proposed.

**Theorem 2**  For given \(\gamma > 0\), if there exist a parametrically affine matrix \(\mathcal{P}(\xi) \in \mathbb{R}^{2n_0}, \) and constant matrices \(\mathcal{Y} \in \mathbb{R}^{n_0 \times N}, \mathcal{L} \in \mathbb{R}^{n \times n_0}, \mathcal{K} \in \mathbb{R}^{n_0 \times n}, \) \(\mathcal{Q}\) structured as in (A.8) and \(S^{-1} \in \mathbb{L}_p\) such that (A.7) holds for all vertices of \(\Xi\), then the observer-based robust controller \(\hat{C}^d\) with \(A_0 + LC_0 = \mathcal{Q}^{-1} \mathcal{Y}, \mathcal{L} = \mathcal{Q}^{-1} \mathcal{L}\) and \(K = \mathcal{B}_2^\dagger \mathcal{Q}_1^{-1} \mathcal{K}\) stabilizes the closed-loop system \(G^d_{\hat{C}}(\xi)\) and satisfies (10) for all possible \(\xi \in \Xi\) with \(S = \mathcal{S}^d\).

\[
\begin{aligned}
\mathcal{P}(\xi) &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\mathcal{P}(\xi) & 0 \\ 0 & 0 & S^{-1} \end{bmatrix} + \text{He} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} \mathcal{Y}_A(\xi) & \mathcal{Q} \mathcal{Y}_B(\xi) \\ 0 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} C(\xi)^T \\ 0 \\ D_{11}(\xi)^T \\ D_{11}(\xi)^T \end{bmatrix} S^{-1} \begin{bmatrix} 0 \\ C(\xi) \\ 0 \\ D_{11}(\xi) \end{bmatrix} > 0, \forall \xi \in \Xi,
\end{aligned}
\]

(A.7)

where \(\mathcal{Y}_A(\xi)\) and \(\mathcal{Y}_B(\xi)\) are defined as follows:

\[
\begin{aligned}
\mathcal{Y}_A(\xi) &= \begin{bmatrix} \mathcal{Y} & \mathcal{Q}_2 A(\xi) + L C_2(\xi) - \mathcal{L} \\ \mathcal{L} \end{bmatrix}, \\
\mathcal{Y}_B(\xi) &= \begin{bmatrix} \mathcal{Q}_1 B_1(\xi) + L D_{21}(\xi) \\ \mathcal{Q}_2 B_1(\xi) \end{bmatrix}
\end{aligned}
\]

The matrix \(\mathcal{Q}\) is structured as follows:

\[
\mathcal{Q} = \begin{bmatrix} Q_1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} Q_1 \\ 0 \\ Q_{11} \\ 0 \\ Q_{22} \end{bmatrix}
\]

(A.8)

where \(Q_1 \in \mathbb{R}^{n_0 \times n_0}, Q_{11} \in \mathbb{R}^{n \times n_0}, Q_{12} \in \mathbb{R}^{n \times (n_0 - n)},\) and \(Q_{22} \in \mathbb{R}^{(n_0 - n) \times (n_0 - n)}\).

The proof is straightforward by using change-of-variables \(\mathcal{L} = \mathcal{Q}_1 L, \mathcal{K} = \mathcal{Q}_1 B_2, \mathcal{Y} = \mathcal{Q}_1 (A_0 + LC_0)\).