Some Properties of SINR Regions for Standard Interference Mappings

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Abstract: In 1995, Yates proposed an axiomatic framework of standard interference mappings and examined the iterative power control algorithm for a system with interference constraints. In the 2000s, Boche and Schubert built on a generalization of the theory of standard interference mappings and considered the feasibility of the constraints by the interference mapping in their sense. In this paper, we consider the signal to interference and noise ratio (SINR) region for any continuous and standard interference mapping, i.e., the set of all attainable values of SINR and make clear some properties of the SINR region. We also show the relations between the SINR regions for any continuous and standard interference mapping and its asymptotic mapping. In addition, we give a new and simple proof of the existence of positive eigenvalues and positive eigenvectors for any continuous and standard interference mapping making use of the properties of the SINR region and show there is an order relation among positive eigenvectors. Furthermore, we discuss an optimization problem with SINR constraints. Under the assumption of the feasibility of the problem, we prove that there exists a unique fixed point which is, at the same time, a unique optimal solution. We also provide a sufficient condition for the feasibility of the problem, based on a unique eigenvalue of the asymptotic mapping.

Key Words: standard interference mapping, SINR region, asymptotic mapping, eigenvalue, eigenvector.

1. Introduction

Interference is one of the main problems in wireless network systems and is investigated for many years. In 1995, Yates [1] proposed an axiomatic framework of interference mappings and examined the iterative power control algorithm for a system with interference constraints. Yates called the interference mappings satisfying the axioms standard. Here, we introduce the axioms given by Yates [1]. First, some notations are as follows.

- \( R \) is the set of real numbers, \( R_+ \) the set of nonnegative reals, and \( R^{++} \) the set of positive reals.
- \( R^n \) is the \( n \)-dimensional Euclidean space, \( R^n_+ \) the nonnegative cone in \( R^n \), and \( R^{++n} \) the positive cone in \( R^{++n} \).
- For \( x, y \in R^n, \ x \leq y \) denotes the ordering on \( R^n \): \( x \leq y \) if and only if \( x_i \leq y_i \) for all \( i = 1, \ldots, n \), or if and only if \( y - x \in R^n_+ \). Here, \( x_i \) is the \( i \)-th component of \( x \). In the same way, \( x < y \) if and only if \( x_i < y_i \) for all \( i = 1, \ldots, n \), or if and only if \( y - x \in R^{++n} \). For \( x \in R^n, x \geq 0 \), i.e., \( x \in R^n_+ \) (resp. \( x > 0 \), i.e., \( x \in R^{++n} \)), then we say \( x \) is nonnegative (resp. positive).

Definition 1 (Axioms of standard interference functions and mappings) We say that \( f : R^n_+ \rightarrow R^n \) is a standard interference function if the following axioms are fulfilled:

1. (positivity) \( f(x) > 0 \) for any \( x \in R^n_+ \),
2. (monotonicity) \( 0 \leq x \leq y \) implies \( f(x) \leq f(y) \),
3. (scalability) \( f(\alpha x) < \alpha f(x) \) for any \( x \in R^n_+ \) and any real \( \alpha > 1 \).

If each component \( f_i \ (i = 1, \ldots, n) \) of a mapping \( f : R^n_+ \rightarrow R^n_+ \) is a standard interference function, then we say \( f \) is a standard interference mapping.

Remark 1

1. In wireless network systems, \( x \in R^n \) is a vector of transmission powers, and \( f(x) \) is the resulting interference at the receiver of user \( i = 1, \ldots, n \).
2. Axiom (1) is implied by nonzero background receiver noise.
3. Axiom (1) follows from Axioms (2) and (3). In fact, from Axiom (3) with \( x = 0 \), we see that \( f(0) < \alpha f(0) \) for any \( \alpha > 1 \). Hence, \( f(0) > 0 \). From Axiom (2), for any \( x \geq 0 \), \( f(x) \geq f(0) > 0 \), which implies Axiom (1).
4. Axiom (2) means increasing transmission power does not reduce interference.
5. If at a transmission power vector \( x \), the inequality \( x_i \geq f_i(x) \) holds, it is said that user \( i \) has an acceptable connection. Axiom (3) implies that if \( x_i \geq f_i(x) \), then \( \alpha x_i \geq \alpha f_i(x) \) for \( \alpha > 1 \). That is, if user \( i \) has an acceptable connection at \( x \), then user \( i \) will have a more than acceptable connection when all powers are scaled up uniformly.
6. Axiom (3) is equivalent to the condition that \( \alpha f(x) < f(\alpha x) \) for any \( x \in R^n_+ \) and any positive real \( \alpha < 1 \), as is easily shown.
It is well-known that any standard interference mapping is continuous on $\mathbb{R}_+^n$ ([2]). Furthermore, there exists a unique continuous and standard interference mapping $F : \mathbb{R}_+^n \to \mathbb{R}_+^n$ such that the restriction $F$ on $\mathbb{R}_+^n$ denoted as $F|_{\mathbb{R}_+^n}$ is equal to $f|_{\mathbb{R}_+^n}$ ([3]).

Hence, in the sequel, we assume that standard interference functions and mappings are continuous on $\mathbb{R}_+^n$.

Example 1

1. Let $A$ be a nonnegative $n \times n$ matrix, i.e., a matrix in which all the elements are greater than or equal to zero, and $b \in \mathbb{R}_+^n$. Then, the mapping $f : \mathbb{R}_+^n \to \mathbb{R}_+^n$ defined by $f(x) = Ax + b$ for all $x \in \mathbb{R}_+^n$ is standard. Especially, the positive constant mapping $f$ given by $f(x) = b$ for all $x \in \mathbb{R}_+^n$ is standard.

2. Suppose $f$ and $g$ are two standard interference mappings on $\mathbb{R}_+^n$. Then, $f + g$, $\max(f, g)$ and $\min(f, g)$ are also standard. Here, $+$, $\max$, and $\min$ mean component-wise operations. For instance, the mapping $\max(f, g)$ is defined by $\max(f, g)(x) = (\max(f_1(x), g_1(x)), \ldots, \max(f_n(x), g_n(x)))$ for all $x \in \mathbb{R}_+^n$, where $f_i$ (resp. $g_i$) is the $i$-th component of $f$ (resp. $g$).

3. Suppose $f$ and $g$ are two standard interference mappings on $\mathbb{R}_+^n$. Then, the compositions $f \circ g$ and $g \circ f$ are also standard. In fact, Axiom (2) for $g \circ f$ follows since $0 \leq x \leq y$ implies $f(x) \leq f(y)$ and hence $g \circ f(x) = g(f(x)) \leq g(f(y)) = g(f(y))$ by the monotonicity of $g$. For any $x \in \mathbb{R}_+^n$ and any real $\alpha > 1$, the inequality $f(\alpha x) < \alpha f(x)$ holds. Hence, we see that $g \circ f(\alpha x) = g(f(\alpha x)) \leq g(\alpha f(x)) < \alpha g(f(x)) = \alpha g(f(x))$, which implies Axiom (3).

After Yates’ approach, other axiomatic frameworks of interference mappings were considered in [4] and [5]. For example, Boche and Schubert [4] built on a framework as a generalization of the theory of standard interference functions:

Definition 2 (Axioms of general interference functions and mappings) We say that $f : \mathbb{R}_+^n \to \mathbb{R}_+$ is a general interference function if the following axioms are fulfilled:

1. (nonnegativity) $f(x) \geq 0$ for any $x \in \mathbb{R}_+^n$,
2. (monotonicity) $0 \leq x \leq y$ implies $f(x) \leq f(y)$,
3. (scale invariance) $f(\alpha x) = \alpha f(x)$ for any $x \in \mathbb{R}_+^n$ and any real $\alpha \geq 0$.

If each component of a mapping $f : \mathbb{R}_+^n \to \mathbb{R}_+$ is a general interference function, then we say $f$ is a general interference mapping.

Remark 2

1. It is well-known that any general interference function is continuous in $\mathbb{R}_+^n$ ([2]).
2. Nonnegativity follows from monotonicity and scale invariance. In fact, from the scale invariance with $x = 0$ and $\alpha = 0$, we see that $f(0) = 0$. From the monotonicity, for any $x \geq 0$, $f(x) \geq f(0) = 0$, which implies the nonnegativity.

3. In order to rule out the trivial case $f(x) = 0$ for all $x \in \mathbb{R}_+^n$, Boche and Schubert made an additional assumption: There exists an $\hat{x} \in \mathbb{R}_+^n$ such that $f(\hat{x}) > 0$. By this additional assumption, we can easily show that $f(x) > 0$ for all $x \in \mathbb{R}_+^n$.

4. Standard interference functions are closely related to general interference functions. In fact, Boche and Schubert [6] showed any standard interference function is representable by some general interference function.

Boche and Schubert [7] considered signal to interference and noise ratio (SINR for short) feasibility problems for general interference functions. Shindoh [8] also investigated SINR feasibility problems for general interference mappings from the point of view of the nonlinear Perron Frobenius theory [9].

There are many researches referring to [1]. However, it seems that there are a few results with respect to theoretical properties of standard interference mappings themselves. As such results, Cavalcante, Shen, and Stanczak [10] proved that any positive concave mapping whose each component is positive and concave is standard (see also [11]). Cavalcante and Stanczak [12] investigated the existence of the asymptotic mapping for any standard interference mapping and applied the results to SINR feasibility problems. Shindoh [13] showed the existence and some properties of eigenvalues for any continuous and standard interference mapping and applied these properties to SINR feasibility problems.

In the conference paper [14], we considered the SINR region for any continuous and standard interference mapping, i.e., the set of all attainable values of SINR and made clear the structure of the SINR region. We also showed the relations between the SINR regions for any standard interference mapping and its asymptotic mapping. In addition, we discussed an optimization problem with SINR constraints. Under the assumption of the feasibility of the problem, we showed that there exists a unique fixed point which is, at the same time, a unique optimal solution.

In this paper, we extend the theorems in [14] and add the following new results:

- In Theorem 2, we describe the SINR region as a system of inequalities, which is closely related to the feasibility of the optimization problem in Section 2.
- In Theorem 5, we verify that any continuous and standard interference mapping has positive eigenvalues and positive eigenvectors. Though the result was previously obtained in [13], we give a new and simple proof making use of the properties of the SINR region. We also show there is an order relation among positive eigenvectors, which is a new result.
- In Theorem 6, we add a sufficient condition for the feasibility of the optimization problem based on the information of a unique eigenvalue of the asymptotic mapping and the properties of the SINR region.
- Example 4 is updated according to our new results.

The paper is organized as follows. Section 2 introduces an optimization problem with SINR constraints. In Section 3, we
show that any continuous and standard interference mapping has an asymptotic mapping which is continuous, monotone, and scale invariant. The results in Sections 2 and 3 are utilized for proving the theorems in Section 4. In Section 4, we state main results of this paper. In Theorem 1, we make clear some properties of the SINR region for any continuous and standard interference mapping. In Theorems 2, we characterize the SINR region by using a system of inequalities. Theorem 3 provides the relations between SINR regions for the standard interference mapping and its asymptotic mapping. In Theorem 4, we show the SINR region for the asymptotic mapping contains a special point. In Theorem 5, we verify that there exist positive eigenvalues and positive eigenvectors for any continuous and standard interference mapping, making use of the properties of the SINR region. We also show there is an order relation among positive eigenvectors. In Theorem 6, we prove optimality and some balancing property for an optimization problem with SINR constraints. In addition, we give a sufficient condition for the feasibility of the problem, based on the unique eigenvalue of the asymptotic mapping and the properties of the SINR region. In Section 5, we give some concluding remarks.

2. SINR Constrained Optimization Problem

The signal to interference and noise ratio (SINR) is a well-known and useful measure in wireless network systems. Let \( f : R^n_+ \to R^n_+ \) be a continuous and standard interference mapping. The SINR of user \( i = 1, 2, \ldots, n \) is defined as

\[
\text{SINR}_i(x) = \frac{x_i}{f_i(x)},
\]

where \( f_i \) is the \( i \)-th component of \( f \). Note that the SINR is well-defined since \( f(x) > 0 \) for all \( x \in R^n_+ \).

We are interested in the following SINR constrained optimization problem for some threshold \( n \)-tuple \( \gamma = (\gamma_1, \ldots, \gamma_n) \in R^n_+ \):

\[
\begin{align*}
\text{min} & \quad ||x|| \\
\text{s.t.} & \quad x \in R^n_+, \\
\text{SINR}_i(x) & \geq \gamma_i (i = 1, \ldots, n).
\end{align*}
\]

Here, \( ||x|| \) denotes the \( l_1 \) norm of \( x \), i.e., \( ||x|| = \sum_{i=1}^{n} |x_i| \). Especially, for \( x \geq 0, ||x|| = \sum_{i=1}^{n} x_i \). The \( l_1 \) norm is monotone since if \( 0 \leq x \leq y \), then \( ||x|| \leq ||y|| \). Note that, if \( 0 \leq x \leq y \) and \( x \neq y \), then \( ||x|| < ||y|| \). The optimization problem above is the minimization of the total power sum given the lower bound of SINR. The optimization problem above was considered in [1],[7],[8],[13].

Given some nonnegative \( n \)-tuple \( \gamma = (\gamma_1, \ldots, \gamma_n) \), define the mapping \( \gamma : f : R^n_+ \to R^n_+ \) by \( \gamma \cdot f(x) = (\gamma_1 f_1(x), \ldots, \gamma_n f_n(x)) \) for all \( x \in R^n_+ \). Then, the problem above is transformed to the following equivalent optimization problem:

\[
\begin{align*}
\text{SINR}(\gamma) & \left\{ \begin{array}{l}
\text{min} \quad ||x|| \\
\text{s.t.} \quad x \in R^n_+, \\
\gamma \cdot f(x) \leq x.
\end{array} \right.
\end{align*}
\]

The next lemma follows from Definition 1:

**Lemma 1** If \( f : R^n_+ \to R^n_+ \) is a standard interference mapping, then, so is \( \gamma \cdot f \) for any \( \gamma \in R^n_+ \).

We are interested in the feasibility and optimality of the optimization problem above. For this aim, we consider the structure of the SINR region, i.e., the set of all attainable values of SINR in Section 4.

3. Asymptotic Mappings

The SINR for any continuous and standard interference mapping \( f : R^n_+ \to R^n_+ \) has a special property. By the scalability of \( f \), the following inequality holds for all \( x \in R^n_+ \) and all \( \alpha > 1 \):

\[
\text{SINR}_i(x) = \frac{x_i}{f_i(x)} < \frac{\alpha x_i}{f_i(\alpha x)} = \text{SINR}_i(\alpha x) \tag{2}
\]

for every \( i = 1, \ldots, n \). This inequality means that the SINR increases when transmission powers are scaled up uniformly.

We are interested in the asymptotic behavior of \( \text{SINR}_i(\alpha x) \) when \( \alpha \to \infty \). For this aim, we consider the mapping \( f_i : R^n_+ \to R^n_+ \) defined by

\[
f_i(x) = \frac{f(tx)}{t}
\]

for all \( x \in R^n_+ \). Here, \( t \) is a fixed positive parameter.

**Lemma 2** For \( 0 < s < t \), the inequality \( f_i(x) < f_s(x) \) holds for all \( x \in R^n_+ \).

**Proof:** Since \( 0 < s < t \), we see \( \frac{s}{t} > 1 \). By the scalability of \( f \), we have

\[
f_i(x) = \frac{f(tx)}{t} = \frac{f(s \cdot x)}{s} < \frac{f(s \cdot x)}{t} = f_s(x).
\]

As a result of Lemma 2, \( \lim_{t \to \infty} \frac{f_i(x)}{t} \) exists for all \( x \in R^n_+ \). We denote this limit by \( f_\infty(x) \). Then, the following lemma holds:

**Lemma 3** Let \( f : R^n_+ \to R^n_+ \) be a continuous and standard interference mapping. Then the mapping \( f_\infty : R^n_+ \to R^n_+ \) defined by \( f_\infty(x) = \lim_{t \to \infty} \frac{f_i(x)}{t} \) for all \( x \in R^n_+ \) has the following properties:

1. \( f_\infty \) is continuous.
2. \( f_\infty \) is monotone.
3. \( f_\infty \) is scale invariant, i.e., \( f_\infty(\alpha x) = \alpha f_\infty(x) \) for all \( x \in R^n_+ \) and all \( \alpha \geq 0 \).

For the proof of Lemma 3, see, for example, [2].

**Definition 3** We call the mapping \( f_\infty : R^n_+ \to R^n_+ \) given above the asymptotic mapping associated to \( f \).

From Lemma 3, we see immediately:

**Lemma 4** The asymptotic mapping \( f_\infty \) associated to \( f \) is a general interference mapping.

**Example 2** Suppose the mapping \( f : R^n_+ \to R^n_+ \) is given by \( f(x, y) = x+y+1 \) and \( f_\infty(x, y) = \sqrt{x} + \sqrt{y} + 2 \). It is easily shown that \( f \) is standard and the asymptotic mapping \( f_\infty \) is given by \( f_\infty(x, y) = (x+y, 0) \).

By the inequality (2) and the properties of the asymptotic mapping \( f_\infty \) associated to \( f \), \( \text{SINR}_i(x) \) satisfies the inequality

\[
\text{SINR}_i(x) < \frac{x_i}{f_\infty(x)} \tag{3}
\]

for any \( i = 1, \ldots, n \) whenever \( x_i \neq 0 \) and \( f_\infty(x) \neq 0 \). Here, \( f_\infty \) is the \( i \)-th component of \( f_\infty \). We also see that \( (\gamma \cdot f_\infty) = \gamma \cdot f \) for any continuous and standard interference mapping \( f \) and any \( \gamma \in R^n_+ \).

**Remark 3** In [12], Cavalcante and Stanczak considered the asymptotic mapping for any continuous and standard interference mapping and discussed the role of the asymptotic mapping for network optimization and feasibility.
4. Structure of the SINR Region

In this section, we make clear the structure of the SINR region for any continuous and standard interference mapping $f: \mathbb{R}_+^n \rightarrow \mathbb{R}_+^n$ and provide some results related to the SINR region.

We define the SINR region of $f$ as

$$
\Gamma(f) = \{y = (y_1, \ldots, y_n) \in \mathbb{R}_+^n : \text{there exists some } x \in \mathbb{R}_+^n \text{ such that } \gamma_i = \text{SINR}_i(x) \text{ for } i = 1, \ldots, n \}.
$$

The SINR region $\Gamma(f)$ is a set of all attainable values of SINR.

Before we state six theorems, we provide a brief overview of the connections between these theorems. The main results are Theorems 1, 5, and 6. Theorems 2 and 3 follow from Theorem 1. Theorem 4 is verified by our previous results in [8]. We lead Theorem 5 from Theorems 1, 3, and 4. Theorem 6 is derived from Theorems 1 and 5.

Now we are in the position to state the following first result:

**Theorem 1** Let $f: \mathbb{R}_+^n \rightarrow \mathbb{R}_+^n$ be a continuous and standard interference mapping. Then, $\Gamma(f)$ has the following properties:

1. $(1)$ $0 \in \Gamma(f)$. Hence, especially $\Gamma(f) \neq \emptyset$.
2. If $y \in \Gamma(f)$, then there exists a unique $x \in \mathbb{R}_+^n$, satisfying $\gamma_i = \frac{1}{f_i(x)}$ for $i = 1, \ldots, n$. Here, $\gamma_i$ is the $i$-th component of $y$.
3. If $y \in \Gamma(f)$ and $0 \leq y' \leq y$, then $y' \in \Gamma(f)$, i.e., $\Gamma(f)$ is downward comprehensive.

**Remark 4** In [15], Mahdavi-Doost et al. considered a standard interference mapping of type of Example 1-1, i.e., $f(x) = Ax + b$ for the $n \times n$ nonnegative matrix $A$ and $b \in \mathbb{R}_+^n$ and gave characterization of the SINR region for interfering links with constrained power. In [16], Chen and Sung also considered the SINR region of the same type and characterized the structure of its region. Theorem 1 is a generalization of their characterizations to the SINR region for any standard interference mapping.

To prove Theorem 1, we need some lemmas.

**Lemma 5** Let $f: \mathbb{R}_+^n \rightarrow \mathbb{R}_+^n$ be a continuous and standard interference mapping.

1. If $f$ has a fixed point $x^*$, i.e., $f(x^*) = x^*$, then the fixed point is unique.
2. If the inequality $f(x) \leq x$ holds for some $x \in \mathbb{R}_+^n$, the sequence $(x(m))$ generated by $x(0) = x$ and $x(m+1) = f(x(m))$ for $m = 0, 1, \ldots$ converges to the positive fixed point of $f$.

**Remark 5** Note that, in (2), the inequality $f(x) \leq x$ holds for some $x \in \mathbb{R}_+^n$, then $x \in \mathbb{R}_+^n$ since $f(x) > 0$ for any $x \in \mathbb{R}_+^n$.

**Proof:** (1) Suppose that there is a fixed point $\tilde{x} \neq x^*$. Without loss of generality, we may assume $x^* \leq \alpha \tilde{x}$ for some $\alpha > 1$ and $x_i^* = \alpha \tilde{x}_i$ for some component $i$. By the monotonicity and the scalability of the $i$-th component $f_i$ of $f$, we have

$$
x_i^* = f_i(x^*) \leq f_i(\alpha \tilde{x}) = \alpha f_i(\tilde{x}) = \alpha x_i.
$$

This contradicts the equality $x_i^* = \alpha \tilde{x}_i$.

(2) By the monotonicity of $f$, the sequence $(x(m))$ satisfies the inequalities

$$
0 \leq \cdots \leq x(m+1) \leq x(m) \leq \cdots \leq x(0) = x.
$$

Hence, the sequence $(x(m))$ is monotone decreasing and bounded below and converges to some $x^* \in \mathbb{R}_+^n$. By the continuity of $f$, we see that $f(x^*) = x^*$, i.e., $x^*$ is a fixed point of $f$. We also know that $x^* \in \mathbb{R}_+^n$, since $x^* = f(x^*) > 0$.

**Proof (of Theorem 1)**

(1) Set $x = 0$. Then $f(0) > 0$ and $\gamma_i = 0 = \frac{1}{f_i(0)}$ for each $i = 1, \ldots, n$. Hence, $0 \in \Gamma(f)$. Especially, we see $\Gamma(f) \neq \emptyset$.

Note that, if $x \geq 0$ and $x \neq 0$, then SINR$(x) > 0$ for at least one $i = 1, \ldots, n$. Hence, $\gamma_i = 0$ is realized by $x = 0$ only.

(2) The case of $\gamma = 0$ is verified by (1). Hence, we assume $\gamma \geq 0$ and $\gamma \neq 0$. Suppose $x, \tilde{x} \in \mathbb{R}_+^n(x \neq \tilde{x})$ realize $\gamma$. Then, the equations $x = \gamma \cdot f(x)$ and $\tilde{x} = \gamma \cdot f(\tilde{x})$ hold. From the equations, we see that, for each $i = 1, \ldots, n$, $\gamma_i = 0$ implies $x_i = \tilde{x}_i = 0$ and $\gamma_i > 0$ implies $x_i > 0$ and $\tilde{x}_i > 0$. Since $x \neq \tilde{x}$, we may assume $x_i \leq \alpha \tilde{x}_i$ for some $\alpha > 1$ and $x_i = \alpha \tilde{x}_i > 0$ for some $i(1 \leq i \leq n)$. By the monotonicity and the scalability of the $i$-th component $f_i$, we have the following inequalities

$$
x_i = \gamma_i f_i(x) \leq \gamma_i f_i(\alpha \tilde{x}) = \alpha \gamma_i \tilde{x}_i.
$$

This contradicts the equality $x_i = \alpha \tilde{x}_i > 0$.

(3) We may assume that $\gamma \neq 0$ since $\gamma = 0$ and $\gamma' \leq \gamma$ imply $\gamma' = 0$.

First, we consider the case of $J = \{j : \gamma_j = 0 \text{ for some component of } \gamma \neq 0\}$. Without loss of generality, we may assume $J = \{1, \ldots, k\}$ ($1 \leq k \leq n - 1$). Set $f = \{k + 1, \ldots, n\}$ and represent the vector $\gamma$ as $\gamma = (\gamma_1, \gamma_2) = (0, \gamma_2)(\gamma_2 > 0)$. Since $\gamma \in \Gamma(f)$ and (2) holds, $\gamma$ is attained by some unique $x' = (x'_2, x'_3) = (0, x'_2) \in \mathbb{R}_+^n(x'_2 > 0)$. Hence, $x'_2 = \gamma_2 f_2(0, x'_2)$. From the condition $\gamma'_2 \leq \gamma_2$, we see that $\gamma'_2 = 0$. Since the mapping $f_2(0, \cdot) : \mathbb{R}_+^1 \rightarrow \mathbb{R}_+^1$ is clearly a standard interference mapping, $\gamma'_2 \in \Gamma(f)$ if and only if $\gamma'_2 \in \Gamma(f(0, \cdot))$. Hence, it is sufficient to show (3) under the condition $\gamma_2 > 0$, i.e., $J = 0$. Furthermore, we may assume $\gamma' > 0$.

Since $\gamma \in \Gamma(f)$, there exists a unique $x^* \in \mathbb{R}_+^n$ with $x^* = \gamma \cdot f(x^*)$. By the condition $\gamma' \leq \gamma$, we see that $\gamma' \cdot f(x) \leq \gamma \cdot f(x) = x'$. Applying Lemma 5 with $x(0) = x$ and $x(m+1) = \gamma \cdot f(x(m))$ ($m = 0, 1, \ldots$), we have a monotone decreasing sequence $\{x(m)\}$ converging to some fixed point $x^* \in \mathbb{R}_+^n$ of $\gamma'' \cdot f$, i.e., $x^* = \gamma'' \cdot f(x^*)$. This equation implies $\gamma'' = \text{SINR}(x^*)$ for each $i = 1, \ldots, n$. Hence, $\gamma'' \in \Gamma(f)$.

The second result shows that the SINR region is defined as a system of inequalities which is closely related to the feasibility of the optimization problem given in Section 2.

**Theorem 2** Let $f: \mathbb{R}_+^n \rightarrow \mathbb{R}_+^n$ be a continuous and standard interference mapping. Define the set $\Gamma'(f) \subset \mathbb{R}_+^n$ as

$$
\Gamma'(f) = \{\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{R}_+^n : \text{there exists some } x \in \mathbb{R}_+^n \text{ such that } \text{SINR}_i(x) \geq \gamma_i \text{ for } i = 1, \ldots, n\}.
$$

Then, $\Gamma'(f) = \Gamma(f)$.

**Proof:** Clearly, $\Gamma(f) \subset \Gamma'(f)$. Hence, we prove $\Gamma'(f) \subset \Gamma(f)$.

Let $\gamma \in \Gamma'(f)$. Then, there exists some $x \in \mathbb{R}_+^n$ satisfying $\text{SINR}_i(x) \geq \gamma_i$ for $i = 1, \ldots, n$. In the same way as the proof of Theorem 1, we have a unique $x^* \in \mathbb{R}_+^n$ with $\text{SINR}_i(x^*) = \gamma_i$ for $i = 1, \ldots, n$, which implies $\gamma \in \Gamma(f)$. \hfill $\Box$
The third result shows relations between SINR regions of a continuous and standard interference mapping \( f \) and the asymptotic mapping \( f_\infty \) associated to \( f \).

Suppose that \( f_\infty(x) > 0 \) for any nonzero \( x \in R^n_+ \). Define the SINR region of \( f_\infty \) as

\[
\bar{\Gamma}(f_\infty) = \{ y = (\gamma_1, \ldots, \gamma_n) \in R^n_+ : \text{there exists some nonzero } x \in R^n_+ \text{ such that } \gamma_i = \frac{x_i}{f_\infty(x)} \text{ for } i = 1, \ldots, n \}. 
\]

Remark 6

1. Since \( f_\infty(0) = 0 \) for the asymptotic mapping \( f_\infty \) by (3) of Lemma 3, \( \frac{f_\infty(x)}{x} \) \((i = 1, \ldots, n)\) is not defined at \( x = 0 \). Hence, under the assumption that \( f_\infty(x) > 0 \) for any nonzero \( x \in R^n_+ \), we define the SINR region \( \bar{\Gamma}(f_\infty) \) for \( f_\infty \) on \( R^n_+ \setminus \{0\} \).

2. The positivity of \( f_\infty \) for any nonzero \( x \in R^n_+ \) does not hold necessarily as Example 2 shows.

3. Note that \( \frac{x}{f_\infty(x)} \) is homogeneous of degree zero since

\[
\frac{tf_\infty(tx)}{tf_\infty(x)} = \frac{tx_i}{f_\infty(x)} \text{ for any positive real } t \text{ and every } i = 1, \ldots, n.
\]

Hence, we see

\[
\bar{\Gamma}(f_\infty) = \{ y \in R^n_+ : \gamma_i = \frac{x_i}{f_\infty(x)} \text{ for some } x \in \Sigma, i = 1, \ldots, n \}.
\]

Here, \( \Sigma = \{ x \in R^n_+ : ||x|| = 1 \} \).

Theorem 3

Let \( f : R^n_+ \to R^n_+ \) be a continuous and standard interference mapping and \( f_\infty \) the asymptotic mapping associated to \( f \). Suppose that \( f_\infty(x) > 0 \) for any nonzero \( x \in R^n_+ \). If \( y \in \bar{\Gamma}(f_\infty) \) and \( 0 \leq \hat{\gamma} < y \), then \( \hat{\gamma} \in \Gamma(f) \).

Proof: Note that \( (y : f_\infty) = y : f_\infty \). Since \( y \in \bar{\Gamma}(f_\infty) \) and \( 0 \leq \hat{\gamma} < y \), we see that \( y \in R^n_+ \). Hence, there is some positive \( x^* \in \Sigma \) with \( y : f_\infty(x^*) = x^* \). By Lemma 2 and the definition of the asymptotic mapping \( f_\infty \), we have the inequality

\[
f_\infty(x^*) < f_\infty(t x^*) < \frac{1}{r} f(t x^*)
\]

for all \( t > 0 \) and \( i = 1, \ldots, n \). Hence, we see that

\[
\gamma_i = \frac{x_i^*}{f_\infty(x^*)} = \frac{x_i}{f_\infty(x)} > \frac{tx_i}{f(t x^*)} = \gamma_i > \hat{\gamma}_i.
\]

Then, by the two inequalities above, there is a sufficiently large positive \( t \) such that

\[
\gamma_i = \frac{x_i}{f_\infty(x)} > \frac{tx_i}{f(t x^*)} > \hat{\gamma}_i
\]

for \( i = 1, \ldots, n \). Hence, we have the inequality

\[
t x^* > \hat{\gamma} : f(t x^*).
\]

Therefore, it follows from Theorem 1 that there exists a unique \( \hat{x} \in R^n_+ \) satisfying \( \hat{x} = \hat{\gamma} : f(\hat{x}) \). This means \( \hat{\gamma} \in \Gamma(f) \).

Example 3

1. Let \( f : R_+ \to R_+ \) be given by \( f(x) = x + 1 \) for all \( x \in R_+ \). This example is used in [1]. It is clear that \( f \) is continuous and standard. The asymptotic function \( f_\infty \) associated to \( f \) is given by \( f_\infty(x) = x \). Then, \( \bar{\Gamma}(f_\infty) = \{1\} \) and \( \Gamma(f) = \{0 \leq \gamma < 1\} \).

2. Let \( f : R^n_+ \to R^n_+ \) be given by

\[
f(x, y) = \left( \begin{array}{cc} 2 & 3 \\ 3 & 2 \end{array} \right) \left( \begin{array}{c} x \\ y \end{array} \right) + \left( \begin{array}{c} 1 \\ 2 \end{array} \right)
\]

for all \( (x, y) \in R^n_+ \). Then, \( f \) is standard and \( f_\infty \) is given by

\[
f_\infty(x, y) = \left( \begin{array}{cc} 2 & 3 \\ 3 & 2 \end{array} \right) \left( \begin{array}{c} x \\ y \end{array} \right)
\]

It is easily shown that

\[
\bar{\Gamma}(f_\infty) = \{ (\gamma_1, \gamma_2) : (x, y) = \left( \begin{array}{c} \gamma_1 \gamma_2 \\ -2 \gamma_1 \gamma_2 \end{array} \right), 0 \leq x \leq 1 \}.
\]

Then, \( \Gamma(f) \) is given as \( \Gamma(f) = \{ (\gamma_1, \gamma_2) : (x, y) = \left( \begin{array}{c} 1 \gamma_2 \\ -2 \gamma_1 \gamma_2 \end{array} \right), 0 \leq \gamma_1 < 1 \} \) and \( \Gamma(f) = \{ (\gamma_1, \gamma_2) : (x, y) = \left( \begin{array}{c} \gamma_1 \gamma_2 \\ -2 \gamma_1 \gamma_2 \end{array} \right), 0 \leq \gamma_2 < 1 \} \).

3. Suppose \( f : R^n_+ \to R^n_+ \) is defined by \( f_1(x, y) = \sqrt{x^2 + y^2} + 1 \) and \( f_2(x, y) = \sqrt{2x^2 + y^2} + 1 \) for all \( (x, y) \in R^n_+ \). Then, \( f \) is standard and \( f_\infty \) is given by \( f_\infty(x, y) = \sqrt{x^2 + y^2} \) and \( f_\infty(x, y) = \sqrt{2x^2 + y^2} \). Hence, \( \bar{\Gamma}(f_\infty) \) is given by \( \bar{\Gamma}(f_\infty) = \{ (\gamma_1, \gamma_2) : (x, y) = \left( \begin{array}{c} \gamma_1 \gamma_2 \\ -2 \gamma_1 \gamma_2 \end{array} \right), 0 \leq \gamma_1 < 1 \} \) and \( \bar{\Gamma}(f_\infty) = \{ (\gamma_1, \gamma_2) : (x, y) = \left( \begin{array}{c} \gamma_1 \gamma_2 \\ -2 \gamma_1 \gamma_2 \end{array} \right), 0 \leq \gamma_2 < 1 \} \).

The fourth result shows that the SINR region for the asymptotc mapping \( f_\infty \) includes a special point:

Theorem 4

Under the assumptions of Theorem 3, the following properties hold:

1. The asymptotic mapping \( f_\infty \) has a unique positive eigenvalue \( \lambda_\infty \) and a unique positive eigenvector \( x_\infty \in \Sigma = \{ x \in R^n_+ : ||x|| = 1 \} \), i.e., \( \lambda_\infty \) and \( x_\infty \) satisfy the equation \( f_\infty(x_\infty) = \lambda_\infty x_\infty \).

2. Suppose \( e \in R^n_+ \) is a vector in which all the elements are equal to one. Then, \( (1/\lambda_\infty)e \in \bar{\Gamma}(f_\infty) \).

Proof: For the proof of (1), see [8]. See also [2] and [9]. Note that the eigenvalue \( \lambda_\infty > 0 \) and its eigenvector \( x_\infty > 0 \) since \( f_\infty(x) > 0 \) for any \( x \in \Sigma \) by the assumption and \( 0 < f_\infty(x_\infty) = \lambda_\infty x_\infty \).

For (2), from the equation \( f_\infty(x_\infty) = \lambda_\infty x_\infty \) and the positivity of \( \lambda_\infty \) and \( x_\infty \), it follows that

\[
\lambda_\infty = \frac{x_\infty}{f_\infty(x_\infty)}
\]

for \( i = 1, \ldots, n \). This implies \( (1/\lambda_\infty)e \in \bar{\Gamma}(f_\infty) \).

Remark 7

The eigenvalue \( \lambda_\infty \) given in Theorem 4 is called the spectral radius of \( f_\infty \). For details of the spectral radius, see [9].
The fifth result states any continuous and standard interference mapping has positive eigenvalues and positive eigenvectors. Though the result is previously verified in [13], we give a new and simple proof using the properties of the SINR region \( \Gamma(f) \). In addition, we show that there is an order relation among positive eigenvectors, which is a new result.

**Theorem 5** Under the assumptions of Theorem 3, the following properties hold:

(1) If \( 0 \leq \gamma < (1/\lambda_\infty)e \), then \( \gamma \in \Gamma(f) \). Here, \( \lambda_\infty \) is a unique eigenvalue of the asymptotic mapping \( f_\infty \), and \( e \in R_{+}^n \) is a vector in which all the elements are equal to one.

(2) If \( \lambda > \lambda_\infty \), then \( \lambda \) is an eigenvalue of the standard interference mapping \( f \). Each eigenvector \( \lambda' \) for \( f \) has a unique eigenvector \( x^* \in R_{+}^n \), i.e., \( \lambda' \) and \( x^* \) satisfy the equation \( f(x^*) = \lambda' x^* \).

(3) Suppose \( \mu > \lambda > \lambda_\infty \) and \( y \in R_{+}^n \) (resp. \( x \in R_{+}^n \)) is a unique eigenvector of \( \mu \) (resp. \( \lambda \)). Then, \( y < x \).

**Proof**: (1) follows from Theorem 3 and (2) in Theorem 4. For (2), set \( \gamma = (1/\lambda')e \in R_{+}^n \). Then, \( \gamma < (1/\lambda_\infty)e \). This means \( \gamma \in \Gamma(f) \), i.e., there exists a unique \( x^* \in R_{+}^n \) such that

\[
\frac{1}{\lambda'} = \frac{x_i^*}{f(x_i)}
\]

for \( i = 1, \ldots, n \). Hence, \( f(x^*) = \lambda' x^* \). This implies that \( \lambda' \) is an eigenvalue of \( f \) and \( x^* \) is an eigenvector associated with the eigenvalue \( \lambda' \).

(3) First, note that \( \mu \) and \( \lambda \) are eigenvalues of \( f \) by (2) of the theorem. Hence, we have the equations \( f(y) = \mu y \) and \( f(x) = \lambda x \).

Set \( \eta = (1/\mu)e \) and \( \gamma = (1/\lambda)e \). Then, we see \( \eta < \gamma \), \( y = \eta \cdot f(y) \), and \( x = \gamma \cdot f(x) \). We also have the strict inequality \( x > \eta \cdot f(x) \) since \( \gamma > \eta \) and \( x = \gamma \cdot f(x) > \eta \cdot f(x) \). The iterations defined by \( x(0) = x \) and \( x(m+1) = \eta \cdot f(x(m)) \) for \( m = 0, 1, \ldots \) generate the decreasing sequence \( x = x(0) > \eta \cdot f(x(0)) = x(1) \geq x(2) \geq \cdots \). The sequence converges to the fixed point \( x^* \). Hence, the uniqueness inequality \( x > y \).

**Example 4** The asymptotic mapping \( f_\infty \) given in Example 3-3 has a unique eigenvalue \( \lambda_\infty = \sqrt{3} \) and its eigenvector \((1/2, 1/2)^T \). If \( \lambda > \sqrt{3} \), \( \lambda \) is an eigenvalue of \( f \) by Theorem 5. It is easily shown that the eigenvector \( z(\lambda) \) for \( f \) is given by \( z(\lambda) = (1/(\lambda - \sqrt{3}), 1/(\lambda - \sqrt{3})) \). Hence, if \( \mu > \lambda > \sqrt{3} \), then \( z(\mu) < z(\lambda) \).

The last theorem provides the unique existence of an optimal solution for the problem \( \text{SINR}(y) \) under the condition \( y \in \Gamma(f) \).

**Theorem 6** Let \( f : R_{+}^n \to R_{+}^n \) be a continuous and standard interference mapping. Suppose \( y > 0 \). Then, the following results hold.

(1) If the feasible set \( \{x \in R_{+}^n : \text{SINR}(x) \geq y_i (i = 1, \ldots, n) \} \neq \emptyset \), then there exists a unique fixed point \( x^* \) of \( y \cdot f \), i.e., \( y \cdot f(x^*) = x^* \). Especially, \( x^* > 0 \). The point \( x^* \) is also a unique optimal solution of \( \text{SINR}(y) \), with the optimal value \( r^* = \|x^*\| \).

**Remark 8** The equalities (4) of Theorem 6 are well-known as the **max min balancing property** for general interference mappings [4,6–8]. Equation (4) shows that this property also holds for standard interference mappings.

5. **Concluding Remarks**

In this paper, we considered the SINR region for any continuous and standard interference mapping and made clear some properties of the region in the general setting. We also showed the relations between the SINR regions for any continuous and standard interference mapping \( f \) and its asymptotic mapping \( f_\infty \). In addition, we give a new and simple proof of the existence of positive eigenvalues and positive eigenvectors for any standard interference mapping making use of the properties of the SINR region. Furthermore, we discuss an optimization problem with SINR constraints. Under the assumption of the feasibility of the problem, we prove that there exists a unique fixed point which is, at the same time, a unique optimal solution.

We note that the results of this paper for any standard interference mapping \( f \) of the type of Example 1-1, given by \( f(x) = Ax + b \) for all \( x \in R_{+}^n \) are closely related to affine eigenvalue problems on the nonnegative orthant considered by Blondel et al. [17].
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