Global Asymptotical Stabilization of Morse-Smale Systems Using Weak Control-Lyapunov Functions

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Abstract: This paper proposes a method of constructing weak control-Lyapunov functions for nonlinear systems by introducing a topological geometric assumption called a Morse-Smale system. A Lyapunov function is one of the most important tools to study stability and stabilization of nonlinear systems. However, a general way of finding Lyapunov functions has not been found yet. First, we confirm there is a weak Lyapunov function for Morse-Smale systems. Next, we define the escapability for singular structures of the weak Lyapunov function. If all singular structures are escapable, then the Morse-Smale system is a globally asymptotically stabilizable one. Finally, we present the method of constructing a set of weak control-Lyapunov functions to achieve global stabilization. The method is described in terms of a recursive sequence of singular structures. We call the sequence a weak Lyapunov filtration.

Key Words: nonlinear systems, Lyapunov functions, iterative modeling and control design.

1. Introduction

A Lyapunov function plays a central role in stability analysis and stabilization of nonlinear control systems. However, in some cases, it is difficult to find a (strict positive definite) Lyapunov function for nonlinear systems. Therefore, we use a weak Lyapunov function [1] or a control-Lyapunov function [2],[3] in certain practical situations. In most of the studies on this topic, a dynamical system is defined on a contractible space with a unique asymptotically stable equilibrium. The contractible space is homotopic to a point and is usually homeomorphic to a Euclidian space. This implies that it is the simplest case in the sense of global topology [4]–[6].

In the more general case, manifolds take various shapes and there are multiple critical points of a potential function on such manifolds. We can study such a complex case by using Morse theory, which is one of the most remarkable results of topological geometry [4]. Morse theory gives a topological decomposition of manifolds under an assumption called a Morse-Smale system. There exists a weak Lyapunov function called a \( \xi \)-function in this case. The \( \xi \)-function generates a structure that is stable against topological deformations. In other words, the critical points of Morse functions represent a topological invariance of manifolds. The Morse function can be identified with a weak Lyapunov function for globally asymptotically stable points, because the time derivative of the Morse function is 0 on critical points.

On the other hand, the global structures of manifolds have attracted a lot of attention in recent years [7]–[11]. The studies have dealt with the relationship between the stability of nonlinear systems and the topology of global manifolds. They extended the above assumption to dynamical systems on the manifold. The extended assumption is called a gradient-like Morse-Smale system (or flow) [6],[7].

In this paper, we discuss the stabilization of nonlinear systems under the original more relaxed assumption, that is, Morse-Smale systems [12]–[16]. The Morse-Smale system consists of critical points and circle sets of critical points. There are many actual systems that have a set of critical points: e.g., oscillating systems, redundant freedom systems, constrained systems, non-holonomic systems, and homogeneous systems. There exists a weak Lyapunov function called a \( \xi \)-function in Morse-Smale systems. \( \xi \)-functions include the set of critical points of Morse-Smale systems as a singular structure. We define escapability for singular structure. If all singular structures are escapable, the Morse-Smale system is globally asymptotically stable. If the structures satisfy a stricter condition, i.e. local escapability, we can construct a set of weak Lyapunov functions for stabilizing the Morse-Smale system. Finally, we propose a recursive method for constructing weak control-Lyapunov functions for Morse-Smale systems. The recursive procedure generates an inclusion sequence of submanifolds for invariant sets of weak control-Lyapunov functions.

Our concept is quite different from classical Lyapunov methods. The objective of classical methods is to find a Lyapunov function in an unknown system with a single critical point. That approach can be considered to be one of finding an appropriate mapping between the systems to a basin of attraction in a Euclidian space. On the other hand, the objective of our method...
is to specify the geometrical conditions such that topologically complicated systems have Lyapunov functions. Upon checking the conditions, we can immediately know whether a Lyapunov function exists for the system. The concept is introduced from the perspective of the Conley's fundamental theorem of dynamical systems \(^1\) [6],[12], which is the basis of global control theory. In the case of Morse-Smale systems, the conditions are as follows: (i) a manifold is closed (compact and without boundaries) and (ii) all sets of critical points are isolated points or closed orbits. The Morse-Smale system is characterized by the set of critical points. In other words, we only have to check local structures around critical points to discuss the global structure of Morse-Smale systems.

This paper is organized as follows. Section 2 is devoted to the mathematical tools that we will need. In section 3, we discuss the stability, controllability and stabilization of Morse-Smale systems with control Lyapunov functions. Section 4 shows a simple example to illustrate the concept of weak Lyapunov filtration. Finally, we discuss the future prospects of this method.

## 2. Mathematical Preliminary

Let \(M\) be an \(m\)-dimensional orientable closed smooth manifold with a Riemannian metric, where a closed manifold means a compact manifold without a boundary.

### 2.1 Invariant Sets

Let us consider a continuous dynamical system \([\varphi^t]_{t \in \mathbb{R}}\), where \([\varphi^t]_{t \in \mathbb{R}}\) is a 1-parameter family of diffeomorphisms generated by \(X\). The state of initial condition \(x\) after time \(t\) is \(x(t) = \varphi^t(x)\). In this case, the positive semi-orbit passing through the point \(x\) is defined by \(O^+_s(x) = \{\varphi^t(x) | t \geq 0, t \in \mathbb{R}\}\). The set of limit points of \(O^+_s(x)\), that is, \(\omega(x) = \cap_{t \geq 0} O^+_s(\varphi^t(x))\), is called an \(\omega\)-limit set. A negative semi-orbit \(O^-_s(x)\) and an \(\alpha\)-limit set such that \(\alpha(x) = \cap_{t \leq 0} O^-_s(\varphi^t(x))\) are defined by the inverse time limit \(t \rightarrow -\infty\) in the same manner.

### 2.2 Lyapunov Functions

A closed invariant set \(I\) is called stable in the sense of Lyapunov if there exists a neighborhood \(T\) involved in any small neighborhood \(Y\) such that \(\forall x \in T\) and \(O^+_s(x) \subset Y\). The C\(^1\)-class function \(V: M \rightarrow \mathbb{R}\) is called a weak Lyapunov function of flow \(\varphi\) if \(V \circ \varphi(t) \leq V(x)\) for \(\forall x \in M\) and \(\forall t \geq 0\). In other words, \(V(x) \leq 0\) for \(\forall x \in M\) and \(\forall t \geq 0\). Moreover, if \(V \circ \varphi(t) < V(x)\), that is \(V(x) > 0\) for \(\forall x \in M\) and \(\forall t \geq 0\), then \(V\) is called a Lyapunov function.

We consider a dynamical system \(\dot{x} = f(x,u)\), where \(x \in M\), \(u \in U\), and \(U\) is an appropriate manifold. If a proper smooth positive function \(V: M \rightarrow \mathbb{R}\) satisfies

\[
\inf_{x \in U} \text{grad } V \cdot f(x,u) < 0
\]

(\(\leq 0\)) for \(\forall x \in M \setminus \{0\}\), then \(V\) on \(M\) is called a control-Lyapunov function (resp. a weak control-Lyapunov function), where the function \(V: M \rightarrow \mathbb{R}\) is called proper if a set \(\{x \in M | V(x) \leq a\}\) is compact for any \(a > 0\).

### 2.3 Morse Theory

Morse theory indentify a topological invariance of manifolds in terms of the critical points of a non-degenerate function called a Morse function [17],[18]. The Morse function is defined as follows.

**Definition 1.** Let \(f: M \rightarrow \mathbb{R}\) be a smooth function. If the derivative \(Df(p): T_pM \rightarrow \mathbb{R}\) is a zero map (resp. a non-zero map), then \(p\) is a critical point (resp. a regular point) of \(f\).

**Definition 2.** \(f\) is called a Morse function if every critical point \(p\) is non-degenerate, that is, det\(Hf(p)\) \(\neq 0\), where \(Hf(p) = \partial^2 f(p)/(\partial x_i \partial x_j)\) is a Hessian.

**Definition 3.** The number of negative eigenvalues of \(Hf(p)\) is called the Morse index of \(p\). ind\((p)\) denotes an index of \(p\).

The local coordinates around the critical points of Morse functions are expressed in quadratic form (see Proposition A.1).

Let us consider a gradient flow \(\dot{x} = -\nabla f(x)\) of a Morse function \(f\) for \(x \in M\). The flow has a solution \(\varphi^t: M \rightarrow M\), which is the generated invertible 1-parameter family of \(-\nabla f\) with respect to \(t\). Note that \(\lim_{t \rightarrow -\infty} \varphi^t(x)\) must be a critical point due to the compactness of \(M\).

**Definition 4.** For all \(p \in M\) of \(f\), we define

\[
W^U(p) = \left\{ x \in M | \lim_{t \rightarrow +\infty} \varphi^t(x) = p \right\},
\]

\[
W^W(p) = \left\{ x \in M | \lim_{t \rightarrow -\infty} \varphi^t(x) = p \right\}
\]

as a stable manifold and an unstable manifold, respectively, where \(W^U(p)\) is an \((m - \lambda)\)-dimensional submanifold of \(M\) and \(W^W(p)\) is a \(\lambda\)-dimensional submanifold of \(M\), where \(\lambda = \text{ind}(p)\).

Because regular points are on the gradient flow, we can see that \(\cup_p W^W(p) = M\) for all critical points \(p\). Note that all points on \(M\) except for critical points are on a single integral curve.

**Remark 2.1.** We can identify the gradient flow with the Morse function. This fact gives us a topological decomposition of the manifolds [4],[19],[20]. The identification between functions and flows is also used in Morse-Smale systems (as is discussed in the next section) in the \(\xi\)-function and a Morse-Smale flow.

**Remark 2.2.** The Morse function \(f\) of the gradient flow is identified with a weak Lyapunov function on \(M\), because the flow is stable to one of the critical points and the critical points of index 0 correspond to global stable points.

### 2.4 Morse-Smale Systems

Next, let us consider a more complex situation of critical points. Thom’s splitting lemma [21],[22] claims the following.

**Definition 5.** Let \(f: M \rightarrow \mathbb{R}\) be a smooth function. We define a nullity \(r\) by the corank \(r = m - \text{rank } Hf(p)\).

**Proposition 2.3** (Thom’s splitting lemma). The local structure around degenerate critical points can be represented as the following differentiable function germ\(^2\):

\(^1\) Conley’s fundamental theorem of dynamical systems states that there exists complete Lyapunov functions outside of a chain recurrent set in dynamical systems, where the chain recurrent set is a set of periodic (\(\epsilon\)-pseudo) orbits.

\(^2\) The germ defined by an equivalence class containing \(f\) itself expresses the behavior of \(f\) in the neighborhood of \(p\).
Consider a Morse-Smale system

\[ f(x) = f(p) - x_1^2 - \cdots - x_\gamma^2 + x_{\gamma+1}^2 + \cdots + x_m^2 + h(x_{\mu+1}, \ldots, x_m), \]  

where \( r > 0 \), \( Hh(p) = 0 \) and \( h \) is the higher order function germ than the second order, \( h \) is called the \textit{residual singularity of} \( f \).

This paper discusses a dynamical system associated with degenerate critical points with nullity 0 or 1 of a weak Lyapunov function, called a Morse-Smale system. A Morse-Smale system

or (flow) \cite{12,16} is a vector field such that \( \alpha \) - and \( \omega \)-limit sets of every trajectory are isolated singular points or closed orbits.

**Definition 6.** A smooth vector field \( X \) is called a \textit{Morse-Smale system} if

i) \( X \) has a finite number of singular points \( \beta_1, \cdots, \beta_n \) and closed orbits \( \beta_{\mu+1}, \cdots, \beta_r \).

ii) For any \( x \in M, \alpha(x) = \beta_i \) and \( \omega(x) = \beta_j \), where \( i \neq j \).

iii) For any closed orbit \( \beta_i \), there is no \( x \in M \setminus \beta_i \) such that \( \alpha(x) = \beta_i \) and \( \omega(x) = \beta_i \).

iv) The stable and unstable manifolds associated with \( \beta_i \) have a transversal intersection.

The set \( \beta_1, \cdots, \beta_n \) is called the \textit{singular elements} of the field \( X \). The Morse-Smale system permits a decreasing function called a \textit{\( \varepsilon \)-function} along trajectories instead of Morse functions.

**Definition 7.** Let \( \delta_i \subset \{ p \} \) be a connected set of critical points of \( f \), where \( i \in \mathbb{N} \). \( \delta_i^{(k)} \) denotes \( \delta_i \) with a nullity \( r \). We define \( \Delta = \bigcup_{i,j} \delta_i^{(k)} \) and \( \Delta' = \bigcup_i \delta'_i \).

**Definition 8.** \( N(x) \) denotaes a neighborhood of \( x \in M \).

**Definition 9.** \( B \) \textit{denotes the open unit ball in} \( \mathbb{R}^3 \), \( S \) \textit{denotes the unit sphere in} \( \mathbb{R}^n \).

**Definition 10.** A smooth function \( f: M \rightarrow \mathbb{R} \) is called a \textit{\( \varepsilon \)-function} for \( M \) if

i) \( \Delta = \Delta^0 \cup \Delta^1 \), where \( \Delta^0 = \bigcup_{i=1}^{r} \delta_i^{(k)} \) and \( \Delta^1 = \bigcup_{i=r+1}^{n} \delta'_i \).

ii) Each \( \delta_i^{(k)} \) is a closed connected \( 1 \)-dimensional submanifolds of \( M \). The Morse index of \( f \) is constant on each \( \delta_i^{(k)} \).

iii) For each \( \delta'_i \), there exists an orientable neighborhood \( N(\delta'_i) \) and a diffeomorphism that maps \( N(\delta'_i) \) into \( \mathbb{R}^{n-1} \times S^1 \). The local coordinates of \( \mathbb{R}^{n-1} \times S^1 \) are \( x_1, \cdots, x_n-1 \) in \( \mathbb{R}^{n-1} \) and \( x_m \) in \( S^1 \). For each point in \( S^1 \), the quadratic part of (4) has the same index of \( f \).

Actually, there exists a \( \varepsilon \)-function corresponding to a Morse-Smale flow as follows (see Definition A.1).

**Theorem 2.4.** If \( X \) is a Morse-Smale system, then there exists a \( \varepsilon \)-function for \( X \).

A Morse-Smale system without closed orbits is called a \textit{gradient-like Morse-Smale system}.

**Definition 11.** Consider a Morse-Smale system \( X \) on \( M \). If \( f(p) = \text{ind}(p) \) for all critical points and closed orbits, then \( f \) is called \textit{self-indexed}.

**Remark 2.5.** According to the compactness of \( M \) and the maximum value theorem, \( f \) will always take a maximum and a minimum on \( M \). If \( f \) is self-indexed, the maximum corresponds to a critical point of the index \( m \) and the minimum corresponds to a critical point of the index 0. Thus, the image of \( f \) exists in the interval \([0, m]\). The critical points \( p_i \) and \( p_j \) can be arranged such that \( f(p_i) > f(p_j) \) implies \( \text{ind}(p_i) \geq \text{ind}(p_j) \) while keeping the topology of \( M \) \cite{18} (see Lemma A.2).

The above shows that \( \varepsilon \)-function can be considered to be a weak Lyapunov function on a closed manifold in the following case.

**Definition 12.** Consider a Morse-Smale system \( X \) on \( M \). Let \( f \) be a positive definite \( \varepsilon \)-function for \( X \). We call \( f \) a \textit{Lyapunov-\( \varepsilon \)-function}.

A self-indexed \( \varepsilon \)-function on \( M \) is a Lyapunov-\( \varepsilon \)-function.

**2.5 Morse-Smale Control Systems**

Next, we add control inputs to the Morse-Smale system.

**Definition 13.** Consider \( M \). Let \( k: M \rightarrow U \) be a function, where \( U \subset \mathbb{R}^n \). Now let us consider the smooth vector field

\[ \dot{x} = f(x, u) \]  

such that \( \dot{x} = f(x, 0) \) is a Morse-Smale system, where \( x \in M \) and \( u = k(x) \in U \). We call (5) a \textit{Morse-Smale control system}.

**3. Main Results**

Here, we present a recursive method for constructing weak control-Lyapunov functions for Morse-Smale control systems. First, we define some basic concepts in global stability. Next, we clarify the requirements of control inputs for global stabilization. Finally, we present the procedure for constructing a finite set of weak control-Lyapunov functions.

**3.1 Problem Statement**

Let us consider an \( m \)-dimensional orientable closed smooth Riemannian manifold \( M \) as a state space. Let

\[ X_u: \dot{x} = f(x, u) \]  

be a Morse-Smale control system for \( x \in M \), where the input \( u = k(x) \in U \) is determined by a control law \( k: M \rightarrow U \). \( X_0 \) denotes the autonomous (Morse-Smale) system of \( X_u \) defined by \( u = 0 \). Now, we assume the following:

- **A1)** \( \{ 0 \} \) is a unique singular point of the index 0. This implies \( \{ 0 \} \) is a global asymptotical stable point.

- **A2)** The control law \( k \) can be designed depending on \( x \in M \). Therefore, \( u \) can be discontinuous.

- **A3)** The system \( X_u \) always has a local Carathéodory solution \cite{23} at each \( x \in M \).

Condition A1) is introduced to simplify the problem. Generally speaking, systems will have many singular points with an index 0. The state transition between such singular points never occurs as a result of local inputs. Thus, we require a singular point rearrangement realized by dynamical compensations or bifurcation to cause the state transition \cite{7}. We avoid such a complex case. In this paper, for the given (6) with A1), we shall clarify the minimum necessary procedure to achieve a global asymptotical stabilization by weak control-Lyapunov
functions. In this case, conditions A2 and A3 arise from the fact that continuous inputs cannot stabilize global systems [3]. This situation is fairly common in comparison with conventional nonlinear systems, because it involves a local system $\dot{x} = F(x) + G(x)u$ around only one critical point of the index 0.

The Lyapunov-$\xi$-function $V_0$ can be used for a semi-global asymptotical stabilization; here ‘semi-global’ means everywhere except invariant sets $V_0 = 0$ for $M \setminus \{0\}$. In the following section, we will state the condition of the input to achieve the global asymptotical stabilization of (6).

**Remark 3.1.** If the Morse-Smale system $F(x,0)$ is given as a controlled system $F(x, 0) = \bar{F}(x, v, 0)$ with a pre-control $v = \bar{k}(x)$, then the Lyapunov-$\xi$-function is a control Lyapunov-$\xi$-function.

### 3.2 Stability

In this section, we will investigate system states staying on invariant sets of Lyapunov-$\xi$-functions that is the main difficulty for stabilization of Morse-Smale systems.

**Definition 14.** Consider $X_0$ on $M$. Let $\Delta' = \Delta \setminus \{0\}$. We define $S = \bigcup_p W^s(p)$ for $p \in \Delta'$. We call $S$ an invariant submanifold.

Any $x \in S$ is locally asymptotically stable to $\Delta'$.

**Definition 15.** Consider $X_0$ on $M$. $R$ denotes a submanifold of $M$ such that $R = M \setminus S$.

Since $R \setminus \{0\}$ contains only regular points, we have the following definition of semi-global asymptotical stability.

**Definition 16.** Consider $X_0$ on $M$. $|_R$ denotes a restriction of a domain to $R \subset M$.

**Lemma 3.2.** Consider $X_0$ on $M$. A restricted flow $X_0|_R$ is asymptotically stable with respect to $\{0\}$ for any $x \in R$.

**Proof.** The solutions on $S$ move toward a point $p \in \Delta'$ through a positive time evolution. Such a solution remains in $\Delta'$ and never converges into $\{0\}$. On the other hand, there exists a $\xi$-function decreasing along a trajectory on $R$ according to Theorem 2.4. Thus, all solutions on $R$ converge to $\{0\}$.

**Lemma 3.3.** Consider $X_0$ on $M$. There exists a Lyapunov-$\xi$-function $V_0$ for $\{0\}$.

**Proof.** $M$ consists of the union of $R$ and $S$. The $\xi$-function on $(R \setminus \{0\}) \cup (S \setminus \Delta')$ decreases along a trajectory. The $\xi$-function has an image in a finite interval, because $M$ is compact. Let $V_0$ be a Lyapunov-$\xi$-function. Thus, $\{0\} \cup \Delta'$ corresponds to the set of $V_0 = 0$.

**Remark 3.4.** Without loss of generality, the $\xi$-function of $M$ can be considered to be a self-indexed Lyapunov-$\xi$-function.

### 3.3 Controllability

**Proposition 3.5.** $R|_{N(x)} = N(x) \setminus \bigcup_p W^s(p)|_{N(x)}$ for $x \in M$, where $p \in \Delta'$.

**Proof.** From Definition 14 and Definition 15, we have $R = M \setminus \bigcup_p W^s(p)$.

The unstable manifold regarding quadratic coordinates $-x^2_i$ in (4) around $p \in \Delta'$ intersects $R|_{N(p)}$, because connecting critical points $p' \in \delta^i_0 \subset N(p)$ are included in the subtracted set $\bigcup_p W^s(p)$ in Proposition 3.5.

**Definition 17.** Let $R_c(p)$ be a reachability set of $X_0$ from $p \in \Delta'$ for a finite time $\tau$. If $R_c(p) \cap R \neq \emptyset$, we call $p$ escapable to $R$.

**Definition 18.** We define a submersion $\pi_\delta: T_pM \to T_pW^s(p)$ such that $(\delta_1, \ldots, \delta_{i-1}, \delta_{i+1}, \ldots, \delta_m) \mapsto (\delta_1, \ldots, \delta_{i-1}, 0, \ldots, 0)$, where $\delta_i = \partial/\partial x_i$ are local coordinates of $W^s(p)$ for $i = 1, \ldots, m$. Let $\delta_i$ is a local coordinate along $\delta^i_0$.

**Definition 19.** Consider $X_0$ on $M$. If $\pi_\delta(x_0) \neq \emptyset$, then we call $p$ locally escapable to $R$.

There is actually a large difference between escapability and local escapability. That is, escapability claims that there is at least one path reaching unstable sets that may exist away from $p \in \Delta'$. We cannot detect such an unstable set only from the local viewpoint. On the other hand, local escapability requires unstabilizable inputs exactly at a critical point $p$. Therefore, there is a case in which $p$ is escapable, but not locally escapable (see Section 4). This fact corresponds to the difference between reachability and local reachability in nonlinear control theory [24],[25]. The Lyapunov method discussed in the next section needs local escapability.

Next, we shall discuss local escapability in terms of control inputs. It is difficult to discuss the stability of a system $X_0$ of the form $\dot{x} = F(x, u)$, because of discontinuous inputs, for example. If a control law $u = k(x)$ is given, we can check the local escapability of a controlled system $\dot{x} = F(x, k(x))$. However, we want to design inputs for certain practical situations. An input affine system $\dot{x} = f(x) + g(x)u$ is one of the simplest theoretical settings to design inputs. We usually assume that the local system around a critical point is input affine. The following is a sufficient condition of locally escapability for input affine systems.

**Definition 20.** If $X_0|_{\delta}$ is $\bar{X}_0|_{\delta}$: $\dot{x} = F(x) + \sum_i G_i(x) u_i$ around $p$, we call $\bar{X}_0|_{\delta}$ a locally input affine system of $X_0$.

**Definition 21.** $L(p)$ denotes a Lie algebra of a locally input affine system $\bar{X}_0|_{\delta}$ that is the minimum involutive distribution of $\bar{X}_0|_{\delta}$ including itself.

**Corollary 3.6.** Consider $X_0$ on $M$. If $\dim L(p) = m$ for a critical point $p \in \Delta'$, then $p$ is locally escapable to $R$.

**Proof.** In this case, $X_0$ at $p$ is locally controllable. Thus, the system state can be moved to $R_{N(p)}$ by controls.

In the same way as for $p \in \Delta'$, we can define local escapability of a set of attracting orbits $W^s(p)$. If all points in $S$ are locally escapable to $R$, the system is globally asymptotically stable to $\{0\}$. However, this requirement is quite strict in certain actual cases. Accordingly, we shall define a realistic situation as follows.

**Definition 22.** Consider $X_0$ on $M$. If $y \in S$ can reach $R$ in a finite time by using $u$, we call $y$ finite time escapable to $R$.

**Definition 23.** Consider $X_0$ on $M$. If all points in $N(x)$ of $x \in M$ are locally escapable to $R$, we call $x$ strongly escapable to $R$.

**Proposition 3.7.** Consider $X_0$ on $M$. If $p$ is strongly escapable to $R$, then $y \in W^s(p)$ is finite time escapable to $R$ through $N(p)$. 

A Lyapunov function strictly decreases along a flow. From Proposition 3.9, trajectories starting from critical points lie in the collar neighborhood. Hence, Lyapunov methods for Morse-Smale systems require local escapability.

**Proposition 3.10.** Consider $X_u$ on $M$. Let $L_0 = \bigcup_i \tilde{L}_i(\delta_i)$ and $L_0^\ast = L_0 \times (0, 1]$. If an invariant submanifold $S$ of $M$ is an escapable submanifold to an interior point of $L_0^\ast$, then $X_u$ is globally asymptotically stable.

If an invariant set $\Delta_1 = \{x \in M \mid V_0 = 0\}$ is an escapable submanifold $Q$, then it is globally asymptotically stable. However, $\Delta_1$ is not homeomorphic to $\mathbb{R}^n$. Accordingly, we construct not (control-) Lyapunov functions, but weak (control-) Lyapunov functions $V_1$ on $V_0 = 0$. Thus, we must consider the escapability of $V_1$. To be precise, we must find a weak control-Lyapunov function $V_1$ converging to an interior point $x_1 \in \sigma_1$ for $x \in \Delta_1$, where $\sigma_1 \subset \Delta_1$ is an escapable submanifold to $L_0^\ast$. However, there exists an invariant set $\Delta_2 = \{x \in \Delta_1 \setminus x_1 \mid V_1 = 0\}$ in general. Thus, we must find another weak control-Lyapunov function $V_2$ converging on an escapable submanifold to $L_1^\ast$ for $x \in \Delta_2$, where $L_1^\ast$ is the collar neighborhood for $\Delta_2$. In the same manner, we have to check the invariant set $\Delta_3 = \{x \in \Delta_2 \setminus x_2 \mid V_2 = 0\}$.

A weak Lyapunov filtration ensures that Lyapunov functions monotonously decrease. Moreover, if the filtration is terminated by 0, the system $X_u$ on $M$ is globally stable. Let us conclude the above construction as follows.

**Algorithm 3.11.** We state the following recursive procedure to construct a weak Lyapunov filtration.

**P0** Let $V_0: M \rightarrow \mathbb{R}$ be a weak control-Lyapunov function converging to $0$ for any $x \in R$.

**P1** Consider $\Delta_1 := \{x \in M \mid V_0(x) = 0\}$. Let $\sigma_1 \subset \Delta_1$ be an escapable submanifold to $L_0^\ast$.

i) If there exists a self-indexed weak control-Lyapunov function $V_1: \Delta_1 \rightarrow \mathbb{R}$ to any interior point $x_1 \subset \sigma_1$ for $x \in \Delta_1$, then go to P2).

ii) If there is no $V_1$, then stop.

**Pj** $(i \geq 2)$ Consider $\Delta_{i,k} := \{x \in \Delta_{i-1} \setminus x_{i-1,k} \mid V_{i-1,k} = 0\}$ and $\Delta_i := \bigcup_k \Delta_{i,k}$, recursively, where $x_{i-1,k}$ is the interior point of $\sigma_{i-1,k} \subset \Delta_{i-1}$, to which $V_{i-1,k}$ converges, $k$ is the number of $\delta$ in $\Delta_{i-1}$.

(a) If $\Delta_i \neq \emptyset$, let $\sigma_{i,k} \subset \Delta_i$ be an escapable submanifold to some point $L_{i-1,k}^\ast$ in $\Delta_{i-1}$, where $L_{i-1,j}(x_{i-1,j}) = V_{i-1,j}(x_{i-1,j})$ for all $j$, $L_{i-1} = \bigcup_j \tilde{L}_{i-1,j}(x_{i-1,j})$, and $L_{i-1,k}^\ast = L_{i-1} \times (0, 1]$. If there exists a self-indexed weak control-Lyapunov function $V_{i,k}: \Delta_k \rightarrow \mathbb{R}$ to any interior point $x_{i,k} \subset \sigma_{i,k}$, then go to P$(i+1)$).

ii) If there is no $V_{i,k}$, then stop.

**Proposition 3.12.** Consider $M$. The weak Lyapunov filtration is a finite degree.
Let $M^n$ be an $m$-dimensional closed manifold. Each level-set $l_0(x)$ is compact, because $V_0$ is a submanifold of $M$, by the implicit function theorem. The transversal intersection between a level-set $l_0(x)$ of $V_0$ and $M^n$ is $M^{n-1}$. Thus, the transversal intersection between a level-set of $V_1$ and $M^{n-1}$ is $M^{n-2}$. In the same manner, we obtain a 0-dimensional intersection between $V_{m-1}$ and $M^0$. Thus, if we find $V_m$, there is no submanifold to stabilize.

Theorem 3.13. Consider $X_u$ on $M$. The weak Lyapunov filtration is $2$ degrees: $M \supset \Delta_1 \supset \Delta_2 \supset \emptyset$.

Proof. The closed orbits $\delta^1$ have two critical points for a gradient flow generated by $V_1$. The critical points $\delta_{0,1}$ and $\delta_{2,1}$ have indexes 0 and 1, respectively. Thus, $V_2$ should make $\delta_{2,1}$ locally escapable to a subset of $\Delta_1 \setminus \Delta_2$. Because $\Delta_2 = \delta_{2,1}$ is a point, the sequence terminates.

4. Example

Let us consider the following system on a torus $M = S^1 \times S^1$ in $\mathbb{R}^2$:

$$
\begin{aligned}
\dot{\theta} &= \frac{15}{8} \sin \theta - \frac{1}{8} \cos \phi \sin \theta + (\cos \phi) u_1, \\
\dot{\phi} &= \frac{1}{8} \sin \phi - \frac{1}{8} \sin \phi \cos \phi + ku_2,
\end{aligned}
$$

where $(\theta, \phi) \in M$ and $(u_1, u_2)$ are inputs. Let us assume $k = 1$. We can find the weak control-Lyapunov function:

$$V_0 = \frac{17}{8} = \frac{15}{8} \cos \theta - \frac{1}{8} \cos \phi - \frac{1}{8} \cos \phi \cos \theta.$$

First, $p(\theta, \phi)_{|_{\theta=\pi}} \in \delta^1_0$ is locally escapable when $\phi \neq \pi/2, 3\pi/2$ because of $u_1$ (Fig. 1, middle-right, $u_1 = 1, u_2 = 0$). Furthermore, the points $\phi = \pi/2, 3\pi/2$ are on escapable submanifolds, because $u_2$ can move the states at these points to a locally escapable region $\phi \neq \pi/2, 3\pi/2$ (Fig. 1, lower-right, $u_1 = 0, u_2 = 1$). On the other hand, $p \in \delta^2_0$ is locally escapable because of $u_2$ (Fig. 1, lower-left, $u_1 = 0, u_2 = 1$).

In the case of $k = \sin \theta$, $\delta^2_1$ is not locally escapable, but escapable, because $W^u(\delta^2_1)$ is an escapable submanifold. Thus, $V_0$ increases along the solution and we cannot construct weak-Lyapunov functions.

5. Conclusion

In this paper, we discussed global asymptotical stability, controllability, and stabilization for Morse-Smale control systems. We presented a recursive method for constructing weak control-Lyapunov functions based on a Morse-Smale flow. We addressed the condition of the nullity $r \leq 1$, because of Morse-Smale. As a result, we found that the procedure could be finished in a finite number of steps. This discussion on a closed manifold can be extended to more complex situations: e.g., manifolds with a boundary, a mixed dynamical system, and a system with more general singularities by using Conley index theory [4],[6],[7]. Furthermore, the points $\phi \in \delta^2_2$ and $\theta \in \delta^1_2$ are not critical points for a gradient flow generated by $V_1$. We presented a recursive method for constructing weak control-Lyapunov functions.

The authors are considering the following future work. On the residual singularity of the local structure around degenerate critical points in Thom’s splitting lemma, for example, in the case of $r = 1$, the singular point $p$ of $f$ is called $A_r$-type if $f^{(p)} = \cdots = f^{(k)}(p) = 0$ and $f^{(k+1)}(p) \neq 0$. For such an $A_r$-type singular point, there exists a formal local coordinate such that $f(x) = f(p) + x^{k+1}$ for $p$. The classification of residual singularities has been advanced by Arnold [26]. The simple singular points, which do not have moduli, are classified by the series of simple Lie algebras: $A_k, k > 1$, $D_k, k > 4$ and $E_6, E_7, E_8$ through a Dynkin diagram. The classification has the capability of dealing with a more unified definition of stability of degenerate critical points. On the other hand, it is known from Hironaka’s resolution of the singularity theorem that there exists a resolution $\psi: X_0 \rightarrow X_0$ of a singularity for any algebraic variety $X_0$. Accordingly, $\psi$ can be obtained by making several blow-ups on the submanifold [27]. This method may be used for changing the degenerate cases into regular problems in Morse theory.

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References

Appendix

A.1 Morse Theory

Proposition A.1 (Morse’s lemma). There exist suitable local coordinates \((x_1, \ldots, x_n)\) in the neighborhood of \(p\) of the Morse index \(\lambda\) such that \(f\) has a standard form given by

\[
f(x) = f(p) - x_1^2 - \cdots - x_{\lambda}^2 + x_{\lambda+1}^2 + \cdots + x_n^2. \tag{A.1}
\]

In fact, the local structure defined by Morse functions represents an integral curve of solutions of gradient flows on \(M\).

A.2 Morse-Smale Systems

Lemma A.2. Let \(X\) be a Morse-Smale system on \(M\). Let \(\beta_i > \beta_j\) mean that there is a trajectory from \(\beta_i\) to \(\beta_j\) whose \(\alpha\)-limit set is \(\beta_i\) and whose \(\omega\)-limit set is \(\beta_j\). Then \(\alpha\) satisfies:

i) \(\beta_i \not= \beta_j\).

ii) If \(\beta_i > \beta_j\) and \(\beta_j > \beta_i\), then \(\beta_i > \beta_j\).

iii) If \(\beta_i > \beta_j\), then \(\dim W^i_j \geq \dim W^j_i\) and equality can occur only if \(\beta_j\) is a closed orbit, where \(W^i_j\) is the unstable manifold associated with \(\beta_i\).

Definition A.1. Let \(X\) be a smooth vector field on \(M\). Then a \(\xi\)-function \(f\) for \(M\) is called a \(\xi\)-function for \(X\) if

i) \(Xf < 0\) for all \(p \in M \setminus \Delta\), i.e., \(f\) is decreasing along the trajectories of \(X\) or the trajectories of \(X\) are transversal to the level lines of \(f\).

ii) If \(p\) is a singular point of \(X\), then \(p \not\in \Delta\). 

iii) There exists a constant \(\kappa > 0\) such that \(-Xf(p) \geq \kappa d(p, \Delta)^2\) for \(p \in N(\delta)\) on each \(N(\delta)\), where \(d\) is the distance function \(d\) taken from a Riemannian metric.

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