A Frequency Domain Approach for Robust Control Design by Fractional Calculus

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Abstract: This paper deals with robust control design for linear time-invariant systems by fractional order control (FOC), in which controllers described by differential equations of non-integer orders. The purpose of this paper is to take advantages of the introduction of control parameters and satisfy additional specifications of design, ensuring a robust performance of the controlled system with respect to gain variations and noises. A method for tuning the controller is proposed to fulfill five different design specifications. The specifications on the gain crossover frequency and the phase margin are readily satisfied, together with the damping property of the time response of the controlled system being kept. Simulation results are given to illustrate the effectiveness of this method.

Key Words: fractional calculus, fractional order control, robust control, gain variations.

1. Introduction

In recent years, the better understanding of the potential of fractional calculus and the increasing number of studies related to the applications of fractional order controllers in many areas of science and engineering have led to the importance of studying aspects such as the analysis, design, tuning and implementation of these controllers, including the authors' study [1]–[3]. Fractional calculus is a generalization of the integration and differentiation to the non-integer order. In theory, fractional order control systems can include both dynamic systems to be controlled and/or controllers which are described by the fractional calculus. However, in control practice, it is more common to consider the fractional order controllers. This is due to the fact that the plant model may have been already obtained as an integer order model in the classical sense. Therefore, in this paper, the fractional order control (FOC) is used for the same meaning as the fractional order controllers.

In this line, the objective of this work is to apply the FOC for industrial applications to improve the performance of the controlled systems. It is important to realize that there is a very wide range of control problems and consequently also a need for a wide range of design techniques. On the research side it appears that the development of design methods for integer order control (IOC) is approaching the point of diminishing returns. There are some difficult problems that remain to be solved. Many researchers have been reported that the theoretical and practical advantage of the FOC over the IOC. Oustaloup [4],[5] studied the fractional order algorithms for the control of dynamic systems and demonstrated the superior performance of the CRONE controller over the conventional PID controller. Podlubny [6] proposed a generalization of the PID controller, namely the $P^\alpha D^\beta$ controller. A frequency domain approach by using fractional order PID controllers was also studied [7]. Further research activities were developed which include the tuning techniques for the FOC by an extension of the classical control theory [8]–[10], the tuning of $H_\infty$ controllers [11], and so on. Although it is useful to extend the practical use of the FOC for various conventional control problems, there are few literatures to give procedures to design it for systems with model uncertainties in a unified manner, i.e., realization, formulation and tuning parameters of the FOC model.

This paper describes a unified way to design the FOC model in a view of a loop shaping design so as to easily apply it for conventional robust control problems. It is one of the advantages of the fractional calculus as an alternative option to solve some of the control problems that can arise when dealing with industrial applications. Since the proposed FOC has five parameters to tune, up to five design specifications for the controlled system can be met. It is essential to study which specifications are more interesting as far as performances and robustness are concerned, since the objective is to obtain a controlled system which is robust against uncertainties of the plant model, load disturbances and high frequency noise. All these constraints will be taken into account in the tuning technique of the controller in order to take advantages of the introduction of the fractional orders.

2. Fractional Calculus

2.1 Mathematical Definitions of Fractional Integral and Derivative

The mathematical definitions of fractional calculus have been the subject of several different approaches [12],[13]. In order to proceed, we first introduce the most encountered definition of the fractional integral. It is called Riemann-Liouville (RL) fractional integral, which is defined on the usual $L_1$ space, written by

$$D^{-\alpha} x(t) \triangleq \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} x(\tau) d\tau. \quad (1)$$

where $q(\in \mathbb{R}^+)$ is the order of integral and $\Gamma(z)$ is the Gamma function.
The fractional derivative can be described using the Laplace transform of the partial conditions for the order $m$

$$\Gamma(z) \approx \int_0^\infty e^{-t} t^{z-1} dt, \quad (z > 0).$$

Then Eq. (1) above is used to define the fractional derivative. The fractional derivative of function $f(t)$ of the order $q \in \mathbb{R}^+$ is given by

$$D^q f(t) \triangleq \frac{d^m}{dt^m} \left( D^{\alpha-\beta} f(t) \right) = \frac{1}{\Gamma(m-p)} \frac{d^m}{dt^m} \int_0^t (t-\tau)^{m-p-1} f(\tau)d\tau,$$  

(3)

where $m \in \mathbb{N}$ satisfies $m-1 < q \leq m$.

For convenience, Laplace domain notion is commonly used to describe the fractional operation. The Laplace transform of the RL fractional derivative/integral Eq. (1), (3) under zero initial conditions for the order $q$ is given by

$$\mathcal{L} \{ D^q x(t) \} = s^q \mathcal{X}(s).$$

(4)

### 2.2 Realization of Fractional Calculus

As can be observed from Eq. (1) and (3), if the FOC would be perfectly realized, all the past inputs from initial time $t_0$ should be memorized. However, it is impossible to realize in practical applications. Therefore proper approximation by a finite difference method [14] which realizes fractional derivative in a frequency domain as illustrated in Fig. 1, is introduced. Then the description of the approximation model is modified so as to be easily implemented in Matlab.

The fractional derivative $s^q (q \geq 0)$ or integral $q < 0)$ within an objective frequency range $[\omega_l, \omega_u]$ is approximated by a rational transfer function of $n$, th-order, denoted by $\hat{s}^q$

$$s^q \approx \hat{s}^q = \frac{k(s-z_1)(s-z_2)\cdots(s-z_n)}{(s-p_1)(s-p_2)\cdots(s-p_n)}$$

(5)

in which the zeros, poles and gain are

$$z_i = -\alpha \frac{\beta_{i-1}}{\beta_i}, \quad p_i = -\alpha \frac{\beta_{i+1}}{\beta_i}, \quad k = \alpha \frac{\beta_{n+1}}{\beta_n},$$

(6)

where

$$\alpha = \omega_\alpha \frac{\alpha}{\beta},$$

$$\beta_i = 2n_i \cdot \log \frac{\omega_i}{\omega_\alpha} + 2i - 1, \quad \text{for } i = 1, 2, \cdots, n_i.$$  

(7)

(8)

Figure 2 shows the Bode plots of an ideal frequency band-limited fractional derivative and its 3rd, 4th and 5th-order approximation which is the right hand side of Eq. (5) where $q = 0.5$, $\omega_\alpha = 0.1 \text{ [rad/sec]}$, $\omega_i = 100 \text{ [rad/sec]}$.

### 3. Problem Formulation

#### 3.1 Description of a Controlled System

A SISO simple feedback controlled system is considered which described by Fig. 3. The FOC denoted by an element $C_F$ will be formulated later in detail. The plant $G$ is supposed to be modeled by a LTI model with uncertainties $\Delta$, an input disturbance $d$ and a sensor noise $n$.

#### 3.2 Design Specifications

As commented in the introduction, the objective of this paper is to design the FOC so that the system fulfills different specifications regarding robustness to plant uncertainties, load disturbances and high frequency noise. For that reason, specifications related to phase margins, sensitivity functions and robustness constraints are going to be considered in this design method, due to their important features regarding performance, stability and robustness. Of course, other kinds of specifications can be met, depending on the particular requirements of the system. Therefore, an ideal open-loop system transfer function $L(s) = C_F(s)G(s)$ should be illustrated by Fig. 4 and the design problem is formulated as follows in details.
1. Phase margin (\( \varphi_m \)) and gain crossover frequency (\( \omega_c \)) specifications: Gain and phase margins have always served as important measures of robustness. It is known that the phase margin is related to the damping of the system and therefore can also serve as a performance measure [15]. The equations that define the phase margin and the gain crossover frequency are

\[
|L(j\omega_c)|_{dB} = 0 \text{ [dB]},
\]

\[
\arg(L(j\omega_c)) = -\pi + \varphi_m \text{ [rad]}.
\]

2. Robustness to variations in the gain of the plant: The next constraint can be considered in this case [16]

\[
\frac{d}{d\omega} \arg(L(j\omega))_{|\omega=\omega_c} \approx 0.
\]

This condition forces the phase of the open-loop system \( L(s) \) to be almost flat at \( \omega_c \) and hence to be nearly constant within an interval around \( \omega_c \). It means that the system is more robust to gain changes and the overshoot of the response is almost constant within a gain range, i.e. damping property of the time response is not deteriorated. It must be remarked that the interval of gains for which the system is robust is not fixed with this condition. That is, the user cannot force the system to be robust for a particular gain range. It depends on the frequency range around \( \omega_c \) for which the phase of the open-loop system keeps flat. This frequency range will be longer or shorter, depending on the resulting controller and the plant.

3. High frequency noise rejection: A constraint on the complementary sensitivity function \( T(j\omega) \) can be established

\[
|T(j\omega)| = \left| \frac{L(j\omega)}{1 + L(j\omega)} \right| \leq W_n, \ \forall \omega \geq \omega_h
\]

with \( W_n \) is the desired noise attenuation for frequency range \( \omega \geq \omega_h \) [rad/sec].

4. Good disturbance rejection: A constraint on the sensitivity function \( S \) can be defined

\[
|S(j\omega)| = \left| \frac{1}{1 + L(j\omega)} \right| \leq W_d, \ \forall \omega \leq \omega_d
\]

with \( W_d \) is the desired output disturbance rejection for desired frequency range \( \omega \leq \omega_d \) [rad/sec].

5. Steady-state error cancellation: The fractional integrator \( 1/s^q \) (\( q \in \mathbb{R}^+ \)) can be, for steady-state error cancellation, more efficient than an integer order integrator. To examine the ability of \( 1/s^q \) upon the conventional integrator \( 1/s^n \) (\( n \in \mathbb{N} \)), consider a unity feedback system with open-loop transfer function

\[
\frac{1}{s^q} G(s) = \frac{K(s-z_1)(s-z_2)\cdots(s-z_m)}{s^n(s-p_1)(s-p_2)\cdots(s-p_n)}
\]

which is said to be type-\( q \) system.

For example, the steady state error for the system with a unit step function input \( 1/s \) is described by

\[
\epsilon_{\text{step}} = \frac{1}{1 + \lim_{s \to 0} \frac{G(s)}{s}}
\]

\[
= \frac{1}{1 + \lim_{s \to 0} \frac{K \prod_{i=1}^{m} (s-z_i)}{s^n \prod_{i=1}^{n} (s-p_i)}}.
\]

The value \( q \) has only to be some value more than zero (\( q > 0 \)) to obtain the error be zero. Therefore the characteristic of the type-\( q \) (\( 0 < q < 1 \)) system is theoretically the same as the type-1 system.

It is also shown that for the system with a ramp function input \( 1/s^2 \), the steady state error is

\[
\epsilon_{\text{ramp}} = \frac{1}{1 + \lim_{s \to 0} \frac{G(s)}{s^2}}
\]

\[
= \frac{1}{1 + \lim_{s \to 0} \frac{K \prod_{i=1}^{m} (s-z_i)}{s^n \prod_{i=1}^{n} (s-p_i)}}.
\]

It is found that the error is zero if \( q \) is more than one (\( 1 < q < 2 \)), which the characteristic is theoretically the same as the type-2 system. So, the conditions to satisfy zero steady-state error of the fractional integrators are easier than the conventional integer integrators.

To sum up the five specifications above, as we already know, the loop gain in a frequency domain has to be fulfilled the shape described by Fig. 4. A good performance of disturbance rejection and less steady-state error require large loop gain in a low-frequency range, while good noise rejection requires small loop gain in a high-frequence range. A good robustness against gain variations requires flat phase around \( \omega_c \); accordingly gentle slope of the loop gain in transition range \( [\omega_d, \omega_h] \) around \( \omega_c \).

4. The FOC Design

This section, which is the main result of this paper, shows a procedure to design the FOC model by a loop shaping approach in a frequency domain.

4.1 Implementation of FOC

Since required to manage the five specifications described in Sec.3.2, the FOC has to be formulated as

\[
C_F(s) = C_0 \left( \frac{s + \omega_d}{s} \right)^{\varphi_t} \cdot \left( \frac{1}{1 + \frac{\omega_d}{s}} \right)^{\varphi_t},
\]

where \( C_0 \) is the gain parameter to tune so as to satisfy the 1st specification. The first parenthesis is an order \( qF(\in \mathbb{R}^+) \) band-limited integrator to fulfill the 4th and 5th specifications. The second parenthesis is band-limited fractional derivative, which is the main element of the FOC, to manage the 1st and 2nd specifications. The third parenthesis is an order \( qF(\in \mathbb{R}^+) \) low-pass filter to satisfy the 3rd specification. The schematic Bode diagrams of this desirable controller is illustrated in Fig. 5. The design objective is to set the parameters so that the enough performance is obtained against model uncertainties while maintaining minimal error due to the disturbance and noise. The parameters adjustment procedure will be described in the following simulation examples in details.
4.2 Design Procedure by Illustrative Examples

Let a controlled plant with model uncertainties and its nominal plant described by

\[ G(s) = \frac{k}{s(1 + \tau s)} e^{-Ls}, \quad G_{\text{nom}}(s) = \frac{k_0}{s(1 + \tau_0 s)} e^{-L_0 s} \]  

where the parameters are \( k_0 = 10, \tau_0 = 1000, L_0 = 0 \) respectively. Following 3 cases of parameters variances are considered.

Case.1: \( 0.1k_0 \leq k \leq 10k_0 \)
Case.2: \( 0.1\tau_0 \leq \tau \leq 10\tau_0 \)
Case.3: \( L_0 \leq L \leq 0.2 \)

The design specifications given in Sec.3.2 lead to the control effort required for this system as follows:

(i) gain crossover frequency of the nominal value:
\( \omega_{gc} = 5 \text{ [rad/sec]} \);
(ii) phase margin: \( \varphi_m = 50^\circ \);
(iii) robustness to variation of Case.1 to 3 should be fulfilled;
(iv) good disturbance rejection and perfect asymptotic tracking to a ramp function signal lead to the sensitivity function:
\[ |S(j\omega)| \leq -40 \text{ [dB]}, \forall \omega \leq \omega_d = 0.01 \text{ [rad/sec]} \];
(v) good noise rejection:
\[ |T(j\omega)| \leq -40 \text{ [dB]}, \forall \omega \geq \omega_0 = 100 \text{ [rad/sec]} \].

The design procedure of the FOC model Eq.(17) is demonstrated as follows.

**Step.1** To ensure the phase margin \( \arg(C_F(j\omega)G_{\text{nom}}(j\omega)) = -130^\circ \), the fractional order \( q = 0.55 \) is calculated. Because the phase which should recover is \( 50^\circ = -130^\circ - (-180^\circ) = \) the phase of the plant around \( \omega_{gc} \), and \( 0.55 \approx 50/90 \). Then the rational transfer function of the 4th-order approximation model of Eq.(5) is adopted.

**Step.2** As the plant gains variations around \( \omega_{gc} \) equals 38.5 [dB], the frequency band \([\omega_l, \omega_h]\) needs at least to cover 1.32 [dec] to ensure the flatness of the phase condition of Eq.(11). Then we roughly account for the margin of the flatness, \( \omega_l = \omega_{gc}/50 \) and \( \omega_h = 50 \cdot \omega_{gc} \) are set.

**Step.3** To reject the disturbance and null steady-state error for a ramp function, it is required to satisfy \( q_I > 0 \). Since we particularly concentrate on managing robustness to the model uncertainties and easy realization of the controller, we set \( q_I = 1 \) and \( \omega_d = \omega_{gc}/200 \).

**Step.4** To avoid the amplification of the high-frequency measurement noise, we set \( q_T = 1 \) and \( \omega_T = 200 \cdot \omega_{gc} \).

**Step.5** To ensure \( |C_F(j\omega)G_{\text{nom}}(j\omega)| = 0 \) [dB], \( C_0 = 900 \) is calculated.

Therefore, the resulting FOC becomes

\[ C_F(s) = 900 \left( \frac{s + 0.025}{s} \right) \cdot \hat{x}_{0.55} \left( \frac{1}{1 + 1000 s} \right), \]  

where the detailed expression for the 4th-order rational transfer function of the approximated fractional derivative is

\[ \hat{x}_{0.55} = 48.6(s + 0.1)(s + 1.09)(s + 11.8)(s + 128) 
\frac{(s + 0.4)(s + 4.3)(s + 46.7)(s + 506)}{(s + 0.04)(s + 4.3)(s + 46.7)(s + 506)} \]  

In order to measure the performance of the obtained FOC and compare with conventional controllers, it is also considered that the IOC (integer order control) denoted \( C_I \) which is the same formula as Eq.(17) but the parameters of orders \( q_I, q_T, q_F \) are integer values. It is similarly tuned the parameters, the IOC is obtained as follows

\[ C_I(s) = 900 \left( \frac{s + 0.025}{s} \right) \left( 1 + \frac{1}{1 + 1000 s} \right) \left( \frac{1}{1 + 1000 s} \right) \]  

The Bode plots of the both controllers Eq.(19) and (21) are shown in Fig. 6. As can be observed, their phases are both set to \( 50^\circ \) at \( \omega_{gc} \). The magnitudes of the functions \( S(s) \) and \( T(s) \) for the nominal plant are also shown in Fig. 7. It is found that the requirements (iv) and (v) are fulfilled on the both controllers.

For the Case.1, the Bode plots of the open-loop system \( L(s) \) are shown in Fig. 8. It is seen that the phase curve of the FOC is flat within an interval around \( \omega_{gc} \). It results that the specification of the phase margin is maintained under parameter variations. On the other hand, it is clear that the phase margin of the IOC is not fulfilled. The step response of the open-loop...
Fig. 8 Bode plots of $L(s)$ for Case.1 where $k = 0.1k_0$, $k = k_0$ (nominal) and $k = 10k_0$.

Fig. 9 Step response for Case.1 where $k = 0.1k_0$, $k = k_0$ (nominal) and $k = 10k_0$.

Fig. 10 Bode plots of $L(s)$ for Case.2 where $\tau = 0.1\tau_0$, $\tau = \tau_0$ (nominal) and $\tau = 10\tau_0$.

Fig. 11 Step response for Case.2 where $\tau = 0.1\tau_0$, $\tau = \tau_0$ (nominal) and $\tau = 10\tau_0$.

Fig. 12 Bode plots of $L(s)$ for Case.3 where $L = 0$ (nominal) and $L = 0.2$.

Fig. 13 Step response for Case.3 where $L = 0$ (nominal) and $L = 0.2$.

The system $L(s)$ are shown in Fig. 9. The system controlled by the FOC is robust to gain perturbations and the overshoot of the response is almost constant. For the Case.2, Fig. 10 shows the Bode plots and Fig. 11 shows the Nichols chart and step response. The results are almost the same as the Case.1 except for the phase variations in the low-frequency range. In these two cases, the better performances of the system with the FOC compared to the IOC are observed. However for the Case.3, the situations shown in Figs. 12 and 13 are not the same as the above two cases. The system with FOC is not so superior in the performance against the system with the IOC under the variation of time delay. It suggests that another adjustment of the FOC controller has to be individually carried out for this type of uncertainties. The improvement of the FOC for the case of time delays is one of the future works.

5. Conclusion

In this paper we presented a robust control design method of fractional order control (FOC) for linear time invariant systems. The procedure to achieve the FOC model can be easily applied to conventional robust control problems. The obtained FOC fulfilled five different requirements which include robustness for some kind of model uncertainties, rejection of noises and disturbances, and cancellation of steady-state error. The
parameters of the FOC were easily tuned to meet sufficient robustness against uncertainties for the type of gain variations. Our further research efforts include adapting the FOC to keep robustness against the type of time delays, exploring the performance of the FOC for non-minimum phase and open loop unstable systems.

References


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