Modeling and Control of 2-D Grasping of an Object with Arbitrary Shape under Rolling Contact

Suguru Arimoto*,** Morio Yoshida*, Masahiro Sekimoto**, and Kenji Tahara***

Abstract: Modeling, control, and stabilization of dynamics of two-dimensional object grasping by using a pair of multi-joint robot fingers are investigated under rolling contact constraints and an arbitrary geometry of the object and fingertips. First, a fundamental testbed problem of modeling and control of rolling motion between 2-D rigid bodies with an arbitrary shape is treated under the assumption that the two contour curves coincide at the contact point and share the same tangent. The rolling constraint induces the Euler equation of motion that is parameterized by a common arclength parameter and constrained onto the kernel space orthogonal to the equations that express evolutions of contact points in terms of the second fundamental form. It is shown that 2-D rolling constraints are integrable in the sense of Frobenius even if their Pfaffian forms are characterized by arclength parameters. A control signal called “blind grasping” is introduced and shown to be effective in stabilization of grasping without using the details of the object shape and parameters or external sensing. An extension of the Dirichlet-Lagrange stability theorem to a class of systems with DOF-redundancy under constraints is suggested by using a Morse-Bott-Lyapunov function.

Key Words: grasping, rolling contact, multi-fingered hand, robot finger, Euler-Lagrange equation.

1. Introduction

This paper aims at tackling a control problem for dextrous multi-fingered hands from computational perspectives, where a complete model of grasping must be developed even under the existence of rolling contacts and an arbitrary geometry of objects. So far the kinematics and geometry of a contact between rigid bodies have been solved by Montanna [1] and a set of velocity relations is given in detail [1]. However, there is a dearth of papers except the papers [2],[3] that attempt to model dynamics of physical interactions between the fingertips and an object under the existence of rollings. Even the exceptional papers [2],[3] do not yet gain physical insights into the constraint forces arising from rolling contacts and show any explicit forms of them in the object wrench space. In the series of papers [4],[5] by the first author of this paper, a set of Lagrange’s equations of motion of the overall fingers/object system under rolling constraints is derived under the assumption that rolling is interpreted as a constraint of the equal velocity of the contact point running on the fingertip sphere relative to running on the object surface [6]. However, all discussions have been restricted to the case of ball-plate rollings. Very recently, a mathematical model of two-dimensional grasping of a rigid object with an arbitrary shape is given as a set of Lagrange’s equations of motion of the overall fingers/object system together with a pair of first order differential equations that update arclength parameters [7]. It should be noted that modeling of the system assumes the full knowledge of the geometry of a given object but design of control signals does neither need the information of the object geometry nor use external sensing of contact points. This paper presents a complete model of grasping in the case that the geometry of fingertips is further arbitrary and discuss a control scheme called “blind control” that need to neither use the geometric information of the fingertips and object nor sense locations of the contact points and object mass center. In this paper, rolling contact is interpreted as the conditions:

1) Two contact points on the contour curves must coincide at a single common point without mutual penetration, and
2) the two contours must have the same tangent at the common contact point.

These two conditions as a whole are equivalent to the Nomizu relation [8] concerning tangent vectors at the contact point and normals to the common tangent. In the either definition of [6] or [8], the rolling constraint is expressed as a Pfaffian form and hence it is originally nonholonomic. Nevertheless, it is known that such a Pfaffian form as a rolling constraint is integrable in the case of ball-plate rollings [4]. More recently, it is shown [7] that the Pfaffian form becomes integrable even in the case that the geometry of an object is arbitrary while the fingertips are spherical. This paper shows that further in the case of arbitrary geometry of both the fingertips and object contours the rolling constraints become integrable in the sense of Frobenius. Hence, 2-D rollings are considered to be holonomic and therefore an explicit form of the Euler equation can be derived as a quotient dynamics on a constraint Riemannian submanifold, on

---

* RIKEN-TRI Collaboration Center, Nagoya, Aichi 463-0003, Japan
** Research Organization of Science and Engineering, Ritsumeikan University, Kusatsu, Shiga 525-8577, Japan
*** Organization for the Promotion of Advanced Research, Kyushu University, Nishi-ku, Fukuoka 819-0395, Japan
E-mail: arimoto@fc.ritsumei.ac.jp
(Received March 30, 2009)
(Revised June 11, 2009)
which the Riemannian metric can be induced in a natural way. Then, a control signal called “blind grasping” and composed of only finger-joint measurements can be regarded as the gradient of a Morse-Bott-Lyapunov function, which is defined on the base Riemannian manifold. An extension of the Dirichlet-Lagrange stability theorem [9] to a class of systems with DOF-redundancy under constraints is suggested by characterizing the Hessian of the Morse-Bott-Lyapunov function based on the Morse lemma [10].

Before discussing the problem of grasping by a pair of robot fingers, a testbed problem of control of a rotational object by a multi-joint finger is investigated in detail from a Riemannian-geometric viewpoint. The problem of modeling and control of two-dimensional grasping is tackled by directly extending the results obtained in the testbed problem.

2. Modeling and Control of Rolling Contact

Consider a modeling and control problem of rolling contact that arises between two contours of a rigid fingerend and a 2-D (2-dimensional) object pivoted at the fixed point $O_0$ in the 2-D Euclidean space $\mathbb{E}^2$ (see Fig. 1). Both the contact and rolling constraints are governed by equalities expressed in terms of arclength parameters. To show this, define the object contour $c_t(s) = (X(s), Y(s))$ by using the arclength parameter $s$ and the local coordinates expression of contact point $P_1$ by $(X(s), Y(s))$ based on the object coordinates $O_0$-XY as shown in Figs. 2 and 3. Similarly, denote the fingerend contour curve by $c_0(s) = (X_0(s), Y_0(s))$ based on local coordinates $O_0$-XY (see Figs. 2 and 3). Since it is assumed that there does not arise any slipping during rolling motion between these two contours, both arclength parameters $s_0$ and $s_1$ shown in Fig. 3 can be equalized and expressed by the single parameter $s$.

Suppose that at point $P_1$ the fingerend contacts with the ob-
ject. Then, the two contact points $P_{1A}$ and $P_{1B}$ shown in Fig. 3 must be coincident with $P_1$. Define

$$\begin{align*}
\theta_1(s) &= \arctan[(X'(s)/Y'(s))]
\psi_0(s) &= \arctan[X'_0(s)/Y'_0(s)]
\end{align*}$$

(1)

where $X'(s)$ denotes the derivative of $X(s)$ with respect to $s$ and $Y'(s), X'_0(s), Y'_0(s)$ denote similar meanings. Further, define unit normals and tangent vectors at $P_1$ as follows:

$$\begin{align*}
\hat{b}_1 &= \begin{pmatrix} \sin(\theta + \theta_1) \\ -\cos(\theta + \theta_1) \end{pmatrix}, \\
\hat{b}_0 &= \begin{pmatrix} -\cos(p_1 + \psi_0) \\ \sin(p_1 + \psi_0) \end{pmatrix}, \\
\hat{n}_1 &= \begin{pmatrix} \cos(\theta + \theta_1) \\ \sin(\theta + \theta_1) \end{pmatrix}, \\
\hat{n}_0 &= \begin{pmatrix} \sin(p_1 + \psi_0) \\ \cos(p_1 + \psi_0) \end{pmatrix}
\end{align*}$$

(2)

where $p_1 = q_1 + q_2$ and we omit writing symbol “v” in $\theta_1$ and $\psi_0$. According to the rolling contact condition 2) mentioned in Section 1, the two contour curves must share the tangent vector at $P_1$, that is, $\hat{b}_0 = \hat{b}_1$. This implies, in turn, $\hat{n}_0 = -\hat{n}_1$. All physical meanings of symbols $q_i$ ($i = 1, 2), \theta, \theta_1(s), \psi_0(s), X, Y, \text{ and } X_0, Y_0$ are defined in Figs. 1–3. Further, we define the following four quantities:

$$\begin{align*}
l_{00} &= X_0(s) \cos \psi_0 - Y_0(s) \sin \psi_0 \\
l_{01} &= -X(s) \cos \theta_1 + Y(s) \sin \theta_1 \\
l_{10} &= X(s) \sin \psi_0 + Y(s) \cos \psi_0 = b^T_0 \hat{O}_{01} \hat{P}_1 \\
l_{11} &= X(s) \sin \theta_1 + Y(s) \cos \theta_1 = b^T_1 \hat{O}_{01} \hat{P}_1
\end{align*}$$

(3)

Suppose that a posture of the system is denoted by the generalized position coordinates $z = (\theta, q_1, q_2)^T$ and all such vectors constitute the configuration space $R^3$. On the other hand, a posture $z$ can be regarded as a point on the 3-D torus $T^3 = S^1 \times S^1 \times S^1$ and the set of all such postures constitutes a Riemannian manifold on which a Riemannian metric can be introduced. Denote this Riemannian manifold by $[M, g_{ij}]$, where $g_{ij}$ denotes the Riemannian metric defined later. Then, contact between the fingertip and object contours induces geometric constraints. According to the rolling contact condition 1) mentioned in Section 1, the two contact locations $r_{1A}$ and $r_{1B}$ corresponding to points $P_{1A}$ and $P_{1B}$ respectively (see Fig. 3) should coincide, where

$$\begin{align*}
\begin{pmatrix} r_{1A} \\ r_{1B} \end{pmatrix} &= \begin{pmatrix} r_0 + l_{00} \hat{b}_0 - l_{10} \hat{n}_0 \\ r_0 + l_{11} \hat{b}_1 - l_{10} \hat{n}_1 \end{pmatrix}
\end{align*}$$

(5)
To differentiate both sides of (6) in \( t \), first note that
\[
\frac{\partial r_{1A}}{\partial s} \bigg|_s = l_{00} \frac{\partial b_0}{\partial s} - l_{00} \frac{\partial n_0}{\partial s} + \left( \frac{\partial l_{00}}{\partial s} \right) b_0 - \left( \frac{\partial l_{00}}{\partial s} \right) n_0
\]
\[= l_{00} \frac{\partial \theta_0}{\partial \theta} (-n_0) - l_{00} \frac{\partial \theta_0}{\partial \theta} \dot{b}_0
\]
\[+ \left( 1 + l_{00} \frac{\partial \theta_0}{\partial \theta} \right) \dot{b}_0 + l_{00} \left( \frac{\partial \theta_0}{\partial \theta} \right) n_0
\]
\[= b_0
\]
(7)
The details of partial derivatives of \( l_{00}, l_{00}, b_0, \) and \( n_0 \) in \( s \) are given in Appendix A. Similarly to (7), it follows that
\[
\frac{\partial r_{1B}}{\partial s} = b_1
\]
(8)
Hence, differentiation of (6) in \( t \) gives rise to
\[
\dot{r}_{1A} = \sum_{i=1,2} \left( \frac{\partial r_{1A}}{\partial q_i} \right) \dot{q}_i + \frac{ds}{dt} b_0
\]
\[= \dot{r}_{1B} = \left( \frac{\partial r_{1B}}{\partial \theta} \right) \dot{\theta} + \left( \frac{ds}{dt} \right) b_1
\]
(9)
Since \( b_0 = b_1, (9) \) yields
\[
\left( \frac{\partial r_{1A}}{\partial q_1} + \frac{\partial r_{1A}}{\partial q_2} - \frac{\partial r_{1B}}{\partial \theta} \right)^T b_1 = 0
\]
(10)
Since \( \frac{\partial b_0}{\partial q_1} = -n_0 \) \((i = 1, 2, 3, \partial b_1/\partial \theta = n_1, \) and \( n_1^T b_1 = 0 \) \((i = 0, 1, 0) \), (10) is reduced to
\[
\dot{r}_{1A}^T b_1 - l_{00}(\dot{q}_1 + \dot{q}_2 - \dot{b}_0) = 0
\]
(11)
This can be written in detail in the following:
\[
\dot{x}_0 \sin(\theta + \theta_1) + \dot{y}_0 \cos(\theta + \theta_1)
\]
\[+ l_{00} \dot{\theta} - \dot{q}_1 - \dot{q}_2 - (l_{01} + l_{10}) \dot{\theta} = 0
\]
(12)
This equality stands for the rolling constraint. For later convenience, define
\[
\begin{align*}
Q_1(z, s) &= (x_0 - x_m) \cos(\theta + \theta_1)
- (y_0 - y_m) \sin(\theta + \theta_1) \\
R_1(z, s) &= (x_0 - x_m) \sin(\theta + \theta_1)
+ (y_0 - y_m) \cos(\theta + \theta_1)
\end{align*}
\]
and note that from geometrical meanings of \( Q_1, R_1, l_{00}, l_{01}, l_{10}, \) and \( l_{00} \) it follows that
\[
\begin{align*}
Q_1(z, s) &= -(l_{00}(s) + l_{01}(s)) \\
R_1(z, s) &= -(l_{00}(s) - l_{01}(s))
\end{align*}
\]
(14)
Further, define
\[
\begin{align*}
Q &= -l_{00}(\theta + \theta_1(s) - q_1 - q_2) + Q_1 \\
R &= l_{00}(\theta + \theta_1(s) - q_1 - q_2) + R_1
\end{align*}
\]
(15)
Then, (12) means
\[
\left( \frac{\partial R}{\partial \theta}, \frac{\partial R}{\partial q_1}, \frac{\partial R}{\partial q_2} \right) \dot{\left( \dot{q}_1, \dot{q}_2, \dot{\theta} \right)}^T = 0
\]
(16)
that is reduced to the constraint form in terms of finitessimally small variation:
\[
\left(-l_{01}, b_1^T J_{01}(q) - l_{00} e^T \right)(d\theta, dq_1, dq_2)^T = 0
\]
(17)
Similarly, from the relation
\[
\left( \frac{\partial Q}{\partial \theta}, \frac{\partial Q}{\partial q_1}, \frac{\partial Q}{\partial q_2} \right) \dot{\left( \dot{q}_1, \dot{q}_2 \right)}^T = 0
\]
(18)
it follows that
\[
\left(-l_{01}, n_1^T J_{01}(q) + l_{00} e^T \right)(d\theta, dq_1, dq_2)^T = 0
\]
(19)
where \( J_{01}(q) \) denotes the Jacobian matrix of \((x_0, y_0)^T\) with respect to \( q_1 \) and \( q_2 \) and \( e = (1, 1)^T \).

Finally, the kinetic energy of the finger and object is given by
\[
K = \frac{1}{2} \dot{q}^T G(q) \dot{q} + \frac{1}{2} \dot{\theta}^2
\]
(20)
where \( G(q) \) and \( I \) stand for the inertia matrix of the finger and the inertia moment of the object around \( O_m \). Thus, by defining the Lagrangian \( L = K \) and assigning tangent vectors \( (\partial b_0, \partial b_1, \partial b_2) \in T_Q M \) at the \( z \in M \) to satisfy (17) and (18) through associating with Lagrange’s multipliers \( \lambda \) and \( \lambda \), we obtain the Euler-Lagrange equation of motion of the system:
\[
\left\{ \begin{array}{l}
\dot{\theta} + f + \lambda \frac{\partial R}{\partial \theta} = 0 \\
G(q) \dot{q} + \left( \frac{1}{2} G(q) + S(q, q) \right) \dot{q} + f \frac{\partial Q}{\partial \theta} + \lambda \frac{\partial R}{\partial q_1} = u
\end{array} \right.
\]
(21)
where \( u \) signifies an external torque for control design. In detail, (21) can be written into
\[
\left\{ \begin{array}{l}
\dot{\theta} - f l_{01} - A l_{10} = 0 \\
G(q) \dot{q} + \left( \frac{1}{2} G(q) + S(q, q) \right) \dot{q} + f \left( J_{01}(q) n_1 + l_{00} e \right) + \lambda l_{01}(q) b_1 = l_{00} e
\end{array} \right.
\]
(22)
This Euler-Lagrange equation is parameterized by the arclength parameter \( s \) that must vary in progress of time \( t \). To derive the formula that governs the evolution of \( s \) in \( t \), note that from the equality \( b_0 = b_1 \) and its derivative in \( t \) it follows that
\[
- \left( \dot{q}_1 + \dot{q}_2 - \kappa_0(s) \frac{ds}{dt} \right) n_0 = \left( \dot{\theta} + \kappa_1(s) \frac{ds}{dt} \right) n_1
\]
(23)
this is reduced to the first order differential equation
\[
\kappa_0(s) + \kappa_1(s) \frac{ds}{dt} = \dot{q}_1 + \dot{q}_2 - \dot{\theta}(t)
\]
(24)
where \( \kappa_0 \) denotes the curvature of the fingertip and \( \kappa_1 \) that of the object (see Appendix A).

3. Stabilization of Rotational Motion Based upon a Morse-Lyapunov Function

Consider now a control signal
\[
u = -c q - \left( \frac{f}{r} \right) J_{01}^T(q) \left( \frac{x_0 - x_m}{y_0 - y_m} \right)
\]
(25)
where \( c, f, \) and \( r \) are appropriate constants having physical units [Nms], [kg], and [m], respectively. The control signal can be constructed from the kinematics of the finger and the fixed location of \( O_m \), without knowing the geometric data on shapes of the fingerend and object contours. First, note that from (2) and (13) it follows that
Lagrangian $L$ can be introduced and the closed-loop system (28) can be rewritten as follows:

$$\dot{\beta} - l_{b1} \Delta f = l_{b1} \Delta \lambda - \frac{f_d}{r} N_1 = 0$$

$$G(q)\ddot{q} + \left(\frac{1}{2} \dot{G} + S\right) q + c q + \Delta f \left[\lambda_0 \mathbf{n}_1 + l_{io} e\right]$$

$$+ \Delta \lambda \left[\lambda_0 \mathbf{n}_1 - l_{io} e\right] + \frac{f_d}{r} N_1 e = 0$$

where

$$N_1 = -l_{io} Q_1 + l_{io} R_1 = l_{io} \mathbf{n}_1 + l_{io} \mathbf{l}_{b1}$$

Now, remind the definitions of $R$ and $Q$ in (15) together with (16) and (18) and take the inner product of $\dot{q}$ and the second equation of (28), and multiply the first one of (28) by $\dot{\beta}$. The sum of these two products leads to the energy relation

$$\frac{d}{dt} E(z, z) = -c||q||^2$$

where $z = (\theta, q_1, q_2)^T$, $\dot{q} = (q_1, q_2)^T$, and

$$E(z, z) = \frac{1}{2} q^T G(q)\dot{q} + \frac{1}{2} \dot{q}^T \dot{q} + \frac{f_d}{2r} ||r_{01} - r_m||^2$$

$$= K + U$$

$$U = \frac{f_d}{2r} ||r_{01} - r_m||^2$$

$$= \frac{f_d}{2r} \left[ (x_01 - x_m)^2 + (y_01 - y_m)^2 \right]$$

The energy can be called the total energy and $U$ the artificial potential energy.

Next, it should be remarked that by defining

$$\tilde{Q} = Q + l_{io} \left( \psi_0 - \frac{\pi}{2} \right) + l_{io} \mathbf{l}_{b1}$$

$$\tilde{R} = R + \left[ l_{io} - l_{b1} \right] - l_{io} \left( \psi_0 - \frac{\pi}{2} \right)$$

it follows (the details are given in Appendix B) that

$$\tilde{Q} = 0, \quad \tilde{R} = 0$$

$$\frac{d}{dt} \tilde{Q} = 0, \quad \frac{d}{dt} \tilde{R} = 0$$

since

$$\begin{align*}
\varphi_1 &= \pi - (q_1 + q_2) + (\theta + \theta_1) \\
\varphi_1 - \psi_0 &= \pi/2
\end{align*}$$

That is, $\tilde{Q} = 0$ means a holonomic constraint of contact and $\tilde{R} = 0$ does a holonomic constraint of rolling. Therefore, the Lagrangian

$$L(z, \dot{z}, s) = K - U - \Delta f \tilde{Q} - \Delta \lambda \tilde{R}$$

can be introduced and the closed-loop system (28) can be regarded as the Euler-Lagrange equation of motion for the Lagrangian $L$ defined in (36). In fact, since $||r_{01} - r_m||^2 = Q_1^2 + R_1^2$, it is possible to see that

$$\begin{align*}
(x_{01} - x_m) = r_{01} - r_m = Q_1 n_1 + R_1 b_1
\end{align*}$$

Hence, by defining

$$\begin{align*}
\Delta f &= f + \frac{f_d}{r} Q_1 = f - \frac{f_d}{r} (l_{io} + l_{b1}) \\
\Delta \lambda &= \lambda + \frac{f_d}{r} R_1 = \lambda - \frac{f_d}{r} (l_{io} - l_{b1})
\end{align*}$$

the Euler-Lagrange equation of (22) can be rewritten into

$$\dot{\beta} - l_{b1} \Delta f - l_{b1} \Delta \lambda - \frac{f_d}{r} N_1 = 0$$

$$G(q)\ddot{q} + \left(\frac{1}{2} \dot{G} + S\right) q + c q + \Delta f \left[\lambda_0 \mathbf{n}_1 + l_{io} e\right]$$

$$+ \Delta \lambda \left[\lambda_0 \mathbf{n}_1 - l_{io} e\right] + \frac{f_d}{r} N_1 e = 0$$

Finally, it is interesting to know that local minimization of the artificial potential $U(z, s)$ under rolling contact constraints $R = 0$ and $\bar{Q} = 0$ must arise at the point $(z, s)$ if and only if $(z, s)$ and $(\Delta f, \Delta \lambda)$ satisfy

$$N_1 = 0, \quad \Delta f = 0, \quad \Delta \lambda = 0$$

Therefore, minimization is attained when $N_1 = 0$, which corresponds to minimization of the length between the two centers $O_{01}$ and $O_m$ under rolling motion (see Fig. 4).

Finally, it is interesting to know that local minimization of the artificial potential $U(z, s)$ under rolling contact constraints $R = 0$ and $\bar{Q} = 0$ must arise at the point $(z, s)$ if and only if $(z, s)$ and $(\Delta f, \Delta \lambda)$ satisfy

$$N_1 = 0, \quad \Delta f = 0, \quad \Delta \lambda = 0$$

To show this, note that $U(z, s)$ can be expressed in terms of a function of $s$ as follows:

$$U(z, s) = \frac{f_d}{2r} (Q_1^2 + R_1^2)$$

$$= \frac{f_d}{2r} \left[ (l_{io} + l_{b1})^2 + (l_{io} - l_{b1})^2 \right] = U(s)$$

Then, it is easy to check that

$$\frac{dU}{ds} = \frac{f_d}{r} (\kappa_0(s) + \kappa(s)) N_1(s)$$

where $\kappa_0$ and $\kappa_1$ denote the curvatures of the fingertip and object contour curved respectively. This shows that $N_1 = 0$ means $U'(s) = dU(s)/ds = 0$ provided that $\kappa_0(s) > -\kappa_1(s)$ for all $s$. The Hessian of $U$ can be calculated as follows

$$\frac{d^2U(s)}{ds^2} \bigg|_{N_1=0} = \frac{f_d(\kappa_0 + \kappa_1)}{r} \left( l_{io} + l_{b1} \right)$$

$$+ \left( \kappa_0 + \kappa_1 \right) \left( l_{io} - l_{b1} \right)$$

In the case that the fingertip is spheric with radius $r$, then

$$\frac{dU(s)}{ds} = \frac{f_d}{r} \left( \kappa_1(s) + \frac{1}{r} \right) l_{b1}$$

$$\frac{d^2U(s)}{ds^2} \bigg|_{N_1=0} = \frac{f_d}{r} \left( \kappa_1(s) + \frac{1}{r} \right) \left( 1 - \kappa_1(s) l_{b1} \right)$$

Finally, we note that a set of all possible postures under rolling movements can be regarded as a single dimensional manifold (that is, a smooth curve) that is homeomorphic to an interval $(a, b)$ of arclength parameter $s$ (see Fig. 5).
4. Pinching of a 2-D Object with Arbitrary Shape by a Pair of Arbitrary Fingertips

Extending the argument previously given for modeling and control of immobilizing a rotational 2-D object by a multi- joint finger to the case of pinching a 2-D object with arbitrary shape by a pair of 2-D robot fingers with arbitrary shape, we derive Euler-Lagrange’s equation of motion of the overall fingers/object system as depicted in Fig. 6. We define physical and geometric variables of the fingers and object in detail as in Fig. 7, similarly to Figs. 2 and 3. In addition, we introduce arclength parameters $s_1$ and $s_2$ along the fingertip contour of the left finger and that of the right finger respectively. The sign of $s_1$ is taken to be positive if it increases in the direction of the tangent vector $b_1$ and the sign of $s_2$ is positive in the direction of $b_2$. According to the definitions of $\theta, \theta_i(s_i)$ ($i = 1, 2$), $n_i$ and $\dot{b}_i$ given in Fig. 7, we have

$$n_i(\theta) = -(-1)^i \begin{pmatrix} \cos(\theta + \theta_i(s_i)) \\ -\sin(\theta + \theta_i(s_i)) \end{pmatrix}, \quad i = 1, 2 \quad (44)$$

and similarly

$$\dot{b}_i = \dot{b}_i(p_i) = (-1)^i \begin{pmatrix} \cos(p_i + \psi_i(s_i)) \\ -\sin(p_i + \psi_i(s_i)) \end{pmatrix} \quad (45)$$

On the other hand, both the tangent vectors $b_1$ and $b_2$ are described in terms of fingertips angles $p_i$ and $\psi_i$ as follows:

$$b_i = b_i(p_i) = (-1)^i \begin{pmatrix} \cos(p_i + \psi_i(s_i)) \\ -\sin(p_i + \psi_i(s_i)) \end{pmatrix} \quad (46)$$

where $p_1 = q_{11} + q_{12} + q_{13}$ and $p_2 = q_{21} + q_{22}$. From differentiation of $b_i(p_i) = b_i(\theta_i)$ in $t$ for $i = 1, 2$ at both the contact points, it follows that

$$(-1)^i \left( \dot{p}_i + (-1)^i \kappa_0(s_i) \frac{ds_i}{dt} \right) n_i$$

$$= -(-1)^i \left( \dot{\theta} - (-1)^i \kappa_i(s_i) \frac{ds_i}{dt} \right) n_i, \quad i = 1, 2 \quad (48)$$

where $\kappa_i(s_i)$ for $i = 1, 2$ denote curvatures of the object contours at contact points $P_1$ and $P_2$ corresponding to arclength parameters $s_1$ and $s_2$ and $\kappa_0(s_i)$ for $i = 1, 2$ denote curvatures of the left fingertip contour at $P_1$ and the right fingertip contour at $P_2$. Equation (48) is reduced to

$$\kappa_0(s_i) + \kappa_i(s_i) \frac{ds_i}{dt} = (-1)^i (\dot{\theta}(t) - \dot{p}_i(t)), \quad i = 1, 2 \quad (49)$$

which constitute the update laws of arclength parameters $s_1$ and $s_2$ with respect to time $t$.

Similarly to the argument developed in Section 2, we define the position constraints at the contact points $P_1$ and $P_2$ as follows:

$$\begin{align*}
\mathbf{r}_A(q_i, s_i) &= \mathbf{r}_{0i} + b_{0i} n_{0i} - n_{0i} n_{hi}, \quad i = 1, 2 \\
\mathbf{r}_{1B}(\theta, s_i) &= \mathbf{r}_{mi} + b_j n_{hi}, \quad i = 1, 2
\end{align*} \quad (50)$$

where $r_{0i}$ ($i = 1, 2$) denote the position vectors of $\overrightarrow{O_0O_i}$ respectively and $r_{mi}$ denotes that of the object mass center $O_m$, all of which are expressed in terms of the frame coordinates $O-xy$ (see Figs. 6 and 7). In (50), $s_j$ denotes the common arclength parameter of the left-hand fingertip contour and the object contour curve at the left hand side, and $s_3$ the common arclength parameter of the two contour curves at the right hand side. The direction of increase of $s_3$ is taken to be coincident with the direction of the tangent $\dot{\mathbf{b}}_2$ and that of $s_1$ is coincident with that of $\mathbf{b}_1$. Then, scalar values $b_{0i}$ and $n_{0i}$ are defined as for $i = 1, 2$

$$\begin{align*}
\mathbf{b}_{0i} &= (\overrightarrow{O_0P_i})^T \mathbf{n}_{0i} \\
n_{0i} &= -(\overrightarrow{O_0P_i})^T \mathbf{n}_{0i}
\end{align*} \quad (51)$$

Similarly to the derivation of (7), it is possible to see that

$$\frac{\partial}{\partial s_i} \mathbf{r}_A = \frac{\partial}{\partial s_i} \mathbf{r}_B = \mathbf{b}_i, \quad i = 1, 2 \quad (52)$$

Hence, the constraint equalities $\dot{r}_A - \dot{r}_B = 0$ ($i = 1, 2$) imply that (in what follows, denote the position of $O_m$ by $x = (x, y, z)^T$...
Since from (44) to (47) it follows that for $i = 1, 2$,

\[
\frac{\partial}{\partial q_i} b_0 = (-1)^{i} n_0 e_1^T \quad \frac{\partial}{\partial \theta} b_i = (-1)^{i} n_i, \quad \frac{\partial}{\partial q_i} n_i = (-1)^{i} b_i \quad (54)
\]

where $e_1 = (1, 1, 1)^T$ and $e_2 = (1, 1, 1)^T$, (53) is rewritten into the form

\[
J_i(q_i) \bar{q}_i - (-1)^i n_0 \bar{q}_i \bar{p}_i + (-1)^i n_0 b_0 \bar{p}_i
\]

\[
= \dot{x} - (-1)^i b_i n_i \dot{\theta} - (-1)^i n_i b_i \dot{\theta} \quad i = 1, 2 \quad (55)
\]

and that of $n_i$ and (55) yields for $i = 1, 2$

\[
-n_1^T \dot{x} + (-1)^i b_i \dot{\theta} + n_1^T J_i(q_i) \bar{q}_i - (-1)^i n_0 \bar{p}_i = 0 \quad (56)
\]

Equation (56) stands for rolling constraints of the Pfaffian form and equation (57) does for contact constraints. Both the constraints can be expressed as for $i = 1, 2$

\[
\begin{pmatrix} -b_i^T, (-1)^i n_i, b_i^T J_i(q_i), (-1)^i n_0 e_i^T \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{\theta} \\ \dot{q}_i \end{pmatrix} = 0 \quad (58)
\]

\[
\begin{pmatrix} -n_i^T, (-1)^i b_i, n_i^T J_i(q_i), (-1)^i b_0 e_i^T \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{\theta} \\ \dot{q}_i \end{pmatrix} = 0 \quad (59)
\]

In light of the argument given in Section 3 and Appendix B for showing integrability of the rolling constraint, we now define the scalar functions:

\[
\bar{R}_i(x, \theta, q_i, s_i) = \begin{pmatrix} (r_{0i} - x)^T b_i (\theta + \theta_i) + (b_0 - b_i) \\ (-1)^i n_0 \theta_i - p_i - \psi_i (s_i) - (-1)^i \frac{\pi}{2} \end{pmatrix}, \quad i = 1, 2 \quad (60)
\]

\[
\bar{Q}_i(x, \theta, q_i, s_i) = \begin{pmatrix} (r_{0i} - x)^T n_i (\theta + \theta_i) + (n_0 - n_i) s_i \\ + (-1)^i b_0 \theta_i - p_i - \psi_i (s_i) - (-1)^i \frac{\pi}{2} \end{pmatrix}, \quad i = 1, 2 \quad (61)
\]

In view of the definitions of angles $\theta, p_i, \theta_i, \psi_i$ for $i = 1, 2$ as shown in Fig. 7, obviously we see that

\[
\theta + \theta_i (s_i) - p_i - \psi_i (s_i) - (-1)^i \frac{\pi}{2} = 0, \quad i = 1, 2 \quad (62)
\]

It should be noted that (62) shows the integrability of (49), that is, the derivative of (62) in time $t$ is reduced to (49). It should also be noted

\[
\begin{pmatrix} (r_{0i} - x)^T b_i = -(b_0 - b_i) \\ (r_{0i} - x)^T n_i = -(n_0 + n_i) \end{pmatrix} \quad i = 1, 2 \quad (63)
\]

\[
\text{Thus, it is possible to show that}
\]

\[
\bar{R}_i = 0, \quad \bar{Q}_i = 0 \quad \text{for} \quad i = 1, 2 \quad (64)
\]

\[
\frac{d}{dt} \bar{R}_i = 0, \quad \frac{d}{dt} \bar{Q}_i = 0 \quad \text{for} \quad i = 1, 2 \quad (65)
\]

In fact, (64) follows from (62) and (63). To verify (65), we first confirm the following formulae:

\[
\frac{\partial}{\partial s_i} \bar{R}_i = 0, \quad \frac{\partial}{\partial s_i} \bar{Q}_i = 0, \quad i = 1, 2 \quad (66)
\]

The details of this verification are similar to (B.4) and (B.5) and hence omitted. Finally, we have

\[
\frac{d}{dt} \bar{Q}_i = \left\{ \frac{\partial \bar{Q}_i}{\partial x^T}, \frac{\partial \bar{Q}_i}{\partial \theta}, \frac{\partial \bar{Q}_i}{\partial q_i} \right\} \begin{pmatrix} \dot{x} \\ \dot{\theta} \\ \dot{q}_i \end{pmatrix} + \frac{\partial \bar{Q}_i}{\partial s_i} \frac{ds_i}{dt} = 0 \quad (67)
\]

in which the last equality follows from (58). Similarly, we have correspondingly to (57) for $i = 1, 2$

\[
\frac{d}{dt} \bar{R}_i = \left\{ \frac{\partial \bar{R}_i}{\partial x^T}, \frac{\partial \bar{R}_i}{\partial \theta}, \frac{\partial \bar{R}_i}{\partial q_i} \right\} \begin{pmatrix} \dot{x} \\ \dot{\theta} \\ \dot{q}_i \end{pmatrix} + \frac{\partial \bar{R}_i}{\partial s_i} \frac{ds_i}{dt} = 0 \quad (68)
\]

In conclusion, both the rolling contact constraints expressed by (56) and (57) are integrable in the sense of Frobenius [9],[10] and can be regarded as holonomic constraints.

5. Equation of Motion of 2-D Pinching under Rolling Constraints with Arbitrary Geometry

On the basis of the arguments concerning the rolling contact constraints given in Section 4, the Lagrangian of the overall fingers-object system depicted in Fig. 6 is defined as

\[
L(X; s_1, s_2) = \sum_{i=1}^{2} \left( \lambda_i \bar{R}_i + f_i \bar{Q}_i \right) \quad (69)
\]

where we denote $X = (x^T, \theta, q_1^T, q_2^T)^T$ and define

\[
K(X) = \sum_{i=1}^{2} \frac{1}{2} \bar{G}_i(q_i) \dot{q}_i + \frac{M}{2} \left( \dot{z}^2 + \dot{y}^2 \right) + \frac{1}{2} \dot{\theta}^2 \quad (70)
\]

and $\lambda_i$ and $f_i$ (for $i = 1, 2$) are Lagrange’s multipliers associated with the holonomic constraints of (64). By regarding the control input torques $u_i$ and $u_2$ exerted from the finger joint actuators as the external torques, Euler-Lagrange’s equation of motion of the system can be derived as follows:

\[
M \ddot{x} - f_1 n_1 - f_2 n_2 - \lambda_1 b_1 - \lambda_2 b_2 = 0 \quad (71)
\]

\[
I \ddot{\theta} - f_1 b_1 + f_2 b_2 - \lambda_1 n_1 + \lambda_2 n_2 = 0 \quad (72)
\]

\[
G_i(q_i) \dot{q}_i + \left( \frac{1}{2} \dot{G}_i + S_i \right) \dot{q}_i + f_i \left\{ J_i(q_i) n_i - (-1)^i b_0 e_i \right\}
\]

\[
+ \lambda_i \left\{ J_i(q_i) b_i + (-1)^i b_0 e_i \right\} = u_i, \quad i = 1, 2 \quad (73)
\]

When both the fingertips are spherical with radius $r_i$ ($i = 1, 2$), $b_0$ for $i = 1, 2$ vanish and $n_{0i} = r_i$ for $i = 1, 2$. Therefore
the dynamics of (73) is reduced to that of such a special case previously treated in [7]. It should be noted that all equations of (71) to (73) are characterized by the two arclength parameters $s_1$ and $s_2$. Therefore, integration of (71) to (73) in time $t$ should be taken simultaneously together with the first-order differential equation of (49) under the constraints (65).

Finally, we show an explicit form of the quotient dynamics developed in the kernel space orthogonally complimentary to the image spanned from the constraint gradients. Let us denote $\Phi = (f_1, f_2, \lambda_1, \lambda_2)^T$. Hence, Euler-Lagrange’s equation of motion of the overall system (equations (71) to (73)) is described by the form

$$\frac{d}{dt} \left( \frac{\partial K}{\partial \dot{q}_i} \right) - \frac{\partial K}{\partial q_i} + \frac{\partial \hat{P}}{\partial \dot{q}_i} \lambda = u$$

(76)

### 6. Quotient Dynamics of the Closed-Loop and Morse’s Function

In order to design a control signal for stable grasping from the practical standpoint, a family of control signals has been introduced, based upon the fingers-thumb opposability that is one of functional characteristics of human hands as discussed in [7]. The family is described by the form

$$u_i = -c_i \dot{q}_i + (1/\beta) J_i^T(q_i) (\rho_0 - \rho_2) - \gamma_i \dot{N}_i, \quad i = 1, 2 \quad (77)$$

where $\beta$ stands for a position feedback gain common for $i = 1, 2$ with physical unit [N/m],

$$\dot{N}_i = e_i^T(q_i(t) - q_i(0)) = p_i(t) - p_i(0), \quad i = 1, 2 \quad (78)$$

and $\gamma_i$ stands for a constant gain with physical unit [Nm] for $i = 1, 2$. Since the sum of inner products of $\dot{q}_i$ and (77) for $i = 1, 2$ is reduced to

$$\sum_{i=1,2} \dot{q}_i^T u_i = -\frac{d}{dt} \left( \sum_{i=1,2} \frac{\gamma_i \dot{N}_i + \beta \parallel \rho_0 - \rho_2 \parallel^2}{2} \right) + \sum_{i=1,2} c_i \parallel \dot{q}_i \parallel$$

(79)

the closed-loop dynamics of (76) in which the control signals of (77) are substituted is described as follows:

$$\hat{G}(X) \ddot{X} + \left( \frac{1}{2} \hat{S} + \hat{S}^T \right) \ddot{X} + C \dot{X} + \frac{\partial \hat{P}}{\partial \dot{q}} \lambda + \frac{\partial P(X)}{\partial X} \lambda = 0 \quad (80)$$

where $S(X, \dot{X})$ denotes the skew-symmetric matrix defined as $S = \text{diag}(0, 0, 0, S_2, S_2)$. $C = \text{diag}(0, 0, 0, c_1 I_2, c_2 I_2)$, and $P(X)$ is the artificial potential inside the bracket $\{ \}$ of (79). Let us define

$$\Psi(X) = \frac{\partial \hat{P}}{\partial \dot{q}} \frac{\partial \hat{P}}{\partial \dot{q}^T} \lambda^{-1/2}, \quad W = (U(X), \Psi(X)) \quad (81)$$

so that the $n \times (n - 4)$ matrix $U$ is orthogonally complemented to the $n \times 4$ matrix $\Psi(X)$ ($n$ denotes the dimension of $X$) and therefore $W(X)$ becomes an orthogonal matrix. Consider the coordinates transformation

$$\dot{X} = W \left( \frac{\dot{q}}{\Phi} \right) \text{ or } \dot{W}^T X = \left( \frac{\dot{q}}{\Phi} \right) \quad (82)$$

Then, as discussed in the previous paper [7], the implicit form of Euler-Lagrange’s equation (80) can be transformed to

$$\hat{G} \dot{q} + \left( \frac{1}{2} \hat{S} + \hat{S}^T \right) \dot{q} + U^T \frac{\partial P(X)}{\partial X} + U^T C \dot{U} = 0 \quad (83)$$

in which Lagrange’s multipliers are deleted, $\hat{G} = U^T GU$, and $\hat{S} = U^T SU - (1/2)(U^T GU - U^T \hat{G} U)$. Then, in a similar way to the stability proof of positioning for a redundant system as shown in [11], we suggest a possible way to show that a solution to (83) tends to an equilibrium submanifold of $(n-4)$-dimension that is constrained by (64) and attains the minimum of the potential $P(X)$. Note that $P(X)$ can be regarded as a Morse-Bott function not only on the Riemannian manifold $[M, \bar{g}]$ but also on the Riemannian submanifold constrained by (64), because the inner product of (83) and $\dot{q}$ yields

$$\frac{d}{dt} \left( K + P \right) = -\dot{q}^T U^T C \dot{U} \dot{q} \quad (84)$$

where $P$ can be regarded as the same potential function (the content of $\{ \}$ of (79)). This argument suggests an extension of the Dirichlet-Lagrange stability theorem to a system redundant in its degrees-of-freedom [5,7], provided $U^T C U$ is fully dissipated with respect to $\dot{q}$ [10].

### 7. Conclusions

Modeling and control dynamics of 2-D object grasping with an arbitrary geometry of the fingertips and object were discussed under the circumstances of rolling contact constraints. It was shown that 2-D rolling constraints expressed by Pfaffian forms are integrable in the sense of Frobenius, from which quotient dynamics of the Euler-Lagrange equation of motion can be derived. Applicability of the Dirichlet-Lagrange stability theorem to a redundant system under constraints was suggested by introducing a Morse-Bott function. Partial results via computer simulation will be reported in [11].

### References


[9] F. Bullo and A.D. Lewis: Geometric Control of Mechanical...
Appendix A

According to Frenet-Serret’s formula (for example, see [12]), we have
\[
\frac{\partial}{\partial s}(b_i, n_i) = (0, -\kappa_i(s), 0), \quad i = 0, 1 \tag{A.1}
\]
where \(\kappa_0(s)\) denotes the curvature of the fingereid contour curve at contact point \(P_1\) and \(\kappa_1(s)\) that of the object contour at \(P_1\). These curvatures are also defined as
\[
\kappa_0(s) = -\frac{d\phi_0(s)}{ds}, \quad \kappa_1(s) = \frac{d\phi_1(s)}{ds} \tag{A.2}
\]
Derivatives of \(l_{b0}\) and \(l_{b1}\) in \(s\) are obtained as follows:
\[
\frac{dl_{b0}}{ds} = X_0'(s) \sin \psi_0 - Y_0'(s) \cos \psi_0 \\
\quad + [X_0(s) \cos \psi_0 - Y_0(s) \sin \psi_0] \frac{d\phi_0}{ds} \\
= 1 - \kappa_0(s)l_{b0} \tag{A.3}
\]
\[
\frac{dl_{b1}}{ds} = X_0'(s) \cos \psi_0 + Y_0'(s) \sin \psi_0 \\
\quad - [X_0(s) \sin \psi_0 + Y_0(s) \cos \psi_0] \frac{d\phi_0}{ds} \\
= \kappa_0(s)l_{b1} \tag{A.4}
\]
In a similar way to these derivations, we have
\[
\frac{dl_{b1}}{ds} = 1 - \kappa_1(s)l_{b1}, \quad \frac{dl_{b2}}{ds} = \kappa_1(s)l_{b2} \tag{A.5}
\]
Appendix B

From (15) and (33) it follows that
\[
\dot{Q} = -l_{b0}(\theta + \theta_1 - q_1 - q_2) + l_{b0}\left(\psi_0 - \frac{\pi}{2}\right) + (l_{b0} + l_{b1}) \\
= (Q_1 + l_{b0} + l_{b1}) - l_{b0}(\theta - q_1 - q_2 - \psi_0 + \frac{\pi}{2}) \\
= 0 \tag{B.1}
\]
where the last equality follows from (14) and (35) and, similarly,
\[
\dot{R} = l_{b0}(\theta + \theta_1 - q_1 - q_2) + R_1 - l_{b0}\left(\psi_0 - \frac{\pi}{2}\right) + (l_{b0} - l_{b1}) \\
= 0 \tag{B.2}
\]
Since all \(l_{b0}, \psi_0, l_{b0},\) and \(l_{b1}\) are of a shape function depending only on \(s\),
\[
\frac{d\dot{Q}}{d\theta} = \left(\frac{\partial \dot{Q}}{\partial \theta}\right) \frac{ds}{d\theta} + \left(\frac{\partial \dot{Q}}{\partial \theta_1}\right) \frac{ds}{d\theta_1} + \left(\frac{\partial \dot{Q}}{\partial \theta_2}\right) \frac{ds}{d\theta_2} + \left(\frac{\partial \dot{Q}}{\partial \theta_3}\right) \frac{ds}{d\theta_3} \\
= \frac{d\dot{Q}}{ds} \tag{B.3}
\]
where the last equality follows from (18). Similarly, from (16) we have \(d\dot{R}/ds = (d\dot{R}/ds)/ds/d\theta\). To show \(d\dot{Q}/ds = 0\), we differentiate \(\dot{Q}\) and \(\dot{R}\) in the following way: