On Almost Orbital Equivalence of Nonlinear Systems

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Abstract: This paper is concerned with equivalence of nonlinear systems from a viewpoint of geometric congruence of system orbits in the state space. The notion of orbital equivalence was originally exploited by Sampei and Furuta in the context of the time-scale transformation approach. They gave a fundamental characterization of orbital equivalence in the form of similarity conditions in the system vector-fields; however, it was not satisfactory in the sense that there remains a significant gap between the necessary condition and the sufficient one, mainly due to its treatment of unavoidable singularity of the time-scale functions. In this paper, we intend to cast a new light on this problem with the aid of a viewpoint of "almost" that neglects the inconsistency on measure zero subsets. By introducing the notions of almost orbital equivalence and almost similarity of the vector-fields, we provide a sole necessary and sufficient conditions connecting them. Furthermore, we discuss the combination of the orbital equivalence analysis and a stability issue based on a recent development of the density function approach for stability analysis. The discussion is also illustrated by numerical examples.

Key Words: nonlinear system, time-scale transformation, orbital equivalence, density function.

1. Introduction

This paper is concerned with equivalence of nonlinear systems from a geometric viewpoint of coincidence of the system orbits as curves in the state space. Two different systems with different time responses may have the same state flow. Once we consider identifying such a system with the others, it leads us to introduce a new relaxed notion of the system equivalence. Sampei and Furuta [1]–[4] studied this idea along the context of time-scale transformation problems. The time-scale transformation was proposed as a tool for system analysis and design by scaling the real time variable \( t \) with a positive scalar function \( \rho(x) \) \( (0 < \rho(x) < \infty) \) to obtain a virtual time scale \( \tau \) satisfying \( \frac{dt}{d\tau} = \rho(x) \). When we see the system with the new time scale \( \tau \), it is guaranteed that there is one-to-one correspondence between the solutions since \( \frac{dt}{d\tau} > 0 \) (namely \( \tau \) does not move backward), and some essential properties of the systems, such as stability, are kept unchanged from the original system. In other words, the time-scale transformation gives us an extra one-degree-of-freedom in control design by arbitrarily choosing the new time scale. This advantage has already been applied to design of a disturbance decoupling controller [4].

The systems that are convertible under time-scale transformation to each other, are called orbitally equivalent \(^1\) to each other (denoted by OE for short). Sampei and Furuta gave a basic characterization of this equivalence by a similarity condition of the system vector-fields. However, their characterization was not a complete one mainly due to the treatment of singularity emerging in the similarity condition.

In this paper, we intend to cast a new light on this problem in the following manner. First, we replace the aforementioned notion of orbital equivalence with a milder one, which we call almost orbital equivalence (AOE for short), by allowing negligible amount of inconsistency on measure zero subsets. Succeedingly we introduce almost similarity and almost feedback similarity for system vector-fields. While AOE concerns equivalence in orbits on the state space, the almost similarity is concerned with equivalence in vector-fields on the tangent space. Based on these definitions, we give necessary and sufficient conditions connecting them for both autonomous and non-autonomous cases.

Another problem left in the past works [1]–[4] is that the time-scale transformation technique itself does not deal with the stability issue. Hence stability of the systems had to be checked independently, e.g., using the conventional Lyapunov method. In this study, we focus on the recent development of the density function approach for the stability analysis [5]–[7]. By virtue of the affinity between AOE and the density function approaches, we show that the orbital equivalence analysis and the stability analysis can be naturally combined.

The rest of this paper is organized as follows. After short preliminaries in Section 2, we introduce the new definition of almost orbital equivalence in Section 3 by relaxing the existing orbital equivalence. In Section 4, we introduce the notion of similarity of the system vector-fields, also in the 'almost' sense. In Section 5, we characterize the almost orbital equivalence in the form of the necessary and sufficient conditions with almost similarity. Combination of the orbital equivalence and the stability analysis based on the density function is discussed in Section 6. Some demonstrative examples are given in Section 7 followed by the conclusion in Section 8.

2. Preliminaries

Consider a pair of autonomous nonlinear systems:
\[ \begin{align*}
\Sigma_1^A & : \frac{dx}{dt} = f(x) \\
\Sigma_2^A & : \frac{dx}{dt} = \xi(x)
\end{align*} \]

and a pair of non-autonomous nonlinear systems with affine control:

\[ \begin{align*}
\Sigma_1 : \frac{dx}{dt} = f(x) + g(x)u \\
\Sigma_2 : \frac{dx}{dt} = \xi(x) + \eta(x)v
\end{align*} \]

where \( x \in M \) is the state, \( M \) is an \( n \)-dimensional manifold of class \( C^\infty \) and \( u, v \in \mathbb{R} \) are the control inputs. In the rest of paper, we use smooth as an alias for being of class \( C^\infty \). We assume all the vector-fields \( f(x), g(x), \xi(x) \) and \( \eta(x) \) are smooth on \( M \), unless stated otherwise. \( T_1 \) denotes the tangent space of the manifold \( M \) at the point \( x \in M \).

Let \( \Phi(\Sigma_i, t, x_0) \) denote the solution of the autonomous system \( \Sigma_i^A \) for \( t \in [0, T) \) starting from \( x(0) = x_0 \in M \). Likewise, \( \Phi(\Sigma_i, t, x_0, u(\cdot)) \) denotes the solution of the non-autonomous system \( \Sigma_i \) starting from \( x_0 \) under the control \( u : [0, \infty) \to \mathbb{R} \). Then, the orbit of an autonomous system \( \Sigma_i^A \) over \( [0, T), T > 0 \) starting from \( x(0) = x_0 \) is defined as

\[ O_{bh}(\Sigma_i^A, x_0, T) := \{ \Phi(\Sigma_i^A, t, x_0) \mid t \in [0, T) \} \]

and its non-autonomous counterpart is defined as

\[ O_{bh}(\Sigma_i, x_0, u(\cdot), T) := \{ \Phi(\Sigma_i, t, x_0, u(\cdot)) \mid t \in [0, T) \} \]

Note the distinction between solutions \( \Phi \) and orbits \( O_{bh} \); a solution is a function of the time \( t \), while an orbit is a collection of points in \( M \) over specified time interval.

For two vectors \( a, b \in \mathbb{R}^n \), the binary relation \( a \parallel b \) implies that \( a \) is parallel to \( b \) and vice versa, i.e., \( \exists k \in \mathbb{R} \) such that \( a = kb, k \neq 0 \). In contrast \( a \nparallel b \) means the vectors \( a \) and \( b \) are not parallel to each other. For two parallel vectors \( a \) and \( b \), we indicate \( \frac{a}{b} \) to imply the scale factor \( k \in \mathbb{R} \) which satisfies \( a = kb \). For a differentiable vector-field \( f : \mathbb{R}^n \to \mathbb{R}^n \), \( \nabla \cdot f \) is defined as

\[ \nabla \cdot f := \frac{\partial f_1}{\partial x_1} + \cdots + \frac{\partial f_n}{\partial x_n}. \]

For a scalar function \( \rho \), \( \nabla \rho \) is defined as

\[ \nabla \rho := \left( \frac{\partial \rho}{\partial x_1}, \cdots, \frac{\partial \rho}{\partial x_n} \right). \]

### 3. Almost Orbital Equivalence

#### 3.1 Background

We begin with the following example to illustrate distinction between solutions and orbits [1].

\[ \begin{align*}
\frac{dx_1}{dt} & = \frac{x_2}{x_1 - \frac{1}{2} x_2} \\
\frac{dx_2}{dt} & = \frac{x_2(\frac{3}{2} + \sin 16x_1)}{(x_1 - \frac{1}{2} x_2)(\frac{1}{2} + \sin 16x_1)}
\end{align*} \]

System (9) is a linear system while (10) is apparently a nonlinear one. Figures 1 and 2 show the time response of both systems from the common initial state \([2, 2]^T\), which are clearly different from each other. Meanwhile, Fig. 3 shows phase portraits of the both systems (9) and (10), i.e., they are identical to each other in the sense of state flow. Now, if we introduce a new time scale to the system (10) using a scalar function \( \rho(x) \):

\[ \frac{dt}{d\tau} = \rho(x) = \frac{1}{\frac{1}{2} + \sin 16x_1} \]

then its new state equation along the time-scale \( \tau \) is:

\[ \frac{dx_1}{d\tau} \bigg|_{t_1} = \begin{bmatrix} x_2 \\ -x_1 - \frac{1}{2} x_2 \end{bmatrix} \]

whose right hand sides of (12) is exactly the same as that of (9). This implies the system (12) behaves exactly the same as the system (9) along the time scale \( \tau \).

#### 3.2 Almost Orbital Equivalence

Now we introduce the almost notion of orbital equivalence by allowing small amount of inconsistency in terms of Lebesgue measure on the state space. Throughout this paper, ‘almost all’ means its complement (i.e., the set of exceptions) has Lebesgue measure 0. The following definition concerns the case of autonomous systems.

**Definition 1.** For almost all initial state \( x_0 \in M \), suppose there exist smooth functions \( T_1, T_2 : [0, \infty) \to [0, \infty) \) which satisfy both

\[ \begin{align*}
O_{bh}(\Sigma_1^A, x_0, T) & = O_{bh}(\Sigma_2^A, x_0, T_1(T)) \\
O_{bh}(\Sigma_2^A, x_0, T) & = O_{bh}(\Sigma_1^A, x_0, T_2(T))
\end{align*} \]

for \( T > 0 \) such that the orbits are defined up to \( \max(T, T_1(T), T_2(T)) \). Then the autonomous systems \( \Sigma_1^A \) and \( \Sigma_2^A \) are called almost orbitally equivalent.

This equivalence can be interpreted as follows. For almost all choice of initial states, the orbit of the system \( \Sigma_1^A \) on \([0, T)\) is identical to that of the system \( \Sigma_2^A \) provided the interval is transformed to \([0, T_1(T)) \).

In the case of non-autonomous systems, transformation of the control input is also a matter of concern.
Definition 2. For almost all initial state $x_0 \in M$ and any smooth input $u(\cdot) : [0, \infty) \to \mathbb{R}$, suppose there exist smooth functions $u_1(\cdot), u_2(\cdot) : [0, \infty) \to \mathbb{R}$ which satisfy

$$O_{\text{orb}}(\Sigma_1, x_0, u(\cdot), T) = O_{\text{orb}}(\Sigma_2, x_0, u_1(\cdot), T_1(T)) \quad (15)$$

$$O_{\text{orb}}(\Sigma_2, x_0, u(\cdot), T) = O_{\text{orb}}(\Sigma_1, x_0, u_2(\cdot), T_2(T)) \quad (16)$$

for $T > 0$ such that the orbits are defined up to $\max[T, T_1(T), T_2(T)]$. Then the non-autonomous systems $\Sigma_1$ and $\Sigma_2$ are called almost orbitally equivalent.

4. Almost Similarity of Vector-Fields

Let us turn to discuss equivalence of system vector-fields (the right-hand side of state equations), in contrast to the equivalence of system orbits discussed above. Simply speaking, we proceed to the equivalence on the tangent space from the equivalence on the state space.

Definition 3 (Almost Similarity; AS). Suppose the autonomous systems $\Sigma_1^A$ and $\Sigma_2^A$ satisfy the following conditions.

If the following conditions hold, we say $\Sigma_1^A$ and $\Sigma_2^A$ are almost similar to each other.

1. $f(x)$ and $\xi(x)$ are parallel almost everywhere, i.e., there exists a smooth function $\rho(x)$ and a subset $S \subseteq M$ having Lebesgue measure zero, such that

$$\rho(x)f(x) = \xi(x), \quad 0 < \rho(x) < \infty \quad (17)$$

holds for all $\forall x \not\in S$.

2. Neither $O_{\text{orb}}(\Sigma_1^A, S, \epsilon)$ nor $O_{\text{orb}}(\Sigma_2^A, S, \epsilon)$ contains any non-empty open subset of $M$ for $\forall \epsilon > 0$, where $O_{\text{orb}}(\Sigma^A, S, \epsilon)$ is the set of local solutions starting from $S$ defined by

$$O_{\text{orb}}(\Sigma^A, S, \epsilon) := \{ \Phi(\Sigma^A, t, x_0) \mid x_0 \in S, t \in [0, \epsilon) \} \quad (18)$$

Moreover, if there exists a neighborhood $U$ of $x_0 \in M$ such that $\Sigma_1^A$ and $\Sigma_2^A$ are almost similar on $U$, then we say $\Sigma_1^A$ and $\Sigma_2^A$ are locally almost similar.

Remark 1. The condition (17) implies that $\rho'(x) := 1/\rho(x)$ is also smooth and $0 < \rho'(x) < \infty$.

Remark 2. If $S$ is a smooth submanifold of $M$ in a neighborhood of $x$, the condition 2 of Definition 3 reduces to

$$\text{span}\{f(x)\} + T_xS \neq T_xM, \quad \text{span}\{\xi(x)\} + T_xS \neq T_xM \quad (19)$$

where $T_xS$ denotes the tangent space of $S$ at $x$. This is a standard condition for non-transversal intersection of submanifolds. Note that it is violated only when $\dim S = n - 1$.

Definition 4 (Almost Feedback Similarity; AFS). Suppose the non-autonomous systems $\Sigma_1$ and $\Sigma_2$ satisfy the following conditions.

1. Both $f(x)$ and $\xi(x)$, $g(x)$ and $\eta(x)$ are parallel via feedback almost everywhere, i.e., there exist smooth functions $\rho(x), \alpha(x), \beta(x)$ and a smooth submanifold $S \subseteq M$ having Lebesgue measure zero such that

$$\rho(x)f(x) + g(x)\alpha(x) = \xi(x) \quad (20)$$

$$\rho(x)\beta(x)g(x) = \eta(x) \quad (21)$$

$$0 < \rho(x) < \infty, \quad 0 < |\beta(x)| < \infty \quad (22)$$

hold for $\forall x \not\in S$.

2. Neither $O_{\text{orb}}(\Sigma_1, S, u(\cdot), \epsilon)$ nor $O_{\text{orb}}(\Sigma_2, S, u(\cdot), \epsilon)$ contains any non-empty open subset of $M$ for $\forall \epsilon > 0$, where $O_{\text{orb}}(\Sigma, S, u(\cdot), \epsilon)$ is defined by

$$O_{\text{orb}}(\Sigma, S, u(\cdot), \epsilon) := \{ \Phi(\Sigma, t, x_0, u(\cdot)) \mid x_0 \in S, t \in [0, \epsilon) \}. \quad (23)$$

Then we say $\Sigma_1$ and $\Sigma_2$ are almost feedback similar to each other. If there exists a neighborhood $U$ of $x_0 \in M$ such that $\Sigma_1$ and $\Sigma_2$ are almost feedback similar on $U$, then we say $\Sigma_1$ and $\Sigma_2$ are locally almost feedback similar.

Remark 3. The condition 1 can be replaced by its alternative version

$$\rho'(x)(\xi(x) + \eta(x)\alpha'(x)) = f(x) \quad (24)$$

$$\rho'(x)\beta'(x)\eta(x) = g(x) \quad (25)$$

$0 < \rho'(x) < \infty, \quad 0 < |\beta'(x)| < \infty \quad (26)$

under the correspondence $\rho'(x) := 1/\rho(x), \alpha'(x) := -\alpha(x)/\beta(x)$ and $\beta'(x) := 1/\beta(x)$.

Remark 4. If $S$ is a smooth submanifold of $M$ in a neighborhood of $x$, the condition 2 of Definition 4 also reduces to

$$\text{span}\{f(x), g(x)\} + T_xS \neq T_xM, \quad \text{span}\{\xi(x), \eta(x)\} + T_xS \neq T_xM \quad (27)$$

Remark 5. The subset $S$ is defined as the collection of points at which (21)–(22) do not hold. Considering $S$ should be of measure zero, possible situations that may occur on $S$ are somewhat limited as follows.

(a). $g(x) \parallel \xi(x)$ or $g(x) \parallel f(x)$. (See Fig. 4)

If $g(x) \parallel \xi(x)$ at some $x \in S$, $f(x) + g(x)\alpha(x)$ cannot be parallel to $\xi(x)$ unless $\alpha(x)$ is infinite. Thus $\rho(x)$ is zero while $\beta(x)$ should be infinite, violating (22). On the other hand, $g(x) \parallel f(x)$ implies $\eta(x) \parallel f(x)$, thus $\rho'(x)$ is zero while $\beta'(x)$ should be infinite, violating (26).

(b). $g(x) = 0$ or $\eta(x) = 0$.

Clearly $\beta'(x)$ or $\beta(x)$ should be zero in these cases.
5. Characterization of Almost Orbital Equivalence and Almost Similarity

In this section, we show that the almost orbital equivalence and the almost similarity are equivalent. Readers may expect that the correspondence between orbital equivalence and similarity of vector-fields is trivial, but it is not true. We stress that introducing ‘almost’ point of view is crucial for complete characterization. The claim is summarized in the following theorems.

**Theorem 1.** The autonomous systems $\Sigma^1$ and $\Sigma^2$ are almost orbitally equivalent if and only if they are almost similar to each other.

**Proof.** **Sufficiency (AOE$\Rightarrow$AS):** Suppose $\Sigma^1$ and $\Sigma^2$ satisfy the conditions for almost similarity in Definition 3. From (17), we can define a new time-scale $\tau$:

$$\frac{dt}{d\tau} = \rho(x) \quad (28)$$

The state equation of $\Sigma^1$ along $\tau$ is given by

$$\frac{dx}{d\tau} = \rho(x)f(x) = \xi(x), \quad \forall x \notin S, \quad (29)$$

which is exactly the same form as the state equation (2) of $\Sigma^2$. Thus the orbits of $\Sigma^1$ and $\Sigma^2$ locally coincide with each other almost everywhere, i.e.,

$$\forall x \notin S, \exists \epsilon_1(x) > 0, \epsilon_2(x) > 0, \quad s.t., \quad O_h(\Sigma^1, x, \epsilon_1(x)) = O_h(\Sigma^2, x, \epsilon_2(x))$$

This does not directly imply that the orbits coincide with respect to all initial conditions. Now, recall the condition 2 of Definition 3 which guarantees $O_h(\Sigma^1, S, u(t), \epsilon)$ does not contain any non-empty open set. Then, from the smoothness assumption of the vector-fields, the set of orbits which intersects $S$ is of measure zero. Therefore the orbits starting from almost all initial conditions do not pass through $S$.

Finally, let us find the conversion formula of the terminal times for corresponding pair of orbits. For an orbit $O_h(\Sigma^1, x_0, u(t), T)$ of the system (1) as shown in Fig. 5, $\rho(\Phi(t, x_0))$ is smooth over $t \in [0, T]$ and satisfies $0 < \rho(\Phi(t, x_0)) < \infty$ unless it encounters $S$. Then,

$$T_1(x_0, T) = \int_0^T \rho(\Phi(\Sigma^1, t, x_0))dt \quad (30)$$

is a smooth function that gives $T_1$ in the right hand side of (15). Likewise, $T_2$ for (16) is given by

$$T_2(x_0, T) = \int_0^T \frac{1}{\rho(\Phi(\Sigma^2, t, x_0))}dt \quad (31)$$

This concludes almost orbital equivalence of the two systems.

**Necessity (AOE$\Rightarrow$AS):** Suppose $\Sigma^1$ and $\Sigma^2$ are almost orbitally equivalent. Take an orbit $O_h(\Sigma^1, x_0, T)$ and the corresponding one $O_h(\Sigma^2, x_0, T_1)$ which satisfy (13). Here $T$ is the terminal time of the orbit measured by the time scale on system $\Sigma^1$, while $T_1$ is the time measured by the time scale on system $\Sigma^2$. Note that $T_1$ is a function of $x_0$ as well as $T$ because of its smooth dependence on initial states, i.e., $T_1 = T_1(x_0, T)$. For almost all $x \in M$ and $T > 0$, there exists $x_0(x, T_0) \in M$ such that $x = \Phi(\Sigma^1, T_0, x_0)$. Thus the time-scale factor $\rho(x)$ at $x$ is constructed as follows

$$\rho(x) = \frac{d}{dT} T_1(x_0(x, T_0), T) \mid_{T=T_0} \quad (32)$$

Finally, suppose $S$ is a subset of $M$ such that the equivalence condition (13)(14) do not hold for any $x_0 \in S$. If either of $S$, $O_h(\Sigma^1, S, \epsilon)$ or $O_h(\Sigma^2, S, \epsilon)$ contains an open subset of $M$, it would conflict with the assumption of almost orbital equivalence that the set of exceptional initial points must be of measure zero. Therefore the condition 1 and 2 of the Definition 3 hold. \(\square\)

Extension of this theorem to non-autonomous case is mostly straightforward.

**Theorem 2.** The non-autonomous systems $\Sigma_1$ and $\Sigma_2$ are almost orbitally equivalent if and only if they are almost feedback similar to each other.

**Proof.** **Sufficiency (AOE$\Rightarrow$AFS):** If the systems $\Sigma_1$ and $\Sigma_2$ satisfy the conditions of almost feedback similarity, then we can define the feedback

$$u = \alpha(x) + \beta(x)v \quad (33)$$

and the time scale transformation

$$\frac{dt}{d\tau} = \rho(x) \quad (34)$$

using the smooth functions $\rho(x), \alpha(x)$, and $\beta(x)$ in Definition 4. These feedback and scaling transform $\Sigma_1$ into

$$\frac{dx}{d\tau} = \rho(x)[f(x) + g(x)\alpha(x)] + \rho(x)g(x)\beta(x)v = \xi(x) + \eta(x)v \quad (35)$$

This implies that the vector fields of the system $\Sigma_1$ in the new time scale $\tau$ corresponds to those of $\Sigma_2$ in the actual time scale $t$. The rest of the proof is the same as in the previous theorem.
6. Discussion: Connection with Stability Analysis Using Density Functions

While Lyapunov method has been a standard tool for stability analysis of nonlinear systems since the dawn of nonlinear control theory, Rantzer [5] suggested an alternative stability analysis tool using density functions, which is dual to Lyapunov method in a sense. In this framework, the density function is a scalar-valued smooth function analogous to ‘density’ of continuum, which is required to be positive and bounded with only exception at the origin (sink) where it may diverge. This motivates us to compare our time-scale function \( \rho(x) \) with the density function, and investigate how the time scale transformation is concerned with the system stability.

Let us reconsider the autonomous systems \( \Sigma_1^A, \Sigma_2^A \) and assume that the origin 0_M \( \in M \) is an equilibrium, namely \( f(0_M) = 0 \) for (1) and \( \xi(0_M) = 0 \) for (2). We also assume for simplicity that the system is complete, i.e., for each initial state \( x_0 \in M \), the solution \( \Phi(\Sigma^2_1, t, x_0) \) is uniquely defined over \( t \in \mathbb{R} \). The following definition is a slightly modified version of asymptotic stability to conform to the ‘almost’ style of discussion.

**Definition 5 (Almost Global Attractivity).** If the solutions \( \Phi(\Sigma^2_1, t, x_0) \) tends to 0_M as \( t \to \infty \) for almost all initial states \( x_0 \in M \), then we say the equilibrium 0_M of the system is almost globally attractive.

Note that almost global attractivity allows the existence of non-convergent solutions having Lebesgue measure zero. Moreover, only the convergence is the matter of concern here as in [5] the local stability (in the sense of Lyapunov) is disregarded (several variations of almost ‘stability’ together with attractivity are discussed in [6],[7]). Now we are ready to state the following observation.

**Theorem 3.** Suppose the nonlinear system \( \Sigma^A_1 \) given by (1) has an almost globally attractive equilibrium at 0_M. Then there exists another system

\[
\Sigma^A_1: \quad \dot{x} = f(x)
\]

which satisfies \( \nabla \cdot \hat{f}(x) > 0 \) almost everywhere and is almost orbitally equivalent to the original system \( \Sigma^A_2 \) on \( M \setminus \{0_M\} \), namely, there exist smooth functions \( T_1, T_2 : [0, \infty) \to [0, \infty) \) which satisfy (13) (14) for almost all \( x_0 \in M \setminus \{0_M\} \) and \( T > 0 \) such that the orbits are defined on \( M \setminus \{0_M\} \) up to \( \max(T, T_1(T), T_2(T)) \).

**Proof.** Since the solutions of \( \Sigma^A_1 \) converge to 0_M for almost all initial state \( x_0 \), the converse theorem of density function approach [8] guarantees that there exists a density function \( \rho : M \to \mathbb{R} \), which is smooth over \( M \setminus \{0_M\} \) satisfying:

1. \( \rho(x) \geq 0 \), \( \forall x \in M \setminus \{0_M\} \)
2. \( \nabla \cdot (f\rho)(x) > 0 \), for almost all \( x \in M \)

Now we see that \( \rho(x) \) plays a role of re-scaling the vector-field \( f \), exactly like the time scale function (28). Both \( f(x) \) and \( \hat{f}(x) := \rho(x)f(x) \) are smooth and almost similar to each other on \( M \setminus \{0_M\} \). Therefore \( \Sigma^A_1 \) and \( \Sigma^A_2 \) are almost orbitally equivalent.

\( \square \)

Note that we consider almost orbital equivalence of systems on \( M \setminus \{0_M\} \). Similar treatment of stability by allowing non-smoothness at the origin is found in [9].

7. Illustrative Example

Consider the following system which was dealt with in [5].

\[
\begin{cases}
\dot{x}_1 = f(x) = \begin{bmatrix} -2x_1 + x_1^2 - x_2^2 \\ -2x_2 + 2x_1x_2 \end{bmatrix} \\
\dot{x}_2 = g(x) = \begin{bmatrix} -2x_2^2 - (x_1 - 2)x_1^2 + x_1 - 4 \\ -2x_2^2 \end{bmatrix}
\end{cases}
\]

This system has two equilibria, at \( 0_{3^2} = [0, 0]^T \) and \( [2, 0]^T \). Consider a candidate of density function \( \rho(x) = |x|^4 \), then we have

\[
\nabla \cdot (f\rho)(x) = \nabla \cdot f + \rho(\nabla \cdot f) = -4|x|^6 \dot{x}_1 + |x|^4(4x_1 - 4) = 4|x|^4 > 0
\]

which implies the system is almost globally attractive.

See Fig. 6 for its phase portrait. Almost all solutions tend to \( 0_{3^2} \) as \( t \to \infty \), except the one passing flow the half segment \( \{x_1 \geq 2, x_2 = 0\} \).

Next, suppose the following system to see how the orbits and attractivity of the original system are preserved.

\[
\begin{cases}
\dot{x}_1 = f(x) = \begin{bmatrix} x_1^2 - x_2^2 \\ -2x_2 + 2x_1x_2 \end{bmatrix} \\
\dot{x}_2 = g(x) = \begin{bmatrix} -2x_2^2 - (x_1 - 2)x_1^2 + x_1 - 4 \\ -2x_2^2 \end{bmatrix}
\end{cases}
\]

\( \square \)

Fig. 6 Phase portrait of the original system (38).

Fig. 7 Phase portrait of (39) where \( \rho(x) = x_2^2 \).
In this case, this system is almost orbitally equivalent to the original system (38). Its phase portrait is shown in Fig. 10. Except $x_1$-axis, all trajectories are equivalent to the original system. Figures 8 and 9 shows time responses of the systems (38) and (39) starting from $(2, 2)^T$.

Finally, consider the following system:

$$
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} = \xi_3(x) = \begin{bmatrix}
-2x_1 + x_1^2 - x_2^2x_1x_2 \\
-2x_2 + 2x_1x_2x_1^2
\end{bmatrix} \quad (39)
$$

If we set $\rho(x) = x_1^2x_2^2$ in this case, then $S = \{(x_1, x_2)^T | x_1 = 0 \text{ or } x_2 = 0\}$ (namely, union of the $x_1$- and $x_2$-axes) intersects the orbits of the system and hence it does not satisfy the non-transversality condition of almost similarity. Thus this system is not almost orbitally equivalent to the original system (38). Its phase portrait is shown in Fig. 10.

8. Conclusion

In this paper, we introduced the notions of almost orbital equivalence and almost similarity of the vector-fields and showed their equivalence. We also pointed out that stability of systems in the sense of Rantzer’s density function is invariant under this equivalence. Future topics include a controller design problem that makes a system almost orbitally equivalent to a given reference system. It is also noteworthy to discuss its relevance with the bisimulation point of view proposed by van der Schaft [10].

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References


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