On Sensitivity Reduction Problems of Sampled-Data Systems: Relationships to the Problems of Discrete-Time Systems

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Abstract: This paper is concerned with the sensitivity reduction (SR) and complementary sensitivity reduction (CSR) problems of sampled-data systems. We begin our study by showing that, given a sampled-data system $\Sigma$, the $H^\infty$ norms of the sensitivity and complementary sensitivity of $\Sigma$ can be expressed as those for an equivalent discrete-time system $\hat{\Sigma}$ called the doubly sensitivity-preserving discretized system. We also consider the conventional ‘hold equivalent’ discretized system $\Sigma_d$, which has generally been believed, due to the ignoring of the intersample behavior, to be irrelevant to the SR/CSR problems of $\Sigma$ (and thus $\hat{\Sigma}$). We then establish that there in fact exists an important relation between the seemingly completely irrelevant discrete-time systems $\hat{\Sigma}$ and $\Sigma_d$. More precisely, we show through the coprime factorization approach that the CSR problem of $\Sigma$ (and thus $\hat{\Sigma}$) is equivalent to a weighted CSR problem of $\Sigma_d$ and that the SR problem of $\Sigma$ is equivalent to a weighted mixed sensitivity reduction problem of $\Sigma_d$. In particular, we show that these properties of $\psi$ can be used to prove some relation between the best achievable performance in the SR (respectively, CSR) problem of $\hat{\Sigma}$ (and thus $\Sigma$) and that of $\Sigma_d$. An interesting property between the SR problem and CSR problem of $\Sigma$ is also provided.

Key Words: digital control, sampled-data systems, sensitivity reduction problem, aliasing factors, coprime factorization approach.

1. Introduction

This paper concerns the sensitivity reduction (SR) and complementary sensitivity reduction (CSR) problems of sampled-data systems. It is well known that these problems are formulated as $H^\infty$ problems of sampled-data systems, whose solution methods have been already established [1]–[3] through some sophisticated so-called $\gamma$-dependent discretization techniques of the continuous-time plant. The $\gamma$-dependent discretization is in sharp contrast with the conventional ‘hold equivalent’ discretization that only views the responses at the sampling instants and ignores completely the intersample behavior. In this paper, we first show that, as far as the SR and CSR problems are concerned, we can have a $\gamma$-independent discretization method to reduce them to equivalent discrete-time problems. It gives a discretized plant that we call the doubly sensitivity-preserving (DSP) discretized plant, which can be used both for the SR problem and CSR problem of sampled-data systems. On the other hand, we refer to the discretized plant obtained by the ‘hold equivalent’ discretization as a naively discretized plant. The present paper is devoted to the study of the SR and CSR problems of sampled-data systems through the DSP discretized plant, including the investigations on some important relationships between the DSP discretized plant and the naively discretized plant.

The study of this paper is motivated by the recent studies about the problems of what determines the best achievable performance of control systems. For example, the $H^2$ tracking performance of continuous-time systems and that of sampled-data systems are studied in [4] and [5], respectively, and these works derive elegant analytic solutions of the best achievable performances of control systems. Concerning the SR/CSR problems of sampled-data systems, however, analytic solutions of the best achievable performances have not been derived yet. For obtaining the analytic solutions of these problems, a “transfer matrix-based approach” would be more appropriate, but such an approach has not been developed because the infinite-dimensionality nature of sampled-data systems makes such a kind of approach extremely difficult. The arguments in this paper enable us to study the SR/CSR problems of sampled-data systems through equivalent discrete-time problems with a fixed $\gamma$-independent discretized plant. Hence, the arguments are not only important on their own, but might hopefully provide a fundamental basis for a further study on the performance limitation about the SR/CSR problems of sampled-data systems.

This paper is organized as follows: In Sec. 2, we first review the FR-operator based representation of sampled-data systems, which can deal with their intersample behavior and aliasing effects. The treatment leading to the naively discretized plant by ignoring the intersample behavior is also summarized there. In Sec. 3, we introduce the DSP discretized plant and show that SR/CSR problems of sampled-data systems can equivalently be reduced to those about the DSP discretized plant. In Sec. 4, we introduce the coprime factorization ap-
proach to the DSP discretized plant and the naively discretized plant, and show some relationship between the SR (respectively, CSR) problem of sampled-data systems (equivalently, that about the DSP discretized plant) and that about the naively discretized plant. More precisely, the equivalent SR and CSR problems about the DSP discretized plant are shown to be equivalent to those about the naively discretized plant with some frequency-dependent weights, where these weights can be regarded as representing the influence of the aliasing in the sampled-data systems when they are to be treated through the seemingly useless naively discretized plant obtained by ignoring of the intersample behavior. These weights are generated by a more fundamental function that we call the aliasing factor. Its analytical property is also studied, through which some relations and properties proved between the best achievable performance for the SR problem and that for the DSP discretized plant. More precisely, the equivalent SR and CSR (respectively, that about the DSP discretized plant) and that about the naively discretized plant, and show some relationship between the SR (respectively, CSR) problem of sampled-data systems (equivalently, that about the DSP discretized plant) and that about the naively discretized plant. Section 5 gives some result about the best achievable performance for the SR problem and CSR problem of sampled-data systems, through the coprime factorization of the DSP discretized plant and the solution of the Nevanlinna problem with the Pick matrix. Section 6 gives some concluding remarks.

Notation: The transpose and complex conjugate transpose of a matrix $A$ are denoted by $A^T$ and $A^*$, respectively. For a continuous-time transfer function $X(s)$ and a discrete-time one $X(z)$, we use the notations $X'(s) := X(-s), X'(z) := X(z^{-1}), Z[X(s)]$ stands for the $z$-transform of $X(s)$, i.e., $Z[X(s)] = \sum_{k=0}^{\infty} x(\pi k) z^{-k}$ where $x(t)$ is the inverse Laplace transform of $X(s)$ and $\pi$ is the sampling period. $I$ is the identity operator on $l_2$. 

2. Preliminary

2.1 FR-Operator Representation of the Sampled-Data System $\Sigma$

In this section, we briefly review the notion of FR-operators introduced in [6] for the frequency-domain treatment of sampled-data systems. Using FR-operators, we can take into account the intersample behavior of sampled-data systems, as well as the influence of the aliasing due to sampling actions.

Consider the sampled-data system $\Sigma$ shown in Fig. 1, consisting of the continuous time plant $P$, the discrete-time controller $C_d$, the ideal sampler $S$ with sampling period $\tau$, and the generalized hold $H$, which works according to $u(k\tau + t) = h(t)u_d[k]$ ($t \in [0, \tau]$) [7], where $u$ and $u_d$ are hold output and input, respectively, and $h(t)$ is referred to as the hold function. The transfer functions of $P$, $H$, and $C_d$ are denoted by $P(s), H(s) = \int_0^\infty h(t)e^{-st}dt$, and $C_d(z)$, respectively.

The sensitivity $S(j\varphi)$ and the complementary sensitivity $T(j\varphi)$ of the sampled-data system are defined as the FR-operators from $r$ to $e$ and $u$, respectively [6]; they are operators on $l_2$ represented as infinite-dimensional matrices given by

$$S(j\varphi) := \left( I + \frac{1}{\tau} H(j\varphi) C_d(e^{j\omega T}) P(j\varphi) \right)^{-1},$$

$$T(j\varphi) := \frac{1}{\tau} H(j\varphi) C_d(e^{j\omega T}) P(j\varphi).$$

where $\varphi \in (-\pi/\tau, \pi/\tau]$ and

$$H(j\varphi) = \begin{bmatrix} \cdots & \vdots & \vdots & \vdots \\ \cdots & H(j\varphi_{-1}) & H(j\varphi_0) & H(j\varphi_1) \\ \cdots & \vdots & \vdots & \vdots \end{bmatrix},$$

$$P(j\varphi) = \begin{bmatrix} \cdots & P(j\varphi_{-1}) & P(j\varphi_0) & P(j\varphi_1) & \cdots \end{bmatrix}.$$
of $\Sigma$ with $\Sigma_d$. Throughout the paper, we will refer to $\Sigma_d$ as the naively discretized system.

The sensitivity function $S_d(e^{j\omega T})$ and the complementary sensitivity function $T_d(e^{j\omega T})$ of the naively discretized system $\Sigma_d$ are defined as the frequency transfer functions from $r_d$ to $e_d$ and $u_d$, respectively:

$$S_d(e^{j\omega T}) := I - K_d(e^{j\omega T})P_d(e^{j\omega T}), \quad (6)$$
$$T_d(e^{j\omega T}) := K_d(e^{j\omega T})P_d(e^{j\omega T}). \quad (7)$$

3. Reduction of SR/CSR Problems of the Sampled-Data System $\Sigma$ to Discrete-Time Problems with the Doubly Sensitivity-Preserving Discretized System $\hat{\Sigma}$

It is generally understood that the naively discretized plant $P_d$ introduced in the preceding section is useless for studying the SR/CSR problems of the original sampled-data system $\Sigma$. For studying such problems directly, we introduce in this section a more elaborate and useful discretized plant that we call the doubly sensitivity-preserving (DSP) discretized plant, denoted by $\hat{P}$. We also introduce what we call the DSP discretized hold, denoted by $\hat{H}$, and consider as in Fig. 3 the closed-loop system consisting of $\hat{P}$, $\hat{H}$ and the same discrete-time controller $C_d$ as in the original sampled-data system $\Sigma$; we call it the DSP discretized system and denote it by $\hat{\Sigma}$ (the rationale for the term DSP will become clear later). What we establish in this section is that $\hat{\Sigma}$ is stable if and only if the original sampled-data system $\Sigma$ is, and that the frequency response gain of the sensitivity (respectively, complementary sensitivity) function of $\hat{\Sigma}$ coincides with that of $\Sigma$ at each angular frequency.

These properties imply that the SR/CSR problems of the sampled-data system $\Sigma$ can be reduced equivalently to the corresponding discrete-time problems for the DSP discretized system $\hat{\Sigma}$. The DSP discretized plant $\hat{P}$ is hence quite important on its own. However, what makes the discussions in the present paper much more significant is that we can in fact reveal some relationship between these equivalent SR and CSR problems in terms of $\Sigma$ and some sort of sensitivity reduction problems about the naively discretized system $\Sigma_d$; what is surprising in such a relationship is that the latter system has been obtained by completely ignoring the intersample behavior of $\Sigma$ so that it is at a glance quite irrelevant to the SR and CSR about the sampled-data system $\Sigma$ (or equivalently, $\hat{\Sigma}$). The discussions on such a relationship between $\Sigma$ (or equivalently, $\hat{\Sigma}$ and $\Sigma_d$ as well as its implication will be the main part of the present paper. However, they will be deferred to Sections 4 and 5, and to prepare some fundamental results, we confine our attention in this section to the derivation of the DSP discretized plant $\hat{P}$ and hold $\hat{H}$.

We begin with a key lemma.

**Lemma 1** Suppose that the state-space realization of the continuous-time plant $P$ is given by

$$P(s) = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix},$$

and the hold function $h(t)$ of the hold circuit $\mathcal{H}$ is given by $h(t) = C_H e^{sH} B_H$. Let $\hat{B}$ and $\hat{C}_H$ be matrices satisfying

$$\begin{bmatrix} \hat{B} \\ \hat{C}_H \end{bmatrix} \begin{bmatrix} \hat{B}^T \\ \hat{C}_H \end{bmatrix} = \int_0^\tau \Gamma(t)\Gamma^T(t) dt,$$

$$\Gamma(t) = \begin{bmatrix} e^{A(t-t)} & 0 \\ 0 & e^{sH} \end{bmatrix} \begin{bmatrix} \hat{B}^T \\ \hat{C}_H \end{bmatrix}.$$ 

Then, for

$$\hat{P}(z) = \begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & 0 \end{bmatrix}, \quad \hat{H}(z) = \hat{C}_H B_H \quad (8)$$

with $\hat{A} = e^{sH}$, the following relation \(^1\) holds:

$$\frac{1}{\tau} \begin{bmatrix} \hat{P} \\ \hat{H} \end{bmatrix} \begin{bmatrix} \hat{P}^T \\ \hat{H} \end{bmatrix} = \begin{bmatrix} \hat{P} \hat{H} \\ \hat{H} \end{bmatrix} \begin{bmatrix} \hat{P}^T \\ \hat{H} \end{bmatrix} \quad \left( \varphi \in \left(-\frac{\pi}{\tau}, \frac{\pi}{\tau}\right) \right). \quad (9)$$

**Proof of Lemma 1:** The proof is based on essentially the same fact as the well-known fact that the right-hand side of (4), known as the impulse modulation of $P(s)H(s)$, is equivalent to the usual discretization of $P(s)H(s)$ given by (5) evaluated on the unit circle. That is, we first note that the left-hand side of (9) can be regarded as a sort of impulse modulation since $P^*(j\varphi_m) = P'(j\varphi_m)$ and $H^*(j\varphi_m) = H'(j\varphi_m)$. Then, we see that it can be represented as the left-hand side of the following equation evaluated at $z = e^{j\varphi}$:

$$\begin{bmatrix} P(s) \\ H'(s) \end{bmatrix} \begin{bmatrix} P(s) & H(s) \end{bmatrix} = \begin{bmatrix} \hat{P}(z) \\ \hat{H}(z) \end{bmatrix} \begin{bmatrix} \hat{P}(z) & \hat{H}(z) \end{bmatrix}. \quad (10)$$

Hence, it suffices to prove the above relation, but it follows readily by applying essentially the same arguments as in [8]. This completes the proof.

In what follows, $\hat{P}(z)$ and $\hat{H}(z)$ given by (8) will be called the doubly sensitivity-preserving (DSP) discretized plant and the DSP discretized hold, respectively, for reasons to become clear shortly. Regarding these $\hat{P}(z)$ and $\hat{H}(z)$, note that $\hat{H}(z)$ is in fact a constant matrix independent of $z$ (see (8)), and that $\hat{P}\hat{H} = P_d$ by (4) and the $(1,2)$-component of (10). From this observation, let us first introduce the system shown in Fig. 4, which is equivalent to that in Fig. 2. We then see that the system $\hat{\Sigma}$ shown in Fig. 3, which we mentioned before, is only slightly modified from the naively discretized system in Fig. 4.

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\(^1\) In the following, we omit $j\varphi$ and $e^{j\varphi}$ when no confusion occurs.
and looks more similar to that in Fig. 1 with respect to the locations of the external inputs. This rearrangement of the naively discretized system \( \Sigma_d \) in terms of the DSP discretized plant \( \hat{P} \) and hold \( \hat{H} \), including the modified treatment of the external inputs, is a very important idea that plays the key role in the overall discussions of the present paper. In the following arguments, the discrete-time system \( \hat{\Sigma} \) shown in Fig. 3 is referred to as the DSP discretized system.

In this section, we begin by establishing that the norm of the sensitivity (respectively, complementary sensitivity) of the sampled-data system \( \Sigma \) at each angular frequency can be computed as that of the DSP discretized system \( \hat{\Sigma} \); this property clearly validates the term DSP. Here, the sensitivity function \( \hat{S} \) (respectively, the complementary sensitivity function \( \hat{T} \)) of \( \hat{\Sigma} \) is defined precisely as the transfer matrix from \( \tau \) to \( \hat{\tau} \) (respectively, \( \tau \); they are given respectively by

\[
\hat{S} = I - \hat{H}C_d(I + \hat{P}\hat{H}C_d)^{-1}\hat{P} = I - \hat{H}K_d\hat{P}, \tag{11}
\]

\[
\hat{T} = \hat{H}C_d(I + \hat{P}\hat{H}C_d)^{-1}\hat{P} = \hat{H}K_d\hat{P}, \tag{12}
\]

where \( K_d \) is given by (3).

**Theorem 1** The DSP discretized system \( \hat{\Sigma} \) is stable if and only if the original sampled-data system \( \Sigma \) is. Furthermore, if \( P(\tau) \) and \( \hat{H}(\tau) \), given by (8) satisfy the condition

\[
\begin{bmatrix} \hat{P} & \hat{H} \end{bmatrix} \text{ is not of full row rank for any } \varphi \in \left(-\frac{\pi}{\tau}, \frac{\pi}{\tau}\right), \tag{13}
\]

then

\[
\|\hat{S}(e^{j\varphi})\| = \|\hat{S}(j\varphi)\| \quad \varphi \in \left(-\frac{\pi}{\tau}, \frac{\pi}{\tau}\right). \tag{14}
\]

\[
\|\hat{T}(e^{j\varphi})\| = \|\hat{T}(j\varphi)\| \quad \varphi \in \left(-\frac{\pi}{\tau}, \frac{\pi}{\tau}\right). \tag{15}
\]

**Remark 1** In [9], a finite-dimensional computation method for \( \|\hat{S}(j\varphi)\| \) and \( \|\hat{T}(j\varphi)\| \) was established, but it involves no viewpoint of “an equivalent discrete-time system” as in the DSP discretized system \( \hat{\Sigma} \). Although such a viewpoint has been provided in other existing method [8] as long as the computation of \( \|\hat{S}(j\varphi)\| \) is concerned, no such viewpoint has been provided in the existing methods, e.g., [9]–[11], that can be applied to the computation of \( \|\hat{S}(j\varphi)\| \). In contrast, Theorem 1 clearly establishes that the (complementary) sensitivity of the sampled-data system \( \Sigma \) can be studied through an “equivalent system” (i.e., the DSP discretized system \( \hat{\Sigma} \)), which is common for the treatment of the sensitivity and that for the complementary sensitivity.

**Proof of Theorem 1:** It is well known that \( \Sigma \) is stable if and only if \( \Sigma_d \) is. Hence, the first assertion is obvious since \( P_d = \hat{P}\hat{H} \). Also, the proof of (15) is essentially the same as the arguments in [8]; it follows readily by applying the equalities about the (1, 1) and (2, 2)-components of \( (9) \) to (2). Hence, the details are omitted. Therefore, it suffices to show that (14) holds under the assumption (13).

From (1), we obtain \( \hat{S} \hat{S} = I + X \) for each \( \varphi \in (-\pi/\tau, \pi/\tau) \), where

\[
X = \frac{1}{\tau} \begin{bmatrix} \hat{P}^* & \hat{H} \end{bmatrix} \begin{bmatrix} K_d^* \cdot \frac{1}{\tau} \hat{H}^* H \cdot K_d & -K_d^* \cdot \frac{1}{\tau} \hat{H}^* H \cdot K_d \cdot 0 \end{bmatrix} \begin{bmatrix} \hat{P} \end{bmatrix} \hat{H}^*. \tag{16}
\]

Since \( X \) is compact (in particular, finite-rank), we have \( \|\hat{S}\|^2 = \max(\lambda_{\max}(\hat{S}^* \hat{S}), 1) = 1 + \max(\lambda_{\max}(X), 0) \). On the other hand, from (11), we obtain \( \hat{S} \hat{S} = I + \bar{X} \) for each \( \varphi \in (-\pi/\tau, \pi/\tau) \), where

\[
\bar{X} = \begin{bmatrix} \hat{P}^* & \hat{H} \end{bmatrix} \begin{bmatrix} K_d^* \cdot \frac{1}{\tau} \hat{H}^* H \cdot K_d & -K_d^* \cdot \frac{1}{\tau} \hat{H}^* H \cdot K_d \cdot 0 \end{bmatrix} \begin{bmatrix} \hat{P} \end{bmatrix} \hat{H}^*. \tag{16}
\]

Thus, we obtain, \( \|\hat{S}\|^2 = \max(\lambda_{\max}(\hat{S}^* \hat{S}), 1) = 1 + \max(\lambda_{\max}(\bar{X}), 0) \). Hence, to establish (14), it is enough to show that \( \max(\lambda_{\max}(\bar{X}), 0) = \lambda_{\max}(X) \) under the assumption (13). To show this, we prove the following three equalities in order:

\[
\max(\lambda_{\max}(\bar{X}), 0) = \max\left\{ \lambda_{\max}\left( \begin{bmatrix} \hat{P}^* & \hat{H} \end{bmatrix} \begin{bmatrix} K_d^* \cdot \frac{1}{\tau} \hat{H}^* H \cdot K_d & -K_d^* \cdot \frac{1}{\tau} \hat{H}^* H \cdot K_d \cdot 0 \end{bmatrix} \begin{bmatrix} \hat{P} \end{bmatrix} \hat{H}^* \right), 0 \right\},
\]

\[
\max\left\{ \lambda_{\max}(\hat{P}^* \hat{H} \cdot \hat{H}^* \hat{P}), \lambda_{\max}(\hat{P}^* \hat{H} \cdot \hat{H}^* \hat{P}) \right\} = \max\left\{ \lambda_{\max}(\hat{P}^* \hat{H} \cdot \hat{H}^* \hat{P}), \lambda_{\max}(\hat{P}^* \hat{H} \cdot \hat{H}^* \hat{P}) \right\} = \lambda_{\max}(X), \tag{17}
\]

The second equality of (17) is a direct consequence of (9), so that it is enough to show the first and third equalities; they both rely on the well-known fact that when \( XY \) and \( YX \) are both well-defined operators or matrices, the spectrum of \( XY \) can possibly be a larger set than that of \( YX \) (or vice versa) only by including just one additional point that can exist at nowhere else than the origin. The first equality of (17) follows immediately from this fact. Similarly, the third equality also follows from this fact if we note that the last quantity in (17) is nonnegative under the assumption (13) since the matrix \( \hat{X} \) given by (16) has an eigenvalue at the origin. Hence, the proof is completed.

**Remark 2** If \( \begin{bmatrix} \hat{P} & \hat{H} \end{bmatrix} \) is of full row rank for some \( \varphi \), we can modify \( \hat{P} \) and \( \hat{H} \) into \( \begin{bmatrix} \hat{P} & 0 \end{bmatrix} \) and \( \begin{bmatrix} \hat{H} & 0 \end{bmatrix} \) within the standing constraint on \( \hat{P} \) and \( \hat{H} \) given by (10). Hence, the condition (13) is not restrictive, but is quite important to make the last quantity in (17) as it is, rather than making it to be \( \max(\lambda_{\max}(X), 0) \). In other words, without the condition (13), we would have to introduce such an “additional max-operation,” which would then prevent us from associating the computation of \( \|\hat{S}(j\varphi)\| \) directly with the DSP discretized system \( \hat{\Sigma} \). Regarding such a max-operation, we give some further remark on the arguments in [9]. Although no such max-operation can be found in [9], this is due to a flaw in the arguments therein. More precisely, the equation (19) of [9] can be validated only under the nonsingularity assumption of the matrix \( M \) therein, but such an assumption is not actually made nor is it satisfied automatically. Since the arguments in [9] are intrinsically irrelevant to such a rank condition as (13) in the present paper, correcting the flaw leads to the same additional max-operation with 0 also in the arguments therein.

**Remark 3** When \( (A, B) \) is controllable, \( \hat{B} \) has full row rank. Hence, when \( C \) is of full row rank, it follows that \( \hat{P}(\tau) \) becomes right-invertible except at a point at infinity, i.e., \( P(\tau) \) has no finite zeros. Similarly, when \( (\hat{C}, \hat{A}_H) \) is observable, \( \hat{C}_H \) has full column rank, and thus if \( B_H \) is of full column rank, then \( \hat{H} \) is left-invertible; since \( \hat{H} \) is independent of \( \varphi \), it means that \( \hat{H} \)
has no zeros. If both $P$ and $\mathcal{H}$ are scalar systems satisfying all the assumptions mentioned above, $\tilde{P}(z)$ and $\tilde{H}$ become a row vector and a column vector, respectively.

By Theorem 1, the DSP discretized system $\tilde{\Sigma}$ shown in Fig. 3 is equivalent to the sampled-data system $\Sigma$ in Fig. 1 when we consider the SR/CSR problems. In this sense, the DSP discretized system $\tilde{\Sigma}$ is much more important than the naively discretized system $\Sigma_d$, which does not take the intersample behavior into account. However, we can easily see that the only difference between $\tilde{\Sigma}$ and $\Sigma_d$ lies in the evaluation points for sensitivity and complementary sensitivity functions. This might suggest that there could be a chance that even the naively discretized system $\Sigma_d$ can be related somehow to the SR and CSR problems of the original sampled-data system $\Sigma$. This is indeed the case, and the discussions in the following part of the present paper are devoted to clarifying such a relationship.

4. Relationship between the SR/CSR Problems of the Sampled-Data System $\Sigma$ and Those of the Naively Discretized System $\Sigma_d$

In this section, we establish that the SR/CSR problems of the sampled-data system $\Sigma$ can be related to some sensitivity reduction problems of the naively discretized system $\Sigma_d$ with some frequency-dependent weights. Even though the discussions on such relationships correspond simply to clarifying the effects caused by the slightly different locations of the external inputs $\check{r}$ and $r_d$ in Figs. 3 and 4, we believe that it is far beyond triviality to derive an explicit relationship, mainly because $\tilde{H}$ is not a scalar system in general. Hence, we employ a coprime factorization approach in this section.

We first introduce in Sec. 4.1 the coprime factorization arguments on the DSP discretized system $\tilde{\Sigma}$ associated directly with the sampled-data system $\Sigma$, as well as on the naively discretized system $\Sigma_d$. On the basis of these coprime factorizations, Sec. 4.2 establishes a link between the CSR problem of the sampled-data system $\Sigma$ and that of the naively discretized system $\Sigma_d$. This is accomplished by introducing a key notion called the aliasing factor, and its role in the study of the performance limitation about the CSR problem of the sampled-data system $\Sigma$ is also discussed there. Then, in Sec. 4.3, we study the SR problem of the sampled-data system $\Sigma$ and establish a similar link to some sort of mixed sensitivity reduction problem of the naively discretized system $\Sigma_d$.

4.1 Coprime Factorization Treatment of $\Sigma$ through $\tilde{\Sigma}$ and That of $\Sigma_d$

In the following, we use $\lambda := 1/z$ instead of $z$ when we describe transfer functions, and consider the SR and CSR problems using the coprime factorization on $\mathbb{R}$, which denotes the set of rational functions analytic on $\mathbb{D}$, the closed unit disc. Here, we use the notation $\mathbb{R}$, also for the set of matrices whose every entry belongs to $\mathbb{R}$.

Here, we review the definitions of inner and outer on $\mathbb{R}$, and related issues [12],[13]. A matrix $G \in \mathbb{R}$ is inner if $G^*G = I$. A matrix $G \in \mathbb{R}$ is outer if $G$ has full row rank for every $\lambda \in \mathbb{D}$, the open unit disc. It is well known that every $G \in \mathbb{R}$ has an inner-outer factorization $G = G^*G'$ where $G^*$ and $G'$ are inner and outer, respectively. A matrix $G \in \mathbb{R}$ is called if it has an inverse in $\mathbb{R}$. A matrix $G$ is said to be co-inner or co-outer if $G^T$ is inner or outer, respectively. A co-inner-outer factorization has the form $G = G^*G'$ where $G^*$ and $G'$ are co-inner and co-outer, respectively. An inner-outer factorization of $G^T$ yields a co-inner-outer factorization of $G$.

In what follows, we set the following assumptions:

**Assumption 1**

(i) $P$ and $\mathcal{H}$ are nonzero scalar systems.

(ii) $P$ is controllable and observable.

(iii) $(C_H,A_H)$ is observable.

(iv) The sampling period $\tau$ is selected such that the discrete-time plant $P_d$ is controllable and observable.

(v) $P$ does not have poles on the imaginary axis.

(vi) $P_d$ does not have zeros on the unit circle.

Next, we express the coprime factorization of $\tilde{P}$ on $\mathbb{R}$ as

$$\tilde{P} = NP/d,$$

where $d$ is a scalar function $^2$, and $NP$ is a row vector (see Remark 3). In this case, from the assumptions (i)–(iv) above,

$$P_d = n/d, \quad n = NP\tilde{H}$$

is a coprime factorization of $P_d (= \tilde{P}\tilde{H})$. Using the solutions $x, y \in \mathbb{R}$ of the Bezout identity

$$nx + dy = 1,$$

it is shown in [12] that the set of all stabilizing controllers $C_d$, denoted by $C$, is given by

$$C = \left\{ C_d = \frac{x + dq}{y - nq} : q \in \mathbb{R}, \quad y - nq \neq 0 \right\}.$$

In what follows, we arbitrarily fix the solutions $x$ and $y$. From (21) together with (3), (11) and (12), we obtain the following equations for the DSP discretized system $\tilde{\Sigma}$.

$$\tilde{S} = I - \tilde{H}(x + dq)NP,$$

$$\tilde{T} = \tilde{H}(x + dq)NP.$$

On the other hand, concerning the naively discretized system $\Sigma_d$, we obtain the following equations from (3), (6) and (7).

$$S_d = 1 - NP\tilde{H}(x + dq) = d(y - nq),$$

$$T_d = NP\tilde{H}(x + dq) = n(x + dq).$$

4.2 Relationship between the CSR Problems of $\Sigma$ and $\Sigma_d$

In this subsection, we clarify that there exists a clear relationship between two seemingly quite irrelevant problems, i.e., the CSR problem of the sampled-data system $\Sigma$ with the intersample behavior completely taken into account, and that of the naively discretized system $\Sigma_d$, which is inherently free from the concept of intersample behavior. This is accomplished by noting the equivalence of $\Sigma$ and the DSP discretized system $\tilde{\Sigma}$ with respect to the CSR problem, and introducing a key notion called the aliasing factor $\psi$, which somehow describes the frequency-dependent effects of the difference in the locations of the external inputs in $\tilde{\Sigma}$ and $\Sigma_d$. The significance of the aliasing factor is also demonstrated through a further study on a relationship in the performance limitation associated with the CSR problems of $\Sigma$ and $\Sigma_d$.

$^2$ We use small letters for scalar functions.
4.2.1 The aliasing factor \( \psi \)

We first introduce the inner-outer factorization of \( \hat{H} \) and the co-inner-outer factorization of \( N_p \) given respectively by

\[
\hat{H} = H' \cdot \hat{h}' = n^p_s \cdot N_s^{n}, \quad (26)
\]

where \( H' \) and \( N_s^n \) are a column vector and a row vector, respectively. Concerning the reason that \( \hat{h}' \) and \( n^p_s \) can be taken to be scalars, see Fact 17 on p. 165 of [12] for details. Note that \( \hat{h}' \) and \( n^p_s \) are units on \( \mathbb{R} \). (see Remark 3). From (23) and (26), the \( H^n \) norm of the complementary sensitivity \( \hat{T} \) is given as follows:

\[
\|\hat{T}\|_\infty = \|\hat{r}'(x + dq)n^p_s\|_\infty = \|n^p_s \hat{r}'(x + dq)\|_\infty \quad (27)
\]

From (27), together with (25), we conclude that the \( H^n \) norm of the complementary sensitivity of the sampled-data system \( \Sigma \) and that of the naively discretized system \( \Sigma_d \) are related by

\[
\|\hat{T}\|_\infty = \left\| \frac{n^p_s \hat{r}'(x + dq)}{n^p_s} T_{o_d} \right\|_\infty = \left\| \frac{n^p_s \hat{r}'(x + dq)}{n^p_s} T_d \right\|_\infty \quad (28)
\]

where \( n' \) and \( n^n \) correspond to the inner-outer factorization \( n = n' n^n \). In (28), \( n^p_s \hat{r}'(x + dq) \) is an important factor that relates the CSR problem of the sampled-data system \( \Sigma \) and that of the naively discretized system \( \Sigma_d \). In other words, this function can be regarded as the frequency-dependent factor with which the influence of aliasing, ignored under the treatment of the sampled-data system \( \Sigma \) as the naively discretized system \( \Sigma_d \), can be recovered in the treatment through \( \Sigma_d \). This observation provides us with a very important viewpoint in the following study of the CSR (or SR) problem of the sampled-data system \( \Sigma \), as well as its performance limitation. We define the inverse of this function as the aliasing factor \(^1\) \( \psi \), which is denoted by \( \psi \):

\[
\psi := \frac{n^n}{n^p_s \hat{r}'}. \quad (29)
\]

Summarizing the above, we obtain the following theorem concerning the relationship between the CSR problem of the sampled-data system \( \Sigma \) and that of the naively discretized system \( \Sigma_d \).

Theorem 2 Concerning the CSR problem, the following two conditions are equivalent:

(i) \( C_d \) is a stabilizing controller for \( \Sigma \) s.t. \( \|\hat{T}\|_\infty < \gamma \).

(ii) \( C_d \) is a stabilizing controller for \( \Sigma_d \) s.t. \( \|\hat{r}' T^d\|_\infty < \gamma \).

To clarify some further relationship between the CSR problems of \( \Sigma \) and \( \Sigma_d \), it is quite important to clarify the properties of the aliasing factor \( \psi \). The following theorem gives an important answer to such an issue.

Theorem 3 The aliasing factor \( \psi \) is a unit such that \( |\psi(\lambda)| \leq 1 \) for \( |\lambda| \leq 1 \), i.e., \( \|\hat{T}\|_\infty \leq 1 \).

Proof of Theorem 3: \( \psi \) is outer because \( n^p_s \) and \( \hat{h}' \) are units (recall Remark 3). Since the outer function \( n^n \) is also a unit by

\(^1\) The reason for defining the aliasing factor as the inverse of the function \( n^p_s \hat{r}' n^n \) is that it is helpful for deriving the analytic properties of the aliasing factor.

From this together with the definition of the aliasing factor \( \psi \), we obtain

\[
\psi \cdot \psi^* = \left( \frac{\hat{P} \cdot \hat{H}}{\hat{P} \cdot \hat{H}} \right) \cdot \left( \frac{\hat{P} \cdot \hat{H}}{\hat{P} \cdot \hat{H}} \right). \quad (29)
\]

Since \( \psi \) is outer, it is analytic on the closed unit disc. Therefore, from the maximum modulus principle, the maximum value of \( |\psi| \) is attained on the unit circle. From the right-hand side of (29) and Schwarz’s inequality, we obtain \( |\psi| \leq 1 \) on the unit circle (since \( \langle \cdot, \cdot \rangle \) is a positive definite inner product on the unit circle). Therefore, when \( |\lambda| \leq 1 \),

\[
|\psi(\lambda)| \leq \max_\psi |\psi(\lambda)|^* \leq 1.
\]

This completes the proof.

It should be noted from (29) together with (10) that the aliasing factor \( \psi \) is determined by \( P(s) \) and \( \hat{H}(s) \) and does not in fact depend on the specific choice of the factors \( \hat{P} \) and \( \hat{H} \) nor on the specific coprime factorizations in (18) and (26). More precisely, \( \psi \) is determined uniquely up to the multiplication by a constant inner function (i.e., a complex number with modulus 1), since \( \psi \) is outer; this indeed validates us to call it the aliasing factor.

It should be noted that the aliasing factor \( \psi \) is closely related to the fidelity index \( \Phi_d \) introduced in [9], defined only on the unit circle by

\[
\Phi_d(\epsilon) := \frac{\sum_{m=-\infty}^{\infty} |P(j\varphi_m)|^2}{\sum_{m=-\infty}^{\infty} |H(j\varphi_m)|^2}. \quad (30)
\]

From (10) and (29), we see that \( \Phi_d(1) = 1/|\psi^* \psi| \) holds on the unit circle. Therefore, the aliasing factor can be regarded as the analytic extension, from the unit circle to the complex plane, of the inverse of the square root of the fidelity index.

Remark 4 Actually, the fidelity index in [9] was introduced for the feedback system in which the external input is located at the output-side of the plant. On the other hand, as shown in Fig. 1, the external input of our system is located at the input-side of the plant. Therefore, the definition (30) of the fidelity index of our system has been slightly modified from the original one to match the context here.

Even though we considered the \( H^n \) norm in (28) or Theorem 2, it is obvious that we can develop parallel arguments on the frequency response gain, by which we have \( \|\hat{T}\| = \|\hat{r}'\| = \|\hat{r}' T_d\| \). This implies that the result in [9] can also be recovered with the use of the aliasing factor.
4.2.2 Relationship in the performance limitation about Σ and Σd

Next, we consider the relationship between the CSR problem of the sampled-data system Σ and that of the naively discretized system Σd through the property of the aliasing factor ψ.

In the naively discretized system Σd, we have

\[
\|T_d\|_\infty = \|n(x + dq)\|_\infty = \|n' n^\prime (x + d'd' q)\|_\infty = \|n' n^\prime (x + d'q')\|_\infty, 
\]

(31)

where d' and d'' correspond to the inner-outer factorization of d = d' d''. Hence, as is well known [12], the CSR problem of Σd, i.e., the problem of minimizing the \(H^\infty\) norm of \(T_d\), can be reduced essentially to the problem of finding an interpolation function that is analytic on the closed unit disc and attains, at the zeros of d', the same value as \(n' x\), while possessing as small \(H^\infty\) norm as possible.

On the other hand, it follows from Theorem 1 that we have the following relation regarding the sampled-data system Σ:

\[
\|T\|_\infty = \|T_d\|_\infty = \|\hat{T}_d\|_\infty = \|\hat{n}' n^\prime (x + d'q')\|_\infty. 
\]

(32)

Due to the difference between (31) and (32) by the factor \(1/\psi\), we can see that the interpolation function in the CSR problem of the sampled-data system Σ should satisfy the same analyticity constraint but has to attain, at an interpolation point, a value whose magnitude is larger (no smaller) than that for Σd by the factor \(1/\psi\) (≥ 1), since \(|d| ≤ 1\) inside the unit circle by Theorem 3. Hence, in general, the \(H^\infty\) norm of the interpolation function would naturally degrade (i.e., becomes larger) in the case of Σ when compared with the case of Σd. We thus arrive at the following theorem immediately.

**Theorem 4** Concerning the SR problem, the following two conditions are equivalent when \(γ > 1\):

(i) \(C_d\) is a stabilizing controller for Σ s.t. \(\|\hat{S}\|_\infty < γ\).

(ii) \(C_d\) is a stabilizing controller for Σd s.t.

\[
\left\| \frac{S_d - \frac{1}{\sqrt{1-\gamma^2}} (1 - \psi^2)^{1/2} T_d}{\sqrt{1-\gamma^2}} \right\|_\infty < γ. 
\]

In the above theorem, we denote by \((1 - \psi^2)^{1/2}\) an outer function \(\xi\) such that \(\xi \xi^\star = 1 - \psi^2\). As stated in the proof of Theorem 1, we have \(\|\hat{S}\|_\infty = \max |\lambda_{\text{max}}(\hat{S}^\prime S)|\) and thus we always have \(\|\hat{S}\|_\infty ≥ \xi\). Hence, the assumption \(γ > 1\) in the above theorem leads to no loss of generality when we refer to the condition \(\|\hat{S}\|_\infty < γ\). The following part of this subsection is devoted to the proof of the above theorem.

Let us observe that \(\|\hat{S}\|_\infty = \|S\|_\infty\) by Theorem 1, and thus consider the condition

\[
\|\hat{S}\|_\infty = \|I - \hat{H}(x + dq) Np\|_\infty < γ. 
\]

(33)

for a given \(γ > 1\), where we used (22) for the representation of \(\hat{S}\). For \(H^\prime\) in (26), we introduce a complementary inner matrix X such that \(\hat{H} = [H^\prime \ X]\) is square and inner. It is shown in [12] that any non-square inner matrix has a complementary inner matrix. \(\hat{H}\) is called a squared inner matrix for \(H^\prime\). Similarly, we can take Y such that \(\hat{N}^i_p = \left[\begin{array}{c} N^i_p \ Y \end{array}\right]\) is square and co-inner. \(\hat{H}\) is referred to as a squared co-inner matrix for \(N^i_p\).

Using \(\hat{H}\) and \(\hat{N}^i_p\), we factorize \(\hat{H}\) and \(N^i_p\), respectively as follows:

\[
\hat{H} = \hat{H} \left[\begin{array}{c} h^r_p \ 0 \end{array}\right], \quad N^i_p = \left[\begin{array}{c} n^{i^\prime}_p \ 0 \end{array}\right] \hat{N}^i_p. 
\]

(34)

Since \(\hat{H}\) is a square matrix satisfying \((\hat{H})^\star \hat{H} = I\), the equality \((\hat{H})^\star = \hat{H}\) also holds. Similarly, we obtain \((\hat{N}^i_p)^\star \hat{N}^i_p = \hat{N}^i_p (\hat{N}^i_p)^\star = I\). These facts will be often used later.

From (34), the condition (33) can be rewritten as

\[
\left\| (\hat{H})^\star (\hat{N}^i_p)^\star - \left[\begin{array}{c} h^r_p \ 0 \end{array}\right] (n^{i^\prime}_p \ 0) \right\|_\infty < γ. 
\]

(35)

Let V and W be the matrices given by

\[
V = \left[\begin{array}{c} 1 \ 0 \end{array}\right] (\hat{H})^\star (\hat{N}^i_p)^\star - h^r (x + dq) n^{i^\prime}_p, 
\]

\[
W = \left[\begin{array}{c} 0 \ 1 \end{array}\right] (\hat{H})^\star (\hat{N}^i_p)^\star, 
\]

(36)

respectively, where the second equality in (36) is obtained by \(n^{i^\prime}_p \ 0 = N_p (\hat{N}^i_p)^\star\), which is derived from (34). Note that the matrix in the left-hand side of (35) is expressed as \(\left[\begin{array}{c|c} V & W \end{array}\right]\).

Here, we rewrite (35) by using the following fact [12]: when \(γ^2 I - W^* W\) is positive definite on the unit circle, \(\left\| \left[\begin{array}{c} V \\ W \end{array}\right] \right\|_\infty < γ\) is

\footnote{Such an outer exists since \(1 - \psi^2\) ≥ 0 on the unite circle. See Theorem 3 of this paper and Lemma 2 on p. 212 of [12].}
Lemma 2 For $W$ given by (37), $\gamma^2 I - WW^*$ is positive definite on the unit circle if and only if $\gamma > 1$.

Lemma 3 When $\gamma > 1$, we have

$$(\overline{H}^\dagger \overline{N}_P^\dagger) (\gamma^2 I - WW^*)^{-1} \overline{N}_P^\dagger \overline{H} = \left[\begin{array}{c} 1 \\ 0 \end{array}\right]$$ \left[\begin{array}{c} 0 \\ 1 \end{array}\right].$$

Proof of Lemma 2: Since $\gamma > 0$, the positive definiteness of $\gamma^2 I - WW^*$ on the unit circle is equivalent to that of $\gamma^2 I - WW^*$ there. From $N_P^\dagger N_P^\dagger = I$ and $\overline{H}^\dagger \overline{H} = I$, it follows from (37) that $\gamma^2 I - WW^* = (\gamma^2 - 1)I$. Hence, the assertion follows immediately.

Proof of Lemma 3: Since $\overline{H}$ is a squared inner matrix and $\overline{N}_P^\dagger$ is a squared co-inner matrix, we obtain the following from (37):

$$(\overline{H}^\dagger \overline{N}_P^\dagger) (\gamma^2 I - WW^*)^{-1} \overline{N}_P^\dagger \overline{H} = (\gamma^2 I - (\overline{H}^\dagger \overline{N}_P^\dagger) W W^\dagger N_P^\dagger \overline{H})^{-1} = (\gamma^2 I - \left[\begin{array}{c} 0 \\ 1 \end{array}\right] \left[\begin{array}{c} 0 \\ 1 \end{array}\right])^{-1}.$$

Hence, the assertion follows immediately.

Combining the above arguments, we can readily see that when $\gamma > 1$, the condition (35) is equivalent to the following condition:

$$\left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & \gamma^2 I - (\overline{H}^\dagger \overline{N}_P^\dagger) W W^\dagger N_P^\dagger \overline{H} \end{array}\right] < 1.$$

Concluding $h^n N_P X$ in the above, we have the following result.

Lemma 4 $h^n N_P XX' (N_P) (h^n)^\dagger = \frac{\psi(n)}{\psi} (1 - \psi^2) (\frac{n}{\psi})^2$.

Proof of Lemma 4: From $\overline{H}^\dagger (\overline{H}^\dagger)^\dagger = I$, we have $H^\dagger (H')^\dagger + (X')(X) = I$. Hence, it follows that

$h^n N_P XX' (N_P) (h^n)^\dagger = h^n N_P (h^n)^\dagger = h^n N_P H (H')^\dagger (N_P) (h^n)^\dagger$

$= h^n n^\dagger (n^\dagger)' (h^n)^\dagger - (N_P H) (N_P H)^\dagger$

$= \frac{n h^n n^\dagger (n^\dagger)' (h^n)^\dagger}{n'} - n^2 = \frac{n}{\psi} (1 - \psi^2) (\frac{n}{\psi})^2$.

By Lemma 4, we can replace $h^n N_P X$ in (39) by $\frac{\psi}{\psi^2} (1 - \psi^2) (\frac{n}{\psi})^2$. Thus, we obtain the equivalent condition

$$\left[\begin{array}{c} \frac{1}{\psi} d(y - n q) \\ \frac{1}{\sqrt{\gamma^2 - 1}} (1 - \psi^2) (\frac{n}{\psi})^2 \end{array}\right] < 1$$

(40)

to the condition $||S_d|| < \gamma$ provided that $\gamma > 1$. Now, the assertion of Theorem 5 follows immediately from (24) and (25).

From Theorem 5, we obtain the following result, a counterpart to Theorem 4.

Theorem 6 $\inf_{C_{dC}} ||S|| < \inf_{C_{dC}} ||S_d||$.

Proof of Theorem 6: From Theorem 5, we see that for every $C_d \in C$ and any $\gamma > 1$, $||S|| < \gamma$ implies $||S_d|| < \gamma$ because

$$||S|| < \gamma \implies ||S_d|| < \gamma.$$

By this, we have $||S|| < \gamma$ for every $C_d \in C$. This yields the assertion of Theorem 6 immediately.

Theorem 6 says that analyzing the best achievable performance with respect to the SR problem of the sampled-data system $\Sigma$ would lead to a “too optimistic” result if it were carried out in terms of the SR problem of the naively discretized system $\Sigma_d$.

It is obvious that we can develop parallel arguments on the frequency response gain. Indeed, we can show that the frequency response gain of $S$ is given by

$$||S|| = A_e + A_\infty,$$

where

$$A_e = \frac{1}{2} \sqrt{\left(1 - \frac{1}{|\psi|^2}\right) ||T_d||^2 + (||S_d|| + 1)^2},$$

$$A_\infty = \frac{1}{2} \sqrt{\left(1 - \frac{1}{|\psi|^2}\right) ||T_d||^2 + (||S_d|| - 1)^2}.$$
that the best achievable performance in the SR problem and that in the CSR problem coincide with each other. Deriving such a result by working directly on $\Sigma$ would be hard, and this demonstrates another aspect of the importance of the DSP discretized system $\Sigma$.

5.1 Nevanlinna Problem

It is well known that the SR/CSR problems of discrete-time systems are closely related to the Nevanlinna problem, and thus those problems of the sampled-data system $\Sigma$ are also obviously related to this problem through the DSP discretized system $\Sigma$. We hence review some basic results about the Nevanlinna problem in this subsection.

Conforming to the forms that we encounter in the arguments in the following subsection, we state the following “canonical” form of the Nevanlinna problem, which will be used to describe the explicit problems dealt with later.

**Problem 1** Suppose that $\alpha_i$ ($i = 1, \cdots, m$) and $\beta_j$ ($j = 1, \cdots, l$) are distinct complex numbers with modulus less than 1. Determine whether there exists $F \in R^n$ satisfying the conditions $\|F\|_\infty < 1$, $F(\alpha_i) = \Xi_i (i = 1, \cdots, m)$, and $F(\beta_j) = \Upsilon_j (j = 1, \cdots, l)$, where $\Xi_i$ and $\Upsilon_j$ are given complex matrices having a norm less than 1.

Obviously, the optimization problem that finds the infimum of $\gamma$ such that there exists $F \in R^n$ satisfying the conditions $\|F\|_\infty < \gamma$, $F(\alpha_i) = \Xi_i (i = 1, \cdots, m)$, and $F(\beta_j) = \Upsilon_j (j = 1, \cdots, l)$ can be reduced to Problem 1 by scaling. It is well known that the following proposition is useful for studying the above problem.

**Proposition 1** Suppose $\lambda_1, \cdots, \lambda_n$ are complex numbers with modulus less than 1 and $\Lambda_1, \cdots, \Lambda_n$ are complex matrices with norm less than 1. Define the Pick matrix $Q$ as

$$Q = \begin{bmatrix} Q_{11} & \cdots & Q_{1n} \\ \vdots & \ddots & \vdots \\ Q_{n1} & \cdots & Q_{nn} \end{bmatrix}, \quad Q_{ij} = \frac{1}{1 - \lambda_i \lambda_j} \cdot (I - \Lambda_i \Lambda_j).$$

(41)

Then, there exists $F \in R^n$ such that $\|F\|_\infty < 1$ and $F(\lambda_i) = \Lambda_i$ ($i = 1, \cdots, m$) if and only if the matrix $Q$ is positive definite.

5.2 Best Achievable Performance in the SR/CSR Problems of the Sampled-Data System $\Sigma$

This subsection is devoted to the proof of the following result, which says that in the sampled-data system $\Sigma$, the best achievable $H^\infty$ norm in the SR problem is equal to that in the CSR problem.

**Theorem 7** $\inf_{C_{elc}} \|S\|_\infty = \inf_{C_{elc}} \|T\|_\infty$

**Proof:** Suppose that $d(\ell)$ has $m$ distinct unstable roots $\lambda = \alpha_i$ ($i = 1, \cdots, m$), and $n(\ell)$ has $l$ distinct unstable roots $\beta = \beta_j$ ($j = 1, \cdots, l$); i.e., $|\alpha_i| < 1$, $d^\ell(\alpha_i) = \cdots = d^\ell(\alpha_0) = 0$ and $|\beta_j| < 1$, $n^\ell(\beta_j) = \cdots = n^\ell(\beta_0) = 0$. (Note that $P_\ell(1/\ell) = n(\ell)/d(\ell)$ has no poles and zeros on the unit circle because of (v) and (vi) in Assumption 1.)

From (40) and (32), together with the inner-outer factorizations $d = d^*d^*$ and $n = n^*n^*$, $\inf_{C_{elc}} \|S\|_\infty$ and $\inf_{C_{elc}} \|T\|_\infty$ can be restated, respectively, as

$$\inf_{\gamma \in \mathbb{R}} \left\{ \gamma > 0 \left| \frac{d}{\sqrt{\gamma + 1}} - \frac{\phi(\alpha)}{\sqrt{\gamma + 1}} \right| < 1 \right\},$$

and

$$\inf_{\gamma \in \mathbb{R}} \left\{ \gamma > 0 \left| \frac{n}{\gamma \phi(\beta)} + n^*d^* \right| < 1 \right\},$$

where $q = d^*d^*q \in \mathbb{R}^n$ and $\phi$ is an outer function satisfying $\phi(\alpha) = (\gamma + 1)^{-1} - 1$; we have also used the fact that $\phi$ is unit. The existence of $\phi$ is due to Theorem 3. Here, note from (20) that $dy = 0$ and $nx = 1$ for $\lambda$ such that $d(\ell) = 0$ while $dy = 1$ and $nx = 0$ for $\lambda$ such that $n(\ell) = 0$. Hence, it follows that the computation of the best achievable performance $\inf_{C_{elc}} \|S\|_\infty$ reduces to Problem 1 with $\Xi_i = \left[ 0 \frac{\phi(\alpha)}{\sqrt{\gamma + 1}} \right] (i = 1, \cdots, m)$ and $\Upsilon_j = \left[ \frac{1}{\gamma} \right] (j = 1, \cdots, l)$. We refer to this problem as Problem-$S$. In a similar fashion, the computation of the best achievable performance $\inf_{C_{elc}} \|T\|_\infty$ reduces to Problem 1 with $\Xi_i = \left[ 0 \frac{\phi(\alpha)}{\sqrt{\gamma + 1}} \right] (i = 1, \cdots, m)$ and $\Upsilon_j = \left[ \frac{1}{\gamma} \right] (j = 1, \cdots, l)$. We refer to this problem as Problem-$T$.

As far as the computation of the infimum of $\gamma$ is concerned, we may replace $\Xi_i$ and $\Upsilon_j$ in Problem-$S$ by their transpose $[12]$. Using this fact and applying Proposition 1, the computation of the infimum of $\gamma$ for Problem-$S$ reduces to the problem of finding the infimum of $\gamma > 1$ such that the Pick matrix

$$Q_S = \begin{bmatrix} X & Y \\ Y^* & (1 - \gamma^{-2})Z \end{bmatrix}$$

is positive definite, where the $(i, j)$-elements of $X$, $Y$, and $Z$, which we denote, respectively, by $X_i$, $Y_i$, and $Z_i$, are given by

$$X_i = \frac{1}{1 - \alpha_i \alpha_j} \left( 1 - \frac{\phi(\alpha_i)}{\sqrt{\gamma + 1}} \right) (i = 1, \cdots, m),$$

and

$$Y_i = \frac{1}{1 - \beta_i \beta_j} \left( 1 - \frac{\phi(\beta_j)}{\sqrt{\gamma + 1}} \right) (j = 1, \cdots, l),$$

respectively. The positive definiteness of $Q_S$ is obviously equivalent to the positive definiteness of the following matrix $Q'_S$:

$$Q'_S = \begin{bmatrix} (1 - \gamma^{-2})X & Y \\ Y^* & Z \end{bmatrix}.$$

Here, note that the $(i, j)$-element of $(1 - \gamma^{-2})X$ can be represented as

$$X_{ij} = \frac{1}{1 - \gamma^2} \left[ \frac{1}{\gamma} \phi(\alpha_i) \right] \left[ \frac{1}{\gamma} \phi(\alpha_j) \right].$$

Observing that the above matrix has essentially the same form as $Q$ in (41), we see from Proposition 1 that the problem of finding the infimum of $\gamma$ such that $Q'_S$ is positive definite is equivalent to the interpolation problem with $\Xi_i = \left[ \frac{1}{\gamma} \right] (i = 1, \cdots, m)$ and $\Upsilon_j = \left[ 0 \right] (j = 1, \cdots, l)$. Note that each value of $\Xi_i$ or $\Upsilon_j$, to be interpolated in this modified interpolation problem is $\phi$ times that of the corresponding scalar value in the interpolation problem associated with Problem-$T$. However, it follows from the relation $\phi = \frac{1}{\gamma^2} - 1$ that $\phi$ is unit. Therefore, the modified interpolation problem derived above for the computation of $\inf_{C_{elc}} \|S\|_\infty$ is essentially equivalent to that associated with Problem-$T$. Hence Problem-$S$ shares the same infimum as Problem-$T$. This completes the proof.

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Note that $d^*$ and $n^*$ are units because of (v) and (vi) in Assumption 1, respectively.
Remark 5 One might suspect that Theorem 7 could be trivial and shown immediately if we used some sort of symmetry between $\tilde{S}$ and $\tilde{T}$, which is evoked by the relation $\tilde{S} + \tilde{T} = I$. More precisely, if the set $\tilde{S} = \{ \tilde{S} = I - \tilde{H}(x + dq) N_p : q \in \mathbb{R} \}$ were to coincide with $\tilde{T} = \{ \tilde{T} = \tilde{H}(x + dq) N_p : q \in \mathbb{R} \}$, then the idea would be indeed correct, but these sets do not in fact coincide for the following reason. Each $\tilde{T} \in \tilde{T}$ has an eigenvalue at the origin for all angular frequencies since $\tilde{H}$ is a column vector and $P$ is a row vector. On the other hand, $\tilde{S} \in \tilde{S}$ has an eigenvalue at the origin if and only if $\tilde{T} = \tilde{H}(x + dq) N_p$ has an eigenvalue at 1, or equivalently, $\tilde{T}_d = N_p \tilde{H}(x + dq) = 1$ by (25) (and thus $S_d = 0$). Therefore, $\tilde{S} \in \tilde{S}$ has an eigenvalue at the origin only at the angular frequencies such that $S_d = 0$. Thus, we have $\tilde{S} \neq \tilde{T}$.

6. Conclusion

We have studied some aspects of the sensitivity reduction (SR) and complementary sensitivity reduction (CSR) problems of sampled-data systems in this paper. We first showed that these problems can be reduced to equivalent discrete-time problems by introducing the doubly sensitivity-preserving (DSP) discretized system $\Sigma$. Through the coprime factorization treatment of $\Sigma$, we further introduced an important function called the aliasing factor. We then showed that the conventional ‘hold equivalent’ discretized system $\Sigma_d$ (called the naively discretized system) can also be used for the SR/CSR problems of the sampled-data system $\Sigma$ provided that appropriate frequency-dependent weights constructed from the aliasing factor are applied on $\Sigma_d$. This should be quite interesting since $\Sigma_d$ has generally been believed to be useless for the study of the SR/CSR problems of the sampled-data system $\Sigma$ when the intersample behavior of $\Sigma$ is completely ignored in $\Sigma_d$. We then showed that some relation between the best achievable performance in the SR (respectively, CSR) problem of $\Sigma$ (and thus $\Sigma$) and that of $\Sigma_d$ can be proved through clarifying an analytic property of the aliasing factor. An interesting property between the best achievable performance about the SR problem of $\Sigma$ and that about the CSR problem of $\Sigma$ was also provided.

We finally remark that similar results can be derived also when the (complementary) sensitivity defined on the input side of the plant is dealt with, as opposed to the treatment in this paper, which is about the (complementary) sensitivity defined on the output side. We hope that the study developed in this paper could provide a fundamental basis for a further study on the performance limitation of sampled-data systems, in addition to such an existing study on the $H^2$ tracking performance of sampled-data systems studied in [5].

References


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