Characterization of Finite Frequency Properties Using Quadratic Differential Forms

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Abstract: Many of practical design specifications are provided by finite frequency properties described by inequalities over restricted finite frequency intervals. A quadratic differential form (QDF) is a useful algebraic tool to characterize energy and power functions when we consider dissipation theory based on the behavioral approach. In this paper, we investigate time domain characterizations of the finite frequency domain inequalities (FFDIs) using QDFs. QDFs allow us to derive a clear characterization of the FFDIs using some inequality in terms of them as a main result. This characterization leads to a physical interpretation in terms of the dissipation inequality with the compensating rate which guarantees dissipativity of a behavior with some rate constraints. Such an interpretation has not been clarified by the previous studies of finite frequency properties. The aforementioned characterization yields an LMI condition whose solvability is equivalent to the FFDIs. This can be regarded as the finite frequency KYP lemma in the behavioral framework.

Key Words: finite frequency property, dissipation theory, behavioral system theory, quadratic differential forms, linear matrix inequality.

1. Introduction

Many of practical design specifications are provided by sets of finite frequency properties which are expressed as inequalities over restricted finite frequency intervals. Hence, the properties play an important role for dynamical system design including plant and controller design integration.

The previous works on characterizations of finite frequency properties are as follows. Iwasaki et al. [1], [2] derived a linear matrix inequality (LMI) characterization for the finite frequency properties, which is a generalized Kalman-Yakubovič-Popov (KYP) lemma. Based on the lemma, a time domain characterization was derived in terms of the inequality for all inputs satisfying the state-space equation and the matrix valued integral quadratic constraint (IQC) for asymptotically stable state-space systems [3]. However, their physical interpretation was not fully satisfactory, when we consider the interaction between supplied power and internal energy of a system. In addition, their characterization was not essential from the view point of dissipation theory, since the characterization was derived through the generalized KYP lemma. For such reasons, it has been desired to characterize the finite frequency properties from the dissipativity viewpoints directly.

Dissipativity is one of the most important properties when we analyze a dynamical system from the energy and power interaction with its outside environment. This interaction is expressed by an inequality called the dissipation inequality. It may be important to consider a dissipativity analysis in frequency domains. This can be verified by the following facts. It is well-known that dissipativity can be equivalently transformed to an inequality over the imaginary axis [4]. Moreover, a stability condition for a feedback system is given in terms of integrals over entire frequencies, called IQC [5]. This paper clarifies how the constraint on the frequency variable appears in the dissipation inequality.

A quadratic differential form (QDF) is a useful algebraic tool to characterize energy and power functions in the dissipation theory based on the behavioral approach [6], because it has a one-to-one correspondence to a two-variable polynomial matrix. Since the behavioral approach is a theoretic framework which does not assume an input-output relationship in advance, we can naturally analyze and design a system described by a nonproper transfer function. Based on QDFs, Willems and Trentelman [7] has proved that a dissipativity of a behavioral system is equivalent to a certain frequency domain inequalities on the entire frequency range. This equivalence is characterized by the dissipation inequality using QDFs. This also leads to an equivalent LMI characterization of the inequalities [8]. However, neither time domain characterization nor LMI characterization of the finite frequency properties has not been derived in the behavioral framework.

In this paper, we consider a characterization of finite frequency properties in the framework of dissipation theory. As a main result, we derive a characterization of the FFDIs in terms of the dissipation inequality described by QDFs. This characterization allows us to understand the significance of the properties directly and yields an equivalent LMI characterization as a natural result of the characterization using the inequality. Figure 1 illustrates a series of these results comparing with the previous works [1], [3]. Theorem 1 and Proposition 3 (black arrows) are the main result of this paper, which is quite a new result. On the other hand, Theorem 2 (gray arrow) derives a relationship with [1].

The organization of the paper is as follows. In Section 2, we review some basic definitions and results about the behavioral...
system theory, QDFs and dissipation theory based on QDFs. The problem formulation is provided in Section 3. In Section 4, we derive a characterization of the finite frequency properties based on the dissipation inequality as a main result. In this result, we characterize the dissipativity properties in terms of some behaviors. Based on the characterization, we give a finite frequency KYP lemma for a numerical checking of the finite frequency properties in Section 5.

The set of $p \times q$ real and complex matrices are denoted by $\mathbb{R}^{p \times q}$ and $\mathbb{C}^{p \times q}$, respectively. We also denote $\mathbb{S}^{p \times q}$ and $\mathbb{H}^{p \times q}$ as the set of $p \times q$ real symmetric and Hermitian matrices, respectively. We denote $\mathbb{R}^{p \times q}[\xi]$ and $\mathbb{R}^{p \times q}[\xi, \eta]$ as the set of $p \times q$ one- and two-variable polynomial matrices, respectively. The set of $p \times q$ complex coefficient one- and two-variable polynomial matrices are denoted by $\mathbb{C}^{p \times q}[\xi]$ and $\mathbb{C}^{p \times q}[\xi, \eta]$, respectively. We denote the set of $q \times q$ Hermitian two-variable polynomial matrices in the indeterminates $\xi$ and $\eta$ by $\mathbb{H}^{q \times q}[\xi, \eta]$. We call $\Phi(\xi, \eta)$ Hermitian if $\Phi(\xi, \eta)^\dagger = \Phi(\eta, \xi)$ holds.

We denote $\mathbb{W}$ as the set of maps from $\mathbb{T}$ to $\mathbb{W}$. Define $C^\infty(\mathbb{R}, \mathbb{V})$ as the set of infinitely differentiable functions from $\mathbb{R}$ to the vector space $\mathbb{V}$. We also define $\mathcal{D}_w^\infty(\mathbb{R}, \mathbb{V}) := \{ \ell \in C^\infty(\mathbb{R}, \mathbb{V}) \mid \ell \text{ has a compact support} \}$. Let $L_2(\mathbb{C}, \mathbb{V})$ denote the set of $L_2$ functions from $\mathbb{C}$ to $\mathbb{V}$.

Finally, the row dimension of the matrix $A$ is denoted by $\text{rowdim}(A)$. We define the range of polynomial matrix $R(\xi)$ and constant matrix $R(\lambda)$ are denoted by $\text{rank} R$ and $\text{rank} R(\lambda)$, respectively. We denote the matrix $[A_1^\top A_2^\top \cdots A_n^\top]$ by col$(A_1, A_2, \cdots, A_n)$. We define diag$(A_1, A_2, \cdots, A_n)$ as the $q \times q$ (block) diagonal matrix with (block) diagonal elements $(A_1 , A_2, \cdots, A_n)$. We also define $\Phi_{\mathbb{H}}(A) = \frac{1}{2}(A + A^\top)$. We denote $Z_t \in \mathbb{R}^{(1+k)p \times q}[\xi]$ ($t = 0, 1, \cdots$) as the polynomial matrix constructed by stacking the polynomial matrices $[I_q, \xi I_q, \cdots, \xi^p I_q]$, i.e. $Z(\xi) = \text{col}(I_q, \xi I_q, \cdots, \xi^p I_q)$.

2. Preliminaries

In this section, we will review the basic definitions and results from the behavioral system and dissipation theory, which are taken from the references [6],[7],[9].

2.1 Linear Continuous-Time Systems and QDFs

In the behavioral system theory, a dynamical system is defined as a triple $\Sigma = (\mathbb{T}, \mathbb{W}, \mathbb{B})$, where $\mathbb{T}$ is the time axis, and $\mathbb{W}$ is the signal space in which the trajectories take their values on. The behavior $\mathbb{B} \subseteq \mathbb{W}^\mathbb{T}$ is the set of all possible trajectories.

In this paper, we will consider a linear time-invariant continuous-time system with $\mathbb{T} = \mathbb{R}$ and $\mathbb{W} = \mathbb{C}^q$. Such a $\Sigma$ is represented by a system of linear differential-algebraic equation as

$$R_0 w + R_1 \frac{d}{dt} w + \cdots + R_L \frac{d^L}{dt^L} w = 0,$$

(1)

where $R_i \in \mathbb{C}^{p \times q}$ ($i = 0, 1, \cdots, L$) and $L \geq 0$. The variable $w \in \mathbb{C}^o(\mathbb{R}, \mathbb{C}^q)$ is called the manifest variable. We call the representation (1) a kernel representation of $\mathbb{B}$. A short hand notation for (1) is

$$R \left( \frac{d}{dt} \right) w = 0,$$

(2)

where $R \in \mathbb{C}^{p \times q}[\xi]$ is given by

$$R(\xi) := R_0 + R_1 \xi + \cdots + R_L \xi^L.$$  

(3)

Then, the behavior is defined as

$$\mathbb{B} := \{ w \in \mathbb{C}^o(\mathbb{R}, \mathbb{C}^o) \mid R \left( \frac{d}{dt} \right) w = 0 \}.$$  

(4)

The representation (2) is said to be a minimal representation of $\mathbb{B}$ if $\text{rowdim} R \leq \text{rowdim} R'$ holds for any other $R' \in \mathbb{C}^{p \times q}[\xi]$ which induces a kernel representation of $\mathbb{B}$.

The behavior $\mathbb{B}$ is called controllable, if for any trajectories $w_1, w_2 \in \mathbb{B}$, there exists a time $T \geq 0$ and a trajectory $w \in \mathbb{B}$ such that $w(t) = w_1(t)$ ($t \leq 0$) and $w(t) = w_2(t-T)$ ($t \geq T$). The behavior $\mathbb{B}$ is controllable if and only if $\text{rank} R(\lambda)$ is constant for all $\lambda \in \mathbb{C}^o[6]$. Whenever $\mathbb{B}$ is controllable, it can be described by an image representation

$$w = M \left( \frac{d}{dt} \right) \ell, \ M \in \mathbb{C}^{p \times q}[\xi],$$  

(5)

where the variable $\ell \in \mathbb{C}^o(\mathbb{R}, \mathbb{C}^o)$ is called the latent variable.

Then, $\mathbb{B}$ is given by

$$\mathbb{B} = \{ w \in \mathbb{C}^o(\mathbb{R}, \mathbb{C}^o) \mid \exists \ell \in \mathbb{C}^o(\mathbb{R}, \mathbb{C}^o) \text{ s.t. } (5) \}.$$  

An image representation in (5) is a special case of the latent variable representation of $\mathbb{B}$. The system of differential equations

$$R \left( \frac{d}{dt} \right) w = M \left( \frac{d}{dt} \right) \ell$$

(6)

is said to be a latent variable representation of $\mathbb{B}$. In terms of the representation, $\mathbb{B}$ can be rewritten as

$$\mathbb{B} = \{ w \in \mathbb{C}^o(\mathbb{R}, \mathbb{C}^o) \mid \exists \ell \in \mathbb{C}^o(\mathbb{R}, \mathbb{C}^o) \text{ s.t. } (6) \text{ holds} \}.$$  

An image representation (5) is called observable if $w = M \left( \frac{d}{dt} \right) \ell = 0$ implies $\ell = 0$. The representation (5) is observable if and only if the constant matrix $M(\lambda)$ is of full column rank for all $\lambda \in \mathbb{C}^o[6]$.

If (5) is an observable image representation, there exists a nonsingular permutation matrix $\Pi \in \mathbb{C}^{p \times q}$ satisfying

$$\Pi M(\xi) = \begin{bmatrix} Y(\xi) \\ \mathcal{U}(\xi) \end{bmatrix}, Y \in \mathbb{C}^{poq}[\xi], \mathcal{U} \in \mathbb{C}^{m \times q}[\xi], p+m = q$$

(7)

with $U(\xi)$ nonsingular [6]. Such a partition is called an input-output partition of $M(\xi)$. We can regard $u := U \left( \frac{d}{dt} \right) \ell$ and $y := Y \left( \frac{d}{dt} \right) \ell$ as input and output, respectively. In this case,
corresponding to the above partition, the transfer function \( G \in C^{p\times m}(\xi) \) from \( x \) to \( y \) is defined by
\[
G(\xi) := Y(\xi)U^{-1}(\xi).
\]

We review the definition and some basic results of QDFs [7] which play a central role in this paper.

We first consider a Hermitian two-variable polynomial matrix in \( \mathbb{R}^{p\times q}[\zeta, \eta] \), described by
\[
\Phi(\zeta, \eta) = \sum_{i,j} \sum_{\ell} i^j \Phi_{ij} c^\ell d^\ell.
\]

The degrees of \( \Phi(\zeta, \eta) \) with respect to \( \zeta \) and \( \eta \) are defined as \( \text{deg}_\zeta \Phi = \max_{i,j} i \) and \( \text{deg}_\eta \Phi = \max_{i,j} j \), where \( I \subseteq \mathbb{Z}^2 \) is defined by \( I := \{(i, j) \subseteq \mathbb{Z}^2 | \Phi_{ij} \neq 0 \} \).

The quadratic differential form (QDF) \( Q_\Phi(\ell) \) is a quadratic form of the variable \( \ell \in C^m(\mathbb{R}, C^q) \) and its derivatives, namely
\[
Q_\Phi(\ell) := \sum_{i=0}^K \sum_{j=0}^K d^i \ell d^j \Phi_{ij}.
\]

We assume that \( \mathfrak{B} \) is controllable in this section. Let \( \mathfrak{B} \in \mathbb{R}^{p\times q}[\zeta, \eta] \) be given. Then, a behavior \( \mathfrak{B} \) is called dissipative with respect to the supply rate \( Q_\Phi(\ell) \) if there holds
\[
\frac{d}{dt} Q_\Psi(w) \leq Q_\Phi(w), \quad \forall w \in \mathfrak{B}.
\]

2.2 Dissipation Theory

We give the definition of dissipativity of a behavior.

**Definition 1** [7] Assume that \( \mathfrak{B} \) is controllable. Let \( \Phi \in \mathbb{R}^{p\times q}[\zeta, \eta] \) be given. Then, a behavior \( \mathfrak{B} \) is called dissipative with respect to the supply rate \( Q_\Phi(\ell) \) if there holds
\[
\int_{-\infty}^{+\infty} Q_\Phi(w) dt \geq 0, \quad \forall w \in \mathfrak{B} \cap D^\infty(\mathbb{R}, C^q).
\]

We may think of \( Q_\Phi(w) \) as the power delivered to the behavior \( \mathfrak{B} \). The dissipativity implies that the net flow of energy into the system is non-negative. This shows the system dissipates energy. Hence, due to this dissipation, the rate of increase of the energy stored inside of the system does not exceed the power supplied to it. This interaction between supply, storage, and dissipation is now formalized in Definition 2 and Proposition 1 below.

We give the definitions of storage function and dissipation rate.
this paper and suppose that an observable image representation of is described by (5) for .

Let in (9) be given. Suppose that this induces the supply rate for . Define the frequency domain in the finite interval by

\[ \Omega := \{ \omega \in \mathbb{R} \mid \tau (\omega - \sigma_1) (\omega - \sigma_2) \leq 0 \}, \quad (13) \]

where \( \sigma_1, \sigma_2 \in \mathbb{R}, \sigma_1 < \sigma_2 \) and \( \tau \in \mathbb{Z} \) is either +1 or -1. The set \( \Omega \) for \( \tau = +1 \) represents the middle frequency interval \([\sigma_1, \sigma_2] \), while \( \Omega \) expresses the high frequency domain \((-\infty, \sigma_1] \) and \([\sigma_2, +\infty) \) in the case of \( \tau = -1 \). Moreover, \( \Omega \) becomes the entire real numbers, i.e. \( \Omega = \mathbb{R} \), if we choose \( \sigma_1 = \sigma_2 := 0 \) with \( \tau = -1 \).

Consider the finite frequency property described by the following finite frequency domain inequality (FFDI)

\[ M^*(j\omega) \Phi(j\omega) M(j\omega) \geq 0, \quad \forall \omega \in \Omega. \quad (14) \]

Our goal is to find a characterization of the above FFDI using QDFs. Namely, we want to give clear answers to the following two questions from the viewpoint of dissipativity when we restrict the frequency domain to the finite interval.

(i) What power function appears in the dissipation inequality (10), or equivalently dissipation equality (11), for compensating the restriction of the interval? Moreover, what role this function plays in (10)?

(ii) What additional property of \( \Phi \) with dissipativity is equivalent to the FFDI (14)?

An interpretation of FFDI (14) from the behavioral approach is the following. Consider the QDF \( Q(\omega) \) induced by \( \Phi \in \mathbb{C}^{\infty} \). Fourier transform of \( Q(\omega) \) is computed as

\[ \hat{w}(j\omega)^* \Phi(j\omega) \hat{w}(j\omega) = \hat{\ell}(j\omega)^* M(j\omega)^* \Phi(j\omega) M(j\omega) \hat{\ell}(j\omega), \]

where \( \hat{w} \in L_2(\mathbb{R}, \mathbb{C}^m) \) and \( \hat{\ell} \in L_2(\mathbb{R}, \mathbb{C}^m) \) are Fourier transforms of \( w \in \mathcal{B} \) and \( \ell \in \mathcal{D}^\omega(\mathbb{R}, \mathbb{C}^m) \), respectively. Since \( \ell(t) \) can be taken as an arbitrarily trajectory in \( \mathcal{D}^\omega(\mathbb{R}, \mathbb{C}^m) \), the inequality

\[ \hat{w}(j\omega)^* \Phi(j\omega) \hat{w}(j\omega) \geq 0, \quad \forall w \in \mathcal{B} \cap \mathcal{D}^\omega(\mathbb{R}, \mathbb{C}^m), \quad \omega \in \Omega \]

is equivalent to FFDI (14). We can regard the above inequality imposes a weighted frequency constraint on \( w \) over the restricted frequency domain \( \Omega \). Hence, it expresses the weighted rate limitation on the trajectories contained in \( \mathcal{B} \), although FFDI (14) is described by using \( M(\xi) \).

**Remark 1** In the state-space setting [1],[3], Iwasaki et al. considered the FFDI

\[ \begin{bmatrix} (j\omega A + \Delta)^{-1} B \end{bmatrix}^\ast \Phi_0 \begin{bmatrix} (j\omega A + \Delta)^{-1} B \end{bmatrix} \leq 0, \quad \forall \omega \in \Omega. \quad (15) \]

where \( \Phi_0 \in \mathbb{C}^{(r+m)(r+m)} \), \( A \in \mathbb{C}^{nxn} \), \( B \in \mathbb{C}^{nxm} \) and \((A,B)\) is a controllable pair. An engineering meaning of FFDI (15) was a system property of a restricted frequency domain. FFDI (14) also has the same meaning once after an input-output relationship is determined. However, we can regard FFDI (14) as a generalization of the FFDI (15) to the behavioral approach. These are explained as follows.

Let \( Y \in \mathbb{C}^{\infty} \) and \( U \in \mathbb{C}^{\infty} \) be defined by a right coprime factorization \((E_n - A)^{-1} B = Y(\xi) U^{-1}(\xi) \). Define \( M(\xi) := \text{col}(Y(\xi), U(\xi)) \) and \( \Phi(\xi, \eta) := -\Phi_0 \in \mathbb{C}^{\infty}, \) then (14) is equivalently rewritten by (15). In addition, define \( \Phi_0 \) as

\[ \Phi_0 := \frac{1}{2} \begin{bmatrix} 0_m & I_m & 0_m \end{bmatrix} \Pi, \quad (16) \]

then FFDI (15) falls to the finite frequency positive realness [2]. Thus, FFDI (14) has the same meaning to (15) and can be considered as a generalization of (15) to the behavioral approach.

**4. Characterization of Finite Frequency Properties**

This section derives a characterization of finite frequency properties in terms of a dissipation inequality and an integral of the supply rate using QDFs as a main result.

**4.1 Main Theorem**

We define \( \sigma_- \in \mathbb{R} \) and \( \sigma_+ \in \mathbb{R} \) by

\[ \sigma_- := \frac{\sigma_2 - \sigma_1}{2} \quad \text{and} \quad \sigma_+ := \frac{\sigma_1 + \sigma_2}{2} \quad (17) \]

and the set \( \mathcal{G} \) by

\[ \mathcal{G} := \left\{ \Gamma \in \mathbb{C}^{\infty}[\xi, \eta] \mid \Gamma(\xi, \eta) := \chi(\xi, \eta)^* \Gamma(\xi, \eta) \right\}, \quad \text{for some } \chi \in \mathbb{C}^{\infty}[\xi, \eta] \text{ s.t. (20)} \quad (18) \]

\[ \chi(\xi, \eta) := \begin{bmatrix} \omega & \sigma_2 - j \sigma_+ & -j \sigma_- \end{bmatrix} \quad (19) \]

and \( \tau Q(\omega) \) for \( \tau \in \mathbb{C}^{\infty}(\mathbb{R}, \mathbb{C}^m) \)

where \( \tau \) is equal to either +1 or -1. We see that there holds

\[ \partial \Gamma(\omega) = -\tau (\omega - \sigma_1)(\omega - \sigma_2) \cdot \sigma_0 \Gamma(j\omega) \geq 0 \quad (21) \]

for all \( \omega \in \Omega \), \( \Gamma \in \mathcal{G} \) and \( \chi \in \mathbb{C}^{\infty}[\xi, \eta] \) such that (20).

We have seen from Proposition 1 that the FDI (12) is equivalent to the dissipation inequality (10). As we consider the FDI (12) restricted to \( \Omega \), we can imagine that an analogous relationship to Proposition 1 holds. This is explained as follows.

Assume that there exist two-variable polynomial matrices \( \Psi \in \mathbb{C}^{\infty}[\xi, \eta] \) and \( \Gamma \in \mathcal{G} \) satisfying

\[ \frac{d}{dt} Q(w) = Q(w) - Q(\omega), \quad \forall \omega \in \mathcal{B}. \quad (22) \]

The above inequality corresponds to the dissipation inequality (10), and it is equivalent to the existence of \( \Delta \in \mathbb{C}^{\infty}[\xi, \eta] \) satisfying a two-variable polynomial matrix equation

\[ (\xi + \eta) M(\xi) \Psi(\xi, \eta) M(\eta) = (\xi) \Phi(\xi, \eta) M(\eta) - M(\xi)^* \Delta(\xi, \eta) M(\eta) \]

and \( Q(\omega) \geq 0, \forall \omega \in \mathcal{B} \). Substituting \( \xi = -j\omega \) and \( \eta = j\omega \) into (23), we obtain the FFDI

\[ (\text{FFDI}) M(j\omega)^* \Phi(j\omega) M(j\omega) \geq 0, \forall \omega \in \Omega. \]
from (21). The above inequality guarantees the FFDI (14).

Inequality (22) also gives a necessary condition for the finite frequency property. Thus, we obtain the following main result which equivalently characterizes the property in terms of QDFs. This theorem gives the answer to the questions which we have proposed in Section 3.

**Theorem 1** Assume that $\mathcal{B}$ in (4) is controllable and that $\mathcal{B}$ is represented by an observable image representation (5). Let $\Phi \in \mathbb{H}^{p \times q}[\zeta, \eta]$ be given. Define $\Omega$ by (13) and $\mathcal{G}$ by (18). Then, the following statements (i), (ii) and (iii) are equivalent.

(i) FFDI (14) holds for all $\omega \in \Omega$.

(ii) There exist $\Psi \in \mathbb{H}^{q \times q}[\zeta, \eta]$ and $\Gamma \in \mathcal{G}$ satisfying inequality (22).

(iii) Inequality

$$
\int_{-\infty}^{\infty} Q_\Phi(w) dt \geq 0
$$

holds for all $w \in \mathcal{B}$ satisfying

$$
\tau \int_{-\infty}^{\infty} \text{He}((\dot{z} - j \sigma z)(\dot{z} - j \sigma z)^*) dt \leq 0,
$$

where $z \in \mathcal{D}^q(\mathbb{R}, C^{N+1})$ is defined by

$$
z := Z_N \left( \frac{d}{dt} \right) w
$$

with some nonnegative integer $N \in \mathbb{Z}$.

**Proof** See Appendix B.1 in [10] for the proof. □

This theorem gives the answer to questions which we have proposed in Section 3, which is explained as follows.

The QDF $Q_\Phi(w)$ satisfying (22) is called a compensation rate for $\mathcal{B}$ with respect to the frequency domain $\Omega$. This is the new function which appears in the dissipation inequality (10) and is the answer to the former part of the question (i). We see that $Q_\Phi(w)$ guarantees dissipativity of some behavior related to $\mathcal{B}$ and $\Omega$, which becomes an answer to the former part of the question (i). The detail of this claim is explained in Subsection 4.2. Statement (iii) gives the answer to question (ii), i.e. the rate constraint inequality (25) is the additional property which is opposed to the dissipativity of $\mathcal{B}$.

**Remark 2** It should be noted that the characterization in Theorem 1 is representation-free. Namely, it does not suppose any particular representation of $\mathcal{B}$ such as state-space systems and transfer functions. In this sense, this theorem gives a more general result than the previous work done by Iwasaki et al. [3].

**Remark 3** The equivalence of (i) and (iii) corresponds to the result if we restrict Theorem 3 in [3] to continuous-time systems. The statement (iii) shows that the integral of the power supplied to the system is nonnegative for the manifest variable which varies in the frequency contained in $\Omega$.

**Remark 4** The two-variable polynomial $\chi(\zeta, \eta)$ in (19) is a real coefficient polynomial if $\Omega$ is symmetric about the origin. Two typical examples are low frequency domain $\Omega_{\text{low}} := \{ \omega \in \mathbb{R} \mid |\omega| \leq \sigma \}$ and high frequency domain $\Omega_{\text{high}} := \{ \omega \in \mathbb{R} \mid |\omega| \geq \sigma \}$, where $\sigma \in \mathbb{R}$ is a given scalar satisfying $\sigma \geq 0$. If $M(\xi)$ and $\Phi(\zeta, \eta)$ are all real polynomial matrices, we can restrict $\Psi(\zeta, \eta)$ and $\Gamma(\zeta, \eta)$ in Theorem 1 to real symmetric two-variable polynomial matrices without loss of generality.

The degree of $\Psi(\zeta, \eta)$ and $\Gamma(\zeta, \eta)$ are unknown in Theorem 1. However, by using $\mathcal{B}$-canonical polynomial matrices, we can obtain the upper bounds by the degree of the polynomial matrix which induces a kernel representation of $\mathcal{B}$. This result becomes a preliminary result for the finite frequency KYP lemma in Section 5. See Appendix C for the definition and basic properties of $\mathcal{B}$-canonical polynomial matrices.

We assume that $R \in \mathbb{C}^{p \times q}[\xi]$ in (3) is row reduced [11] in the remainder of this subsection. This assumption does not lose any generality, because there always exists a unimodular polynomial matrix $U \in \mathbb{C}^{p \times q}[\xi]$ satisfying $R_{\text{real}}(\xi) = U(\xi)R(\xi)$, where $R_{\text{real}} \in \mathbb{C}^{p \times q}[\xi]$ is row reduced. It should be noted that $R_{\text{real}}(\xi)$ may be obtained by the command rowred of Polynomial Toolbox [12] for MATLAB. In addition, we set the following degree constraint without loss of generality.

$$
\deg R \geq \deg \xi, \Phi = \deg \phi, \Phi
$$

(27)

If (27) does not hold, i.e. $\deg R < \deg \xi, \Phi = \deg \phi, \Phi$, we can reduce it to (27) by taking $R_{l+1} = R_{l+2} = \cdots = R_K = 0_{p \times q}$. Hence, it is sufficient to prove under the assumption (27).

From Theorem 1 and Lemma 3, we obtain a characterization for the finite frequency property using $\mathcal{B}$-canonical polynomial matrices.

**Proposition 3** Assume that $\mathcal{B}$ in (4) is controllable and that $R \in \mathbb{C}^{p \times q}[\xi]$ is row-reduced. Suppose that $\mathcal{B}$ is represented by an observable image representation (5). Let $\Phi \in \mathbb{H}^{p \times q}[\zeta, \eta]$ be given by (9) and satisfy (27). Define $\Omega$ by (13) and $\mathcal{G}$ by (18). Then, the following statements (i), (ii) and (iii) are equivalent.

(i) FFDI (14) holds for all $\omega \in \Omega$.

(ii) There exist unique $\mathcal{B}$-canonical $\Psi \in \mathbb{H}^{q \times q}[\zeta, \eta]$ and $\Gamma \in \mathcal{G}$ with $\mathcal{B}$-canonical $\Gamma \in \mathbb{H}^{q \times q}[\zeta, \eta]$ satisfying (22).

(iii) Inequality (24) holds for all $w \in \mathcal{B}$ satisfying (25), where $z \in \mathcal{D}^q(\mathbb{R}, C^{N+1})$ is defined by (26) with nonnegative integer such that $N \leq \deg R - 1$.

**Proof** See Appendix B.5 in [10] for the proof. □

Proposition 3 shows that the upper bounds of the degree of $\Psi(\zeta, \eta)$ and $\Gamma(\zeta, \eta)$ are determined by that of $R(\xi)$.

**Remark 5** Consider the case where a kernel representation of $\mathcal{B}$ is described by the state-space equation

$$
\dot{x} = Ax + Bu, A \in \mathbb{C}^{n \times n}, B \in \mathbb{C}^{n \times m},
$$

(28)

where $x \in \mathbb{C}^n(\mathbb{R}, C^n)$ and $u \in \mathbb{C}^m(\mathbb{R}, C^m)$ are the state and input variable of $\mathcal{B}$, respectively. See Appendix A for the definition of state in the behavioral approach. Iwasaki et al. [3] proved that FFDI (14) holds for all $\omega \in \Omega$ if and only if there holds inequality

$$
\int_{-\infty}^{\infty} x^* u \Phi_0 x \phi u dt \leq 0, \Phi_0 \in \mathbb{R}^{(n+m)(n+m)}
$$

(29)
for all $u \in \mathcal{L}_2(\mathbb{R}, \mathbb{C}^m)$ satisfying (28) and
\[
\tau \int_{-\infty}^{\tau} \text{He}((\dot{x} - j\sigma_1 v)(\dot{x} - j\sigma_2 w)) \, dt \leq 0. \tag{30}
\]
We show that Proposition 3 (iii) includes the above characterization in the following.

Define $w := \text{col}(x, u, R(\xi)) := [A - \xi I, B]$ and $\Phi(\xi, \eta) := -\Phi_0 \in \mathbb{E}^{(m\times n)(m\times m)}$. Then, (29) is equivalent to (24), and the integer $N$ in Proposition 3 (iii) becomes $N = 0$ from $\deg R = 1$. We easily see that the minimal state map is induced by a constant matrix $X := [I_n \ 0_{m\times n}]$. Pre- and post-multiplying (25) by $X$ and $X'$, respectively, we get (30). This shows that Proposition 3 (iii) includes the characterization in [3].

4.2 Physical Interpretation

We clarify the physical interpretation of Theorem 1 from the viewpoint of dissipation theory in this subsection.

Define the subbehavior $\mathcal{B}_\Omega \subset \mathcal{B}$ by
\[
\mathcal{B}_\Omega := \left\{ w \in \mathcal{B} \mid \text{w satisfies (25)} \text{ for } z \in D^\infty(\mathbb{R}, \mathbb{C}^{N+1}) \text{ in (26)} \right\}. \tag{31}
\]
Since $Q_1(w)$ can be rewritten by
\[
Q_1(w) = -z^* T z + j\sigma_1 \left( z^* \dot{T} z - z^* \dot{T} z \right) - \sigma_1 \sigma_2 z^* \dot{T} z,
\]
we have
\[
Q_1(w) = \text{tr} \left[ \dot{T} \left[ -z \ddot{z} + j\sigma_1 \left( z \ddot{z} - z \ddot{z} \right) - \sigma_1 \sigma_2 z \dot{z} \right] \right] = \text{tr} \left[ \dot{T} \left[ -\tau \text{He} ((\dot{z} - j\sigma_1 \dot{z}) (\dot{z} - j\sigma_2 \dot{z})) \right] \right],
\]
where $\dot{T} \in H^{N+1}(\mathbb{R}, N+1)$ is the coefficient matrix of $T(\xi, \eta)$ defined in Appendix B. Hence, we can regard $\mathcal{B}_\Omega$ as the set of all trajectories in $\mathcal{B}$ which vary in the frequency contained in $\Omega$. Namely, $\mathcal{B}_\Omega$ has a rate constraint determined by $\Phi(\xi, \eta)$ and $\Omega$.

We can obtain the following corollary from Proposition 1 and Theorem 1, which shows a physical interpretation of Theorem 1 in terms of the dissipation inequality.

Corollary 1 Assume that $\mathcal{B}$ in (4) is controllable and that $\mathcal{B}$ is represented by an observable image representation (5). Let $\Phi \in H^{m\times n}(\xi, \eta)$ be given. Define $\Omega$ by (13), $\mathcal{G}$ by (18) and $\mathcal{B}_\Omega$ by (31). Then, the following statements (i), (ii), and (iii) are equivalent.

(i) FFDI (14) holds for all $\omega \in \Omega$.

(ii) There exists $\Psi \in H^{m\times n}(\xi, \eta)$ satisfying the dissipation inequality
\[
\frac{d}{dt} Q_\Psi(w) \leq Q_\Phi(w), \quad \forall w \in \mathcal{B}_\Omega. \tag{32}
\]

(iii) There exist $\Psi \in H^{m\times n}(\xi, \eta)$, $A \in H^{m\times n}(\xi, \eta)$ and $\Gamma \in \mathcal{G}$ satisfying the dissipation equality
\[
\frac{d}{dt} Q_\Psi(w) = Q_\Phi(w) - Q_{A^\dagger}(w) \tag{33}
\]
and
\[
Q_{A^\dagger}(w) \geq 0\]
for all $w \in \mathcal{B}_\Omega$. This implies that the QDF $Q_{A^\dagger}(w)$ is a dissipation rate for $\mathcal{B}_\Omega$.

(iv) The behavior $\mathcal{B}_\Omega$ is dissipative with respect to the supply rate $Q_\Phi(w)$.

Proof See Appendix B.2 in [10] for the proof. □

Corollary 1 provides us a physical interpretation of Theorem 1. An analogous inequality to (22) appears in the reference [3] as an intermediate result derived from generalized KYP lemma [1] (see the proof of Theorem 3 in [3]). However, an interpretation from dissipativity viewpoint was not provided such as a role of the compensating rate and a class of subbehavior where dissipativity is guaranteed. The following description gives a clear answer to the latter part of the question (ii) we have proposed in Section 3.

It is not difficult to see that $\mathcal{B}$ is not necessarily dissipative with respect to the supply rate $Q_\Phi(w)$ from Proposition 1. However, Corollary 1 (iv) states that, if we concentrate ourselves to the trajectories to those varying in the frequency contained in $\Omega$, then $\mathcal{B}_\Omega$ becomes dissipative with respect to the supply rate $Q_\Phi(w)$. This shows that the compensating rate guarantees dissipativity of the subbehavior which has a constraint on the rate of change. We describe this interpretation after an intuitive example. Such an observation has not been done in the previous work by Iwasaki et al. [3].

Consider the latent variable $\ell_\omega \in C^\infty(\mathbb{R}, \mathbb{C}^m)$ by $\ell_\omega(t) := e^{\omega t}v, v \in \mathbb{C}^m$ for a given $\omega \in \Omega$. We easily get
\[
w(t) = M \left( \frac{d}{dt} e^{\omega t}v = e^{\omega t} M(j\omega)v, \right. \tag{34}
\]
which implies $w \in \mathcal{B}_\Omega$. Since $\tau Q_\Psi(w) \geq 0, \forall w \in \mathcal{B}$, we have
\[
Q_\Psi(w) = -\tau (\omega - \sigma_1)(\omega - \sigma_2) \cdot \tau Q_\Psi(w) \geq 0, \forall w \in \mathcal{B}_\Omega \text{ s.t. (34)}. \tag{35}
\]
From (22) and the above inequality, there holds
\[
Q_\Phi(w) \geq Q_\Psi(w) + \frac{d}{dt} Q_\Psi(w) \geq \frac{d}{dt} Q_\Psi(w), \forall w \in \mathcal{B}_\Omega \text{ s.t. (34)}. \tag{35}
\]
On the other hand, we have $Q_\Psi(w) \leq 0$ for all $w \in \mathcal{B}_\Omega$ such that (34) if $\omega \in \Omega$. Hence, (35) does not always hold, which concludes the intuitive explanation.

We generalize the above intuitive explanation to a physical interpretation in dissipation theory. We can see from Theorem 1 and Corollary 1 that QDF $Q_\Psi(w)$ satisfies the equality
\[
\int_{-\infty}^{\tau} Q_{A^\dagger}(w) \, dt = \int_{\tau}^{\infty} Q_\Psi(w) \, dt, \quad \forall w \in \mathcal{B}_\Omega \cap D^\infty(\mathbb{R}, \mathbb{C}^m).
\]
Also, we observe that
\[
Q_{A^\dagger}(w) = Q_\Psi(w) + Q_\Psi(w) = \text{tr} \left[ \left[ \begin{array}{c|c} \Delta & \ddot{z} \\ \hline \ddot{z} & \tau T \end{array} \right] \cdot \text{He} ((\dot{z} - j\sigma_1 \dot{z})(\dot{z} - j\sigma_2 \dot{z})) \right]
\]
holds. This implies that $Q_{A^\dagger}(w)$ satisfies $Q_{A^\dagger}(w) \geq 0, \forall w \in \mathcal{B}_\Omega$. If we regard QDFs $Q_\Phi(w)$
and \(Q_\Psi(w)\) as the supply rate and the storage function in (33) along the line of Definition 2, the above observation shows that QDF \(Q_{\Psi \Gamma}(w)\) becomes the dissipation rate of \(\Psi_{\Gamma}\) for supply rate \(Q_\Psi(w)\) from Definition 2 (ii). Therefore, \(Q_{\Psi}(w)\) can be regarded as a compensation power which guarantees dissipativity of \(\Psi_{\Gamma}\).

**Remark 6** Corollary 1 (iii) also gives a time domain characterization of sum-of-squares (SoS) decomposition by similar discussion made by Hara and Iwasaki [13]. See Subsection 4.2 in [10] for the detailed discussion.

### 4.3 Finite Frequency Positive Realness

In this subsection, we apply Theorem 1 to the finite frequency positive-realness [2] under an input-output setting.

Let (7) be an input-output partition of \(M(\xi)\) in (5), where \(Y \in \mathbb{R}^{p \times m}[\xi]\) and \(U \in \mathbb{R}^{m \times m}[\xi]\) is nonsingular. Then, \(w\) is partitioned as \(w = \text{col}(y, u)\), where \(u := U \left( \frac{d}{d\xi} \right) \ell\) and \(y := Y \left( \frac{d}{d\xi} \right) \ell\) are an input and output, respectively. Such a partition always exists by the observability assumption of (5). Then, the transfer function \(G \in \mathbb{C}^{p \times m}[\xi]\) from \(u\) to \(y\) is given by (8). Define the low frequency domain by

\[
\Omega_{\text{low}} := \{\omega \in \mathbb{R} | |\omega| \leq \sigma \text{ and } \det(U(j\omega)) \neq 0\} \tag{36}
\]

for a given \(\sigma \in \mathbb{R}, \sigma \geq 0\).

In the following, we characterize the finite frequency positive realness of \(G(\xi)\) in the case where \(G(\xi)\) is square, i.e. \(p = m\). Then, \(G(\xi)\) is called finite frequency positive real (FFPR) with bandwidth \(\sigma\) if

\[
G(j\omega) + G(j\omega)^* \geq 0, \quad \forall \omega \in \Omega_{\text{low}} \tag{37}
\]

holds. This property is one of the key properties for the integrated design [2].

Suppose that \(\Phi \in \mathbb{H}^{2m \times 2m}[\zeta, \eta]\) is described by (16), where \(\Pi \in \mathbb{C}^{p \times m}\) is a nonsingular permutation matrix in (7). Then, we obtain the following corollary from Theorem 1.

**Corollary 2** Assume that \(\Psi\) in (4) and that \(\Psi\) is represented by an observable image representation (5). Let \(\Phi \in \mathbb{H}^{2m \times 2m}[\zeta, \eta]\) be given by (16). Let (7) be an input-output partition of \(M(\xi)\), where \(Y \in \mathbb{R}^{p \times m}[\xi]\) and \(U \in \mathbb{R}^{m \times m}[\xi]\) is nonsingular. Define \(G \in \mathbb{C}^{p \times m}[\xi]\) by (8). Let \(\Phi \in \mathbb{H}^{p \times q}[\xi]\) be given by (16). Define \(\Omega_{\text{low}}\) by (36). Then, the following statements (i), (ii), (iii) and (iv) are equivalent.

(i) The transfer function \(G(\xi)\) is FFPR with bandwidth \(\sigma\).

(ii) FFDI (15) holds for all \(\omega \in \Omega_{\text{low}}\).

(iii) There exist \(\Psi \in \mathbb{H}^{2m \times 2m}[\zeta, \eta]\) and \(\Gamma \in \mathcal{G}\) satisfying

\[
Q_{\Psi}(w) + \frac{d}{dt}Q_{\Psi}(w) \leq u'y, \quad \forall w \in \Psi.
\]

(iv) Inequality

\[
\int_{-\infty}^{+\infty} u'ydt \geq 0
\]

holds for all \(u, y \in \mathcal{D}^{p\times m}[\mathbb{R}, \mathbb{C}^{m}]\) satisfying

\[
\int_{-\infty}^{+\infty} \ddot{z}zdt \leq \sigma^2 \int_{-\infty}^{+\infty} \dddot{z}zdt,
\]

where \(z \in \mathbb{C}^{m}[\mathbb{R}, \mathbb{C}^{m+N+1}]\) is defined by (26) with \(w = \text{col}(y, u)\) for some nonnegative integer \(N \in \mathbb{Z}\).

**Proof** See Appendix B.4 in [10] for the proof.

### 5. Finite Frequency KYP Lemma

In this section, we give an LMI characterization of FFDI (14) or the finite frequency KYP lemma in the behavioral framework for a numerical checking of the finite frequency properties.

We assume that \(R \in \mathbb{C}^{p \times p}[\xi]\) in (3) is row reduced and the degree constraint (27) holds throughout this section. These assumptions do not lose any generality as we have explained.

We first transform the kernel representation (2) into a latent variable representation with a first-order differential-algebraic equation along the same line in [8]. Let \(r_i \in \mathbb{C}^{p \times p}[\xi] (i = 1, \ldots, p)\) denote the \(i\)th row of \(R(\xi)\), i.e. \(R(\xi) = \text{col}(r_1(\xi), r_2(\xi), \ldots, r_p(\xi))\). For these \(r_i\)’s, define \(R_0 \in \mathbb{C}^{p \times p}[\mathbb{R}(\xi)]\) by

\[
R_0(\xi) := \text{col}(R_1^*(\xi), R_2^*(\xi), \ldots, R_p^*(\xi))
\]

where \(R_i(\xi) := \text{col}(r_1(\xi), \ldots, r_i(\xi), \ldots, r_p(\xi))\).

For the finite frequency KYP lemma, we obtain the following characterization.

\[
\begin{align*}
[I_{Lq} & \quad 0_{q \times q}] 
\begin{bmatrix}
dw \\
0_{q \times q} & I_{Lq}
\end{bmatrix} 
\begin{bmatrix}
R_0(\xi) \\
R_1(\xi) \\
\ldots \\
R_p(\xi)
\end{bmatrix} \leq 0, \\
\begin{bmatrix}
0_{q \times q} & I_{Lq}
\end{bmatrix} 
\begin{bmatrix}
R_0(\xi) \\
R_1(\xi) \\
\ldots \\
R_p(\xi)
\end{bmatrix} \leq 0,
\end{align*}
\]

where \(R_0 \in \mathbb{C}^{p \times p}[\mathbb{R}(\xi)]\) and \(R_i(\xi) \in \mathbb{C}^{q \times q}[\xi, \mathbb{R}]\) for a numerical checking of the finite frequency properties.

This implies that \(\tilde{R}_0\) can define the first-order latent variable representation with manifest variable \(w\) and the latent variable \(k\) as

\[
w = \begin{bmatrix} I_q & 0_{q \times q} \end{bmatrix} \tilde{R}_0 k, \quad E \frac{d}{dt} k = F k,
\]

where

\[
E := \begin{bmatrix} I_{Lq} & 0_{q \times q} \end{bmatrix} \tilde{R}_0, \quad F := \begin{bmatrix} 0_{q \times q} & I_{Lq} \end{bmatrix} \tilde{R}_0 \in \mathbb{C}^{q \times q}.
\]

Using (38), (39) and (40), \(\Psi\) in (4) coincides with the set of trajectories given by (40) (see page 287 in [8]), i.e.

\[
\Psi = \{ w \in \mathbb{C}^{m} | \exists k \in \mathbb{C} \text{ s.t. } (40) \}.
\]

Using \(\tilde{R}_0\) and \(\Phi\), we define \(\Phi_0\) in \(\mathbb{H}^{2m \times 2m}[\xi]\) by

\[
\Phi_0 := \begin{bmatrix} \tilde{R}_0^* \end{bmatrix} \begin{bmatrix} 0_{(L-Kp)(L+1)q} & 0_{L-Kp}(L+1)q & 0_{L-Kp}(L+1)q \end{bmatrix} \tilde{R}_0,
\]

where \(\tilde{R}_0 \in \mathbb{H}^{p \times p}[\mathbb{R}(\xi)]\) is the coefficient matrix of \(\Phi_0(\xi, \eta)\). Consequently, we obtain the finite frequency KYP lemma in the behavioral framework. This is a natural result which follows from Proposition 3 and Lemma 1.
Theorem 2 Assume that $\mathcal{B}$ in (4) is controllable and that $R \in \mathbb{C}^{p \times q}[\xi]$ is row reduced. Suppose that $\mathcal{B}$ is represented by an image representation (5). Let $\Phi \in \mathbb{H}^{p \times q}[\zeta, \eta]$ be given by (9) and satisfy (27). Define $\Omega$ by (13). Then, the following statements (i) and (ii) are equivalent:

(i) FFDI (14) holds for all $\omega \in \Omega$.

(ii) There exist $\tilde{\Psi} \in \mathbb{H}^{L \times L}$ and $\tilde{\Upsilon} \in \mathbb{H}^{L \times L}$ satisfying

$$r \tilde{\Upsilon} \geq 0,$$

$$E^* \tilde{\Psi} F + F^* \tilde{\Psi} E + (\sigma_1^2 - \sigma_2^2)E^* \tilde{\Upsilon} E - F^* \tilde{\Upsilon} F + \left( (j \sigma_1 F^* \tilde{\Upsilon} E) + (j \sigma_2 F^* \tilde{\Upsilon} E) \right)^* \leq \Phi_0,$$

where $E, F \in \mathbb{C}^{L \times d}$ and $\Phi_0 \in \mathbb{H}^{d \times d}$ are defined by (41) and (42), respectively, and $\sigma_1, \sigma_2 \in \mathbb{R}$ are defined by (17).

Proof See Appendix B.6 in [10] for the proof. $\square$

We now explain relationship between Theorem 2 and the previous work [1],[8] associated with KYP lemma.

(i) Theorem 2 is not a new result because this is a special case of the generalized KYP lemma [1] if we restrict ourselves to continuous-time systems and the curve in the complex plane to $\Omega$. The lemma was derived based on the input-output setting, however, Theorem 2 does not assume such a relation in advance.

(ii) Theorem 2 includes the KYP lemma derived in [8] in a sense that we can deal with the LMI s over the restricted frequency domain. See [10] for the detailed discussion.

6. Numerical Example

In this section, we demonstrate a simple numerical example to show how FFDI (14) is characterized in terms of QDFs based on Theorems 1, 2 and Corollary 1. See Subsection 5.3 of the reference [10] for readers who are interested in an application of the finite frequency positive realness to a mechanical example, although we do not include the material to this paper due to the limitation of pages.

Consider the behavior $\mathcal{B}$ given by a kernel representation

$$\frac{d^2}{dt^2} + 2 \Omega w_1 + \frac{d}{dt} \Omega w_2 + \Omega w_3 = 0,$$

where $w := \text{col}(w_1, w_2, w_3)$ is the manifest variable. This representation is induced by a polynomial matrix

$$R(\xi) = \begin{bmatrix} \xi^2 + 2 & \xi & -1 \\ 0 & 1 & \xi^2 \end{bmatrix}.$$

We see that $\mathcal{B}$ has an observable image representation

$$w = M \frac{d}{dt} \ell, \quad M(\xi) = \begin{bmatrix} \xi^3 + 1 & -\xi^2 (\xi^2 + 2) \\ \xi^2 + 2 \\ \xi^2 \end{bmatrix}, \quad \ell \in \mathbb{C}^{\infty}(\mathbb{R}, \mathbb{R}).$$

We introduce a two-variable polynomial matrix $\Phi \in \mathbb{H}^{3 \times 3}[\zeta, \eta]$ defined by

$$\Phi(\zeta, \eta) := \begin{bmatrix} 1 - \zeta \eta & \zeta \eta + \eta^2 & 0 \\ \zeta \eta + \zeta^2 & 2 \zeta + 2 \eta + 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

which induces the supply rate for $\mathcal{B}$ given by

$$Q_\Phi(w) = w_1^2 - w_1^2 + 2 w_1 w_2 + 2 w_2 w_2 + 2 w_2^2 + w_2^2.$$

We analyze a finite frequency property based on the above $M(\xi)$ and $\Phi(\zeta, \eta)$, where we set the (low) frequency domain $\Omega := [-1, 1]$. We have the following FFDI

$$M(\jmath \omega)^* \partial \Phi(\jmath \omega) M(\jmath \omega) = -3 \omega^6 + 5 \omega^4 - 5 \omega^2 + 5 \geq 0, \quad \forall \omega \in \Omega,$$

since $M(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$ and

$$\partial \Phi(\jmath \omega) = \text{diag}(1 - \omega^2, 1, 1) \geq 0, \quad \forall \omega \in \Omega.$$

Define two-variable polynomial matrices $\Psi, \Delta \in \mathbb{H}^{3 \times 3}[\zeta, \eta]$ and $\Gamma \in \mathbb{H}^{3 \times 3}[\zeta, \eta]$ by

$$\Psi(\zeta, \eta) := \begin{bmatrix} 0 & \eta & 0 \\ \zeta & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \Delta(\zeta, \eta) := \text{diag}(0, 1, \zeta \eta),$$

$$\Gamma(\zeta, \eta) := (1 - \zeta \eta) \Psi(\zeta, \eta), \quad \Psi'(\zeta, \eta) := \text{diag}(1, 0, 1).$$

Then, QDFs $Q_\Psi(w)$ and $Q_\Gamma(w)$ are computed as

$$Q_\Psi(w) = 2 w_1 w_1 + 2 w_2 w_2$$

$$Q_\Gamma(w) = w_1^2 - w_1^2 + w_2^2 + w_2^2,$$

respectively, and hence we have

$$Q_\Phi(w) - \frac{d}{dt} Q_\Psi(w) = Q_\Delta(w) + Q_\Gamma(w) = w_1^2 - w_1^2 + w_2^2 + w_2^2.$$

We easily see that the dissipation inequality (32) does not hold. In addition, if we add $Q_\Gamma(w)$ to the left-hand side of the above inequality, we get

$$Q_\Gamma(w) + \frac{d}{dt} Q_\Psi(w) \leq Q_\Phi(w), \quad \forall w \in \mathcal{B},$$

because $Q_\Psi(w) = w_1^2 + w_2^2 \geq 0$ holds for all $w \in \mathcal{B}$. Hence, we can see from Theorem 1 (ii) that FFDI (14) holds.

Moreover, focusing on inequality

$$Q_\Delta(w) + Q_\Gamma(w) = w_1^2 - w_1^2 + w_2^2 + w_2^2 \geq 0, \quad \forall w \in \mathcal{B}\Omega,$$

yields the dissipation inequality

$$\frac{d}{dt} Q_\Psi(w) \leq Q_\Phi(w), \quad \forall w \in \mathcal{B}\Omega.$$

This shows that $\mathcal{B}\Omega$ is dissipative with respect to the supply rate $Q_\Phi(w)$ from Corollary 1 (ii). This is guaranteed by the existence of the compensation rate $Q_\Gamma(w)$. In the remainder of this example, we will see how to check the finite frequency property based on the LMI conditions in Theorem 2. We also consider the behavior $\mathcal{B}$ whose kernel representation is induced by $R(\xi)$ in (45). We see that $R(\xi)$ coincides with $R(\xi)$. This polynomial matrix has the coefficient matrix $R_\xi \in \mathbb{R}^{2 \times 2}$ which is given by

$$R_\xi = \begin{bmatrix} 2 & 0 & -1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$


Appendix A  State Map

We introduce the notion of state map [16] in this appendix. A polynomial matrix $X \in C^{n \times n}[\mathbb{C}]$ is said to induce a state map $\Psi$ and a latent variable

$$x = X \left( \frac{d}{dt} \right) w$$

(A.1)

called a state variable for $\Sigma$, if $x$ satisfies the axiom of state

$$\left[ \begin{array}{c} w_1 \\ x_1 \\ x_2 \\ w_2 \end{array} \right] \in \mathbb{B}_{\text{full}} \\
\text{and } x_1(0) = x_2(0) \Rightarrow \left[ \begin{array}{c} w_1 \\ x_1 \\ w_2 \\ x_2 \end{array} \right] \in \mathbb{B}_{\text{full}},$$

(A.2)

where $\mathbb{B}_{\text{full}}$ is a full behavior defined by

$$\mathbb{B}_{\text{full}} := \left\{ \left[ \begin{array}{c} w \\ x \end{array} \right] \in C^{n \times n}(\mathbb{R}, C^{n \times n}) \left| \begin{array}{c} x \text{satisfies (2) and (A.1)} \end{array} \right. \right\}.$$

In (A.2), $(v_1 \land v_2)(t)$ denotes $(v_1 \land v_2)(t) = v_1(t)$ for $t < 0$ and $(v_1 \land v_2)(t) = v_2(t)$ for $t \geq 0$. It is easily seen that the state map

$$\mathbb{B}_{\text{full}} = \left\{ \left[ \begin{array}{c} w \\ x \end{array} \right] \in C^{n \times n}(\mathbb{R}, C^{n \times n}) \left| x \text{satisfies (2) and (A.1)} \right. \right\}.$$
\(X \left( \frac{1}{2} \right)\) is not unique. A state map \(X \left( \frac{1}{2} \right)\) is said to be minimal, if \(\text{rowdim}(X) = \text{rowdim}(X')\) for any other \(X' \in \mathbb{C}^{m \times q}[\xi]\) which induces a state map for \(\Sigma\) [16].

Let \(X \in \mathbb{C}^{m \times q}[\xi]\) induce a minimal state map for \(\Sigma\), and let \(x = X \left( \frac{1}{2} \right) \xi\). Then, there exist matrices \(A \in \mathbb{C}^{m \times m}\) and \(B \in \mathbb{C}^{m \times q}\) satisfying \(\dot{x} = Ax + Bu\) from Proposition 9.2 in [6].

Appendix B  Coefficient Matrices

We define the coefficient matrix of a polynomial matrix, which is mainly used in Section 5.

The coefficient matrix of (one-variable) polynomial matrix \(R(\xi)\) in (3) is defined by \(R := [R_0 \ R_1 \ \cdots \ R_l] \in \mathbb{C}^{p \times (L+1)q}\).

The polynomial matrix \(R(\xi)\) is expressed as \(R(\xi) = RL(\xi)\) in terms of \(R\).

We give the definition of the two-variable version of the above coefficient matrix. With every \(\Phi \in \mathbb{H}^{p \times q}[\xi,\eta]\) in (9), we define its coefficient matrix \(\Phi \in \mathbb{H}^{K \times p(K+1)q}\) by

\[
\Phi := \begin{bmatrix}
\Phi_{0,0} & \Phi_{0,1} & \cdots & \Phi_{0,K} \\
\Phi_{1,0} & \Phi_{1,1} & \cdots & \Phi_{1,K} \\
\vdots & \vdots & \ddots & \vdots \\
\Phi_{K,0} & \Phi_{K,1} & \cdots & \Phi_{K,K}
\end{bmatrix}
\]

Then, \(\Phi(\xi,\eta)\) is expressed \(\Phi(\xi,\eta) = Z_0(\xi)F_0 Z_0(\eta)\) using \(\Phi\).

Lemma 1 [17] Let \(\Phi \in \mathbb{H}^{p \times q}[\xi,\eta]\) be given. Then, we have \(Q_0(\ell) \geq 0\) for all \(\ell \in \mathbb{C}^{m,\mathbb{C}}\) if and only if \(\Phi \geq 0\) holds.

Appendix C  \(\mathcal{B}\)-canonical Polynomial Matrices

We introduce \(\mathcal{B}\)-canonicity of polynomial matrices in this appendix, which are taken from the references [11],[18]. This relates LMIs with QDFs and plays an important role in Section 5.

We assume that \(R \in \mathbb{C}^{m \times q}[\xi]\) in (2) is row reduced [11] in this section. The assumption does not lose the generality as we have explained in Subsection 4.1.

Definition 3 [18] Let \(\mathcal{B}\) be represented by a kernel representation (2) for \(R \in \mathbb{C}^{m \times q}[\xi]\). Assume that \(R(\xi)\) is row reduced. Let \(D \in \mathbb{C}^{p \times q}[\xi]\) be given. Let \(r_i \in \mathbb{C}^{1 \times q}[\xi]\) and \(d_i \in \mathbb{C}^{1 \times q}[\xi]\) \((i = 1, \cdots , p)\) denote the \(i\)th rows of \(R(\xi)\) and \(D(\xi)\), respectively.

A polynomial matrix \(D(\xi)\) is called \(\mathcal{B}\)-canonical if \(\deg d_i \leq \deg r_i - 1, \forall i = 1, \cdots , p\) holds.

Lemma 2 [18] Let \(\mathcal{B}\) be represented by a kernel representation (2) for \(R \in \mathbb{C}^{m \times q}[\xi]\). Assume that \(R(\xi)\) is row reduced. For any \(D \in \mathbb{C}^{m \times q}[\xi]\), there exists a unique \(\mathcal{B}\)-canonical \(D' \in \mathbb{C}^{m \times q}[\xi]\) satisfying \(D \left( \frac{1}{2} \right) = D' \left( \frac{1}{2} \right) F\), \(\forall F \in \mathbb{C}^{m \times q}[\xi]\).

For \(\Phi \in \mathbb{H}^{p \times q}[\xi,\eta]\) in (9), there exist \(F \in \mathbb{C}^{\text{rank}\Phi \times (K+1)q}\) satisfying \(\Phi = F \Sigma F^\dagger\), where \(\Sigma \Phi = \mathbb{S}^{\text{rank}\Phi \times \text{rank}\Phi}\), \(F\) is of full row rank, and \(\det \Sigma \Phi \neq 0\). In this case, we get \(\text{rank}\Sigma \Phi = \text{rank}\Phi\).

With such a factorization of \(\Phi\), we obtain a canonical factorization of \(\Phi(\xi,\eta)\) as

\[
\Phi(\xi,\eta) = F(\xi)\Sigma \Phi F(\eta),
\]

where \(F \in \mathbb{C}^{\text{rank}\Phi \times q}[\xi]\) is defined by \(F(\xi) := F Z_0(\xi)\).

Definition 4 [18] Let \(\mathcal{B}\) be represented by a kernel representation (2) for \(R \in \mathbb{C}^{m \times q}[\xi]\). Assume that \(R(\xi)\) is row reduced. Let \(\Phi \in \mathbb{H}^{p \times q}[\xi,\eta]\) be given by (9). Let \(F \in \mathbb{C}^{\text{rank}\Phi \times q}[\xi]\) be defined by the canonical factorization (C.1). Then, \(\Phi(\xi,\eta)\) is called \(\mathcal{B}\)-canonical if \(F(\xi) = \mathcal{B}\)-canonical.