Gain-Scheduled Output-Feedback Controllers Using Inexactly Measured Scheduling Parameters for Linear Parametrically Affine Systems

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Abstract: This note considers the design problem of Gain-Scheduled (GS) controllers for Linear Parameter-Varying (LPV) systems whose state-space matrices are supposed to be parametrically affine with some mild constraints. To be precise, we consider two types of GS controllers, GS $H_\infty$ and GS $H_2$ controllers. The controllers to be designed are supposed to have parametrically affine state-space matrices. In sharp contrast to conventional GS controller design methods, the scheduling parameters are supposed to be inexact. Using parameter-independent Lyapunov functions, in which some conservatism is admitted, we give sufficient conditions for designing GS $H_2$ and GS $H_\infty$ output-feedback controllers which are robust against measurement errors of the measured scheduling parameters. The proposed methods are formulated in terms of parametrically affine Linear Matrix Inequalities (LMIs) with single line search parameters. The methods recover the design methods of conventional GS output-feedback controllers, in which scheduling parameters are supposed to be exactly measured, with an appropriate choice of the line search parameters. A simple numerical example is included to illustrate our results.

Key Words: Gain-Scheduling, $H_2$ control, $H_\infty$ control, LPV systems, uncertain scheduling parameters.

1. Introduction

During the last fifteen years, many researchers have studied Gain-Scheduled (GS) controller design for Linear Parameter-Varying (LPV) systems. See, e.g., [1]–[7] and references therein. Addressing GS controller design for LPV systems generally involves solving Parameter-Dependent Linear Matrix Inequalities (PDLMIs). Although this was considered a very challenging problem, several powerful methods have recently been proposed and their effectiveness demonstrated in many papers, e.g., Sum-Of-Squares (SOS) relaxation [8]–[10], the Slack Variable (SV) approach [11], and the coefficient check approach using Pólya’s theorem [12]. Thus, in this sense an obstacle to designing GS controllers has been removed, and for some particular problems we can design GS controllers without introducing conservatism if sufficiently large numerical complexity is admitted for solving the related PDLMIs, i.e., GS stabilizing state-feedback controller design [5],[7].

However, another issue must be addressed when applying GS controllers to real systems; that is, generally speaking, the scheduling parameters cannot be measured exactly. The design problem of GS controllers which are robust against measurement errors of the measured scheduling parameters has been addressed in several papers [13]–[17]. In [13], only some scheduling parameters are supposed to be available for controllers, and the problem setup is not practical because measurement errors are not addressed. On the other hand, in [14],[15] measurement errors are considered by introducing an additional uncertainty block in the $H_\infty$ control problem ¹. Generally speaking, an $H_\infty$ problem with multiple uncertainty blocks produces conservative controllers, and GS controllers designed using their methods inevitably introduce conservatism. In [16], scheduling parameters are supposed to be available with measurement errors which are proportional to the true values of the scheduling parameters. This problem setup is slightly impractical because it cannot address bias errors, which appear in many measurement systems. On the other hand, the method is very attractive because when there are no measurement errors, it recovers the conventional GS controller design in which exactly measured scheduling parameters are supposed to be available. In [17], Sato et al. successfully proposed design methods of GS $H_2$ and GS $H_\infty$ controllers which are robust against bias-type measurement errors using Parameter-Dependent Lyapunov Functions (PDLFs); however the designed controllers are supposed to be state-feedback controllers, which is not practical when designing controllers for real systems.

In summary, when designing GS output-feedback controllers under the condition that measurement errors are supposed to be in given intervals, we have no choice but to use the methods in [14],[15] which admit some conservatism due to multiple uncertainty blocks in the $H_\infty$ control problem.

In this note, we tackle the design problem of GS $H_2$ and GS $H_\infty$ output-feedback controllers in which the measured scheduling parameters have some measurement errors ². In contrast to [16], measurement errors are supposed to lie within intervals defined a priori and to vary with time. This problem setup is very natural and practical because the bias error case is included.

On the other hand, our formulation imposes Quadratic

¹ Rigorously speaking, “$H_\infty$ performance” cannot be defined in the problem because the closed-loop system is an LPV system. We use this terminology and a terminology “$H_2$ performance” so that readers can easily grasp our addressed problems while admitting that these terminologies are slightly abused.

² The rigorous definitions of our problems will be shown in section 2.
Stability (QS) on the closed-loop system, which is one of the drawbacks of our methods. On this topic, Masubuchi and Kurata have recently proposed a design method for GS output-feedback controllers which require no derivatives of the scheduling parameters \[18\]. Thus, merging their method with ours is one of our future research topics. As another feature, our methods are formulated in terms of parametrically affine LMIIs with single line search parameters, which indicates that our formulations are not affine with respect to decision variables. However, thanks to the parameters, our methods recover conventional design methods in which scheduling parameters are supposed to be exactly measured.

This paper is organized as follows: In section 2, we first define LPV plant systems and the GS output-feedback controllers to be designed, and define our problems. In section 3, we first show our design methods, then give some remarks on them. In section 4, design results for a simple numerical example are presented to show the method's feasibility.

In this note, we use the following notations: \((X)\) is a shorthand for \(X + X^T\), \(0_{m,n}\) and \(0\) respectively denote sets of \(n \times m\) dimensional zero matrix, an \(n \times n\) dimensional identity matrix and an appropriately dimensional zero matrix; \(\mathbb{R}^{m,n}\) and \(S^n\) respectively denote sets of \(n \times m\) real matrices and \(n \times n\) symmetric real matrices; \(\otimes\) denotes Kronecker product; \(\ast\) denotes an abbreviated off-diagonal block in a symmetric matrix; and \(\text{diag}(X_1, \ldots, X_k)\) denotes a block-diagonal matrix composed of \(X_1, \ldots, X_k\).

### 2. Preliminaries

#### 2.1 System Definitions

We consider the following parametrically affine LPV plant system \(G(\theta)\) with \(k\) independent scalar parameters \(\theta = [\theta_1 \cdots \theta_k]^T\).

\[
\begin{align*}
\dot{x} &= A(\theta)x + B_1(\theta)w + B_2u \\
\dot{z} &= C_1(\theta)x + D_{11}(\theta)w + D_{12}u \\
y &= C_2x + D_{21}w
\end{align*}
\]

where \(x \in \mathbb{R}^n\) is the state vector with \(x = 0\) at \(t = 0\), \(w \in \mathbb{R}^m\) is the disturbance input vector, \(u \in \mathbb{R}^r\) is the control input vector, \(z \in \mathbb{R}^q\) is the performance output vector and \(y \in \mathbb{R}^p\) is the measurement output vector. The parameters \(\theta_i\), which represent the changes of plant dynamics, are supposed to be time-varying (potentially time-invariant).

In (1), the following assumptions are made.

**Assumption 1** Matrices \(B_2, C_2, D_{12}\) and \(D_{21}\) are constant.

Similarly to [1], this assumption can be satisfied by incorporating strictly proper Linear Time-Invariant (LTI) filters to both \(u\) and \(y\).

The state-space matrices in (1) are supposed to be given as follows:

\[
\begin{align*}
A(\theta) &= A_0 + \bar{A}(\theta \otimes I_n), & \bar{A} &= [A_1 \cdots A_3] \\
B_1(\theta) &= B_{1n} + \bar{B}_1(\theta \otimes I_n), & \bar{B}_1 &= [B_{11} \cdots B_{13}] \\
C_1(\theta) &= C_{1n} + \bar{C}_1(\theta \otimes I_n), & C_{1n} &= [C_{11} \cdots C_{13}] \\
D_{11}(\theta) &= D_{11n} + \bar{D}_{11}(\theta \otimes I_n), & \bar{D}_{11} &= [D_{111} \cdots D_{113}]
\end{align*}
\]

The dimensions of matrices \(A_i, B_{1i}, \ldots, C_{11}, D_{11i}\), etc. are as follows: \(A_i \in \mathbb{R}^{n,n}\), \(B_{1i} \in \mathbb{R}^{n,m}\), \(C_{1} \in \mathbb{R}^{q,n}\) and \(D_{11i} \in \mathbb{R}^{q,q}\), \((i = 0, \ldots, k)\).

The parameters \(\theta_i\) are assumed to lie in a hyper-rectangle \(\Omega_k\) which is known in advance: \(\theta(t) \in \Omega_\theta, \forall t \geq 0\). The vertex set of \(\Omega_k\) is denoted by \(\partial(\Omega_k)\).

For the LPV system (1), we consider a parametrically affine full-order output-feedback controller. We suppose that we cannot obtain the true values of the scheduling parameters, but instead obtain the scheduling parameters with some measurement errors; that is, the \(i\)-th scheduling parameter is given as \(\theta_i + \delta_i\) with its measurement error \(\delta_i\). Vector \(\delta = [\delta_1 \cdots \delta_k]^T\) denotes the measurement errors in the measured scheduling parameters. The measurement errors \(\delta_i\) are assumed to lie in a hyper-rectangle \(\Omega_i\) which is known in advance: \(\delta(t) \in \Omega_i, \forall t \geq 0\). The vertex set of \(\Omega_i\) is denoted by \(\partial(\Omega_i)\). In this practical situation, we define GS controller \(C(\theta + \delta)\) as follows:

\[
C(\theta + \delta) = \begin{bmatrix}
\hat{x}_c &= A_c(\theta + \delta)x_c + B_c(\theta + \delta)y \\
u &= C_c(\theta + \delta)x_c + D_c(\theta + \delta)y
\end{bmatrix},
\]

where \(x_c \in \mathbb{R}^n\) denotes the state vector with \(x_c = 0\) at \(t = 0\), and \(A_c(\theta + \delta), B_c(\theta + \delta), \ldots\), etc., which are to be designed, are appropriately dimensioned affine matrices with respect to \(\theta_i\) and \(\delta_i\).

The closed-loop system \(G_c(\theta, \theta + \delta)\) comprising \(G(\theta)\) and \(C(\theta + \delta)\) is given as follows:

\[
\begin{align*}
\dot{x}_{cl} &= A_{cl}(\theta, \theta + \delta)x_{cl} + B_{cl}(\theta, \theta + \delta)w \\
y &= C_{cl}(\theta, \theta + \delta)x_{cl} + D_{cl}(\theta, \theta + \delta)w
\end{align*}
\]

where \(x_{cl} = [x^T \ x_c^T]^T\), and

\[
\begin{align*}
A_{cl}(\theta, \theta + \delta) &= [A(\theta) + B_1(\theta)D_{11}(\theta, \theta + \delta)C_{11}\ C_{12}] \\
B_{cl}(\theta, \theta + \delta) &= [B_1(\theta)D_{11}(\theta, \theta + \delta)D_{12}\ C_{12}] \\
C_{cl}(\theta, \theta + \delta) &= [C_1(\theta) + D_{12}(\theta, \theta + \delta)C_{12}\ D_{12}(\theta, \theta + \delta)C_{13}] \\
D_{cl}(\theta, \theta + \delta) &= D_{11}(\theta) + D_{12}(\theta, \theta + \delta)D_{21}
\end{align*}
\]

#### 2.2 Problem Definition

We are now ready to define our problems.

**Problem 1 (H_2 controller design)** Suppose that \(D_{11}(\theta) = 0\) holds for all \(\theta\) and that the scheduling parameters \(\theta_i\) are measured as \(\theta_i + \delta_i\) with measurement errors \(\delta_i\). For a given positive number \(\gamma_2\), find a controller \(C(\theta + \delta)\) with \(D_{cl}(\theta, \theta + \delta) = 0\) which stabilizes \(G_c(\theta, \theta + \delta)\) and satisfies (4) with a random variable \(w_0\) satisfying \(E(w_0^2) \geq I_n\) for all the combinations of admissible trajectories \(\theta\) and uncertainties \(\delta\).

\[
E(\int_0^\infty z^T z \ dt) < \gamma_2^2\ \text{for}\ t \neq 0 \quad \text{and} \quad w_0 (t = 0) = 0 \quad t \neq 0
\]

**Problem 2 (H_\infty controller design)** Suppose that the scheduling parameters \(\theta_i\) are measured as \(\theta_i + \delta_i\) with measurement errors \(\delta_i\). For a given positive number \(\gamma_\infty\), find a controller \(C(\theta + \delta)\) which stabilizes \(G_c(\theta, \theta + \delta)\) and satisfies (5) for all the combinations of admissible trajectories \(\theta\) and uncertainties \(\delta\).

\[
\sup_{w_0 \in L_2, w_\delta \neq 0} \|e_\|_2 < \gamma_\infty
\]
In sharp contrast to most of the conventional GS controller design problems in the literature, Problems 1 and 2 look for GS output-feedback controllers which are scheduled with inexactely measured scheduling parameters.

2.3 Basic Lemmas

Suppose that some full-order controller $C(\theta + \delta)$ as in (2) is given. Then, the following two lemmas are directly derived from the well known results, e.g. [2],[19].

Lemma 1 [17] Suppose that $D_{ij}(\theta, \theta + \delta) = 0$ holds for all $\theta$, $\delta$. For a given positive number $\gamma_1$, if there exist a positive definite matrix $X_{cl} \in S^{2n}$ and a parameter-dependent symmetric matrix $N(\theta) \in S^{2n}$ such that (6), (7), and (8) hold, then the closed-loop system $G_{cl}(\theta, \theta + \delta)$ is exponentially stable and satisfies (4) for all the combinations of admissible trajectories $\theta$ and uncertainties $\delta$.

$$
\begin{bmatrix}
(A_{cl}(\theta, \theta + \delta)X_{cl}) & * \\
C_{cl}(\theta, \theta + \delta)X_{cl} & -I_n
\end{bmatrix} < 0, \quad \forall(\theta, \delta) \in \Omega_\theta \times \Omega_\delta 
$$

(6)

$$
\begin{bmatrix}
N(\theta) & * \\
B_{cl}(\theta, \theta + \delta)X_{cl}
\end{bmatrix} > 0, \quad \forall(\theta, \delta) \in \Omega_\theta \times \Omega_\delta
$$

(7)

$$
\gamma_2^2 - \text{Tr}(N(\theta)) > 0, \quad \forall \theta \in \Omega_\theta
$$

(8)

Lemma 2 [17] For a given positive number $\gamma_\infty$, if there exist a positive definite matrix $X_{cl} \in S^{2n}$ such that (9) holds, then the closed-loop system $G_{cl}(\theta, \theta + \delta)$ is exponentially stable and satisfies (5) for all the combinations of admissible trajectories $\theta$ and uncertainties $\delta$.

$$
\begin{bmatrix}
(A_{cl}(\theta, \theta + \delta)X_{cl}) & * & * \\
C_{cl}(\theta, \theta + \delta)X_{cl} & -\gamma_\infty I_n & -\gamma_\infty I_n
\end{bmatrix} < 0, \quad \forall(\theta, \delta) \in \Omega_\theta \times \Omega_\delta
$$

(9)

We apply Lemmas 1 and 2 to Problems 1 and 2, respectively, in the next section.

3. Main results

In Lemmas 1 and 2, a candidate Lyapunov function is set as $x_{cl}^TJx_{cl}$ using a positive definite matrix $X_{cl}$.

A similar argument to Lemma B in [20] can be made for $X_{cl}$ even for GS controller design for LPV systems. This consequently implies that $X_{cl}$ can be set as follows without loss of generality:

$$
X_{cl} = \begin{bmatrix}
X & Y \\
Y & Y
\end{bmatrix}
$$

(10)

where $X, Y \in S^n$. For completeness, we give our claims and their proofs in the appendix.

We tackle Problems 1 and 2 using the $X_{cl}$ in (10).

3.1 Proposed Methods

3.1.1 $H_2$ controller design

When considering Problem 1, it is supposed that $D_{1n}(\theta) = 0$ holds for all $\theta$ and we set $D_{i}(\theta + \delta) = 0$. We propose the following theorem for Problem 1.

Theorem 1 For a given positive number $\gamma_2$, suppose that there exist a positive number $\gamma$, positive definite matrices $X \in S^{n}$ and $Z \in S^{n}$, and parametrically affine matrices $A_{cl}(\theta + \delta) \in R^{2n}$, $B_{cl}(\theta + \delta) \in R^{2n}$, $C_{cl}(\theta + \delta) \in R^{n}$, $D_{cl}(\theta) \in R^{n}$, $T(\delta) \in R^{2n}$, $S(\delta) \in S^{n}$ and $N(\theta) \in S^{n}$ such that (11), which is at the top of the next page, (12), (13), and (14) or (15) hold. Then, the controller $C(\theta + \delta)$ whose space-state matrices are given as in (16), in which $A(\theta + \delta)$ denotes the matrix $A(\theta)$ in (1) with $\theta + \delta$ instead of $\theta$, makes the closed-loop system $G_{cl}(\theta, \theta + \delta)$ exponentially stable and satisfies (4) for all the combinations of admissible trajectories $\theta$ and uncertainties $\delta$.

$$
\begin{bmatrix}
N(\theta) & * \\
B_{cl}(\theta) & * \\
ZB_{cl}(\theta) - B_{cl}(\theta + \delta)D_{cl} & I_n Z
\end{bmatrix} > 0, \quad \forall(\theta, \delta) \in \text{ver}(\Omega_\theta) \times \text{ver}(\Omega_\delta)
$$

(12)

$$
\gamma_2^2 - \text{Tr}(N(\theta)) > 0, \quad \forall(\theta, \delta) \in \text{ver}(\Omega_\theta)
$$

(13)

$$
\begin{bmatrix}
-R(\delta) & * & * & * \\
-T(\delta) & S(\delta) & * & * \\
-\hat{A}(\delta \otimes I_n) & X & \text{ver}(\Omega_\delta) & 0
\end{bmatrix} < 0, \quad \forall(\theta, \delta) \in \text{ver}(\Omega_\theta)
$$

(14)

$$
\begin{bmatrix}
-A_{cl}(\theta + \delta) & Z(\hat{A}(\delta \otimes I_n))X - B_{cl}(\theta + \delta)C_{cl}X + ZB_{cl}(\theta + \delta) - A_{cl}(\theta + \delta)Y^{-1} & B_{cl}(\theta + \delta) & 0
\end{bmatrix} < 0, \quad \forall(\theta, \delta) \in \text{ver}(\Omega_\theta)
$$

(15)

In (16), $Y$ is defined as $X - Z^{-1}$.

Proof First, note that all the inequalities are affine with respect to the associated parameters, i.e. $\theta$ and/or $\delta$. Thus, if they hold at all vertices associated with the parameters then they also hold for all the possible values of the parameters.

The following inequality is derived from (14).

$$
\begin{bmatrix}
R(\delta) & * \\
T(\delta) & S(\delta)
\end{bmatrix} - \begin{bmatrix}
0 & * \\
Z\hat{A}(\delta \otimes I_n) & X
\end{bmatrix} \leq 0, \quad \forall(\theta, \delta) \in \text{ver}(\Omega_\theta)
$$

(17)

Thus, the following inequality holds.

$$
\begin{bmatrix}
R(\delta) & * \\
T(\delta) & S(\delta)
\end{bmatrix} - \begin{bmatrix}
0 & * \\
Z\hat{A}(\delta \otimes I_n) & X
\end{bmatrix} \leq 0, \quad \forall(\theta, \delta) \in \text{ver}(\Omega_\theta)
$$

(17)

Similarly, the same inequality as (17) is also derived from (15). Thus, if one of (14) or (15) holds, (17) holds.

From (17) and (11), inequality (18) holds.

$$
\begin{bmatrix}
A(\theta)X + B_{cl}C_{cl}(\theta + \delta) & A(\theta) \\
\hat{A}(\delta \otimes I_n)X - B_{cl}(\theta + \delta)C_{cl} & -I_n
\end{bmatrix} < 0
$$

(18)

From (16), the matrices $\hat{A}_{cl}(\theta + \delta)$, etc. are represented as
After substituting $A_\epsilon(\theta,\theta+\delta)$ and $C_\epsilon(\theta,\theta+\delta)$ into (18), we obtain the inequality

$$
\begin{bmatrix}
A_\epsilon(\theta,\theta+\delta) & A(\theta) \\
A_\epsilon(\theta,\theta+\delta) & ZA(\theta) - B_\epsilon(\theta,\theta+\delta) C_2 \\
C_\epsilon(\theta,\theta+\delta) & D_\epsilon(\theta,\theta+\delta) & C_1(\theta)
\end{bmatrix}
\begin{bmatrix}
R(\delta) & * \\
T(\delta) & * \\
-C_1(\theta)
\end{bmatrix}
\begin{bmatrix}
Y \\
S(\delta) \\
-I_n
\end{bmatrix}
< 0, \ \forall (\theta,\delta) \in \text{ver}(\Omega_0) \times \text{ver}(\Omega_0)
$$

(11)

$A_\epsilon(\theta,\theta+\delta) = ZA(\theta+\delta)X - ZB_\epsilon(\theta,\theta+\delta) C_2 X + ZB_\epsilon C(\theta+\delta) Y - ZA(\theta+\delta) Y$,

$B_\epsilon(\theta+\delta) = ZB_\epsilon(\theta,\theta+\delta)$,

$C_\epsilon(\theta+\delta) = C_\epsilon(\theta,\theta+\delta) Y$.

As the proof is very similar to that of Theorem 1, it is omitted here.

**Remark 1** In Theorems 1 and 2, if $\epsilon$ is fixed then the conditions are formulated in terms of parametrically affine LMIs. However, in our proposed methods, $\epsilon$ is also a decision variable. Thus, a line search for $\epsilon$ is required to solve them.

### 3.2 Recovery of Conventional Design Methods

In this subsection, we claim that our methods recover the conventional GS controller design methods when $\delta = 0$ holds constantly, i.e. $\Omega_0 = \{0\}$ holds.

Before showing our claim, we show a conventional design method for parametrically affine GS $H_2$ controllers using a parameter-independent Lyapunov matrix (10).

**Lemma 3** Suppose that $\Omega_0 = \{0\}$ holds. For a given positive number $\gamma_2$, suppose that there exist positive definite matrices $X_1 \in S^p$ and $Z_\epsilon \in S^q$, and parametrically affine matrices $A_\epsilon(\theta) \in \mathbb{R}^{n_{x_\epsilon} \times n_{x_\epsilon}}$, $B_\epsilon(\theta) \in \mathbb{R}^{n_{x_\epsilon} \times n_{u_\epsilon}}$, $C_\epsilon(\theta) \in \mathbb{R}^{n_{y_\epsilon} \times n_{x_\epsilon}}$ whose state-space matrices are given as in (24), (25), and (26) hold. Then, the controller $C(\theta)$ whose state-space matrices are given as in (27) makes the closed-loop system $G_\epsilon(\theta)$ exponentially stable and satisfies (4) for all admissible trajectories $\theta$.

$$
\begin{bmatrix}
A(\theta)X_1 + B_2 C_\epsilon(\theta) & A(\theta) \\
A_\epsilon(\theta) & Z_\epsilon A(\theta) - B_\epsilon^T(\theta) C_2 \\
C_\epsilon(\theta) & D_\epsilon(\theta) & C_1(\theta)
\end{bmatrix}
\begin{bmatrix}
X_1 \\
Y \\
S(\theta)
\end{bmatrix}
< 0, \ \forall \theta \in \text{ver}(\Omega_0)
$$

(24)

$$
\begin{bmatrix}
N_\epsilon(\theta) \\
B_1(\theta) \\
Z_\epsilon B_\epsilon(\theta) - B_\epsilon^T(\theta) D_2
\end{bmatrix}
\begin{bmatrix}
X_1 \\
I_n \\
Z_\epsilon
\end{bmatrix}
> 0, \ \forall \theta \in \text{ver}(\Omega_0)
$$

(25)

$$
\begin{cases}
A_\epsilon(\theta) = Z^{-1}(ZA(\theta)X + ZB_\epsilon C_\epsilon(\theta)) \\
-\epsilon A_\epsilon(\theta) + \epsilon (ZB_\epsilon D_\epsilon(\theta) - B_\epsilon(\theta) C_2 X) Y^{-1} \\
B_\epsilon(\theta) = Z^{-1}(ZB_\epsilon D_\epsilon(\theta) - B_\epsilon(\theta) C_2 X) Y^{-1} \\
C_\epsilon(\theta) = (C_\epsilon(\theta) - D_\epsilon(\theta) C_2 X) Y^{-1} \\
D_\epsilon(\theta) = D_\epsilon(\theta)
\end{cases}
$$

(23)

In (23), $Y$ is defined as $X - Z^{-1}$.

In (27), $Y_\epsilon$ is defined as $X_\epsilon - Z_\epsilon^{-1}$.

We make the following assertion on the $H_2$ problem when $\Omega_0 = \{0\}$ holds.

**Lemma 4** Suppose that the measurement errors in the measured scheduling parameters are constantly zeros, i.e. $\Omega_0 = \{0\}$ holds. For a given positive number $\gamma_2$, if the condition of Theorem 1 is satisfied then the condition of Lemma 3 is always satisfied. Furthermore, the converse also holds.
Proof If the condition of Theorem 1 is satisfied, then the condition of Lemma 3 is satisfied with the following matrices.

\[
X_c = X, \quad Z_c = Z, \quad A_c(\theta) = A_c(\theta + \delta), \\
B_c(\theta + \delta) = B_c(\theta) + \delta C_c(\theta), \quad C_c(\theta + \delta) = C_c(\theta) + \delta D_c(\theta), \\
N_c(\theta + \delta) = N_c(\theta).
\]

This is because the non-negativity of \( R(\delta) * T(\delta) S(\delta) \) is derived from either (14) or (15) and (11) holds, (24) always holds with the foregoing matrices. The remaining part is obvious.

Next, we present the proof of the converse part below. Suppose that the condition of Lemma 3 is satisfied. As the left-hand side of (24) is strictly negative, there exists a positive number \( \tilde{e} \) satisfying (24) in which

\[
\begin{pmatrix}
A(\theta)X_c + B_c(\theta + \delta)A(\theta)C_c(\theta)
\end{pmatrix} * 
\begin{pmatrix}
R(\delta) * T(\delta) S(\delta)
\end{pmatrix} < 0, \quad \forall \theta \in \vartheta(\Omega_0).\]

From the fact that \( \delta = 0 \) holds, we now set \( R(\delta) = R_0, T(\delta) = T_0 \) and \( S(\delta) = S_0 \). If we set \( R_0 = T_0 = 0, S_0 = \tilde{e} I \) and \( \varepsilon \) as a positive number satisfying \( e^{1/2}(Z_c, Z_c) \geq \varepsilon \), then \( -\tilde{e} I + \varepsilon Z_c Z_c \leq 0 \) holds. That is, the following inequality holds.

\[
-\tilde{e} I + \varepsilon Z_c Z_c \leq 0.
\]

Then, obviously, (14) holds with \( R(\delta) = T(\delta) = 0 \) and \( S(\delta) = \tilde{e} I \). Thus, the condition of Theorem 1 is satisfied with the following matrices.

\[
X = X_c, \quad Z = Z_c, \quad A_c(\theta + \delta) = A_c(\theta), \\
B_c(\theta + \delta) = B_c(\theta), \quad C_c(\theta + \delta) = C_c(\theta), \\
N_c(\theta + \delta) = N_c(\theta), \\
R(\delta) = 0, \quad T(\delta) = 0, \quad S(\delta) = \tilde{e} I.
\]

This completes the proof.

Remark 2 Although we only show the proof using (14), the counterpart using (15) can also be proved with following matrices: \( R(\delta) = \tilde{e} I, T(\delta) = S(\delta) = 0, \) and \( \varepsilon \) as a positive number satisfying \( e^{1/2}(X_c, X_c) \geq \varepsilon \) for a sufficiently small positive number \( \tilde{e} \).

A similar assertion holds for the \( H_{\infty} \) problem when \( \Omega_0 = \{0\} \) holds. Before showing our claim, we show a conventional design method for parametrically affine GS \( H_{\infty} \) controllers using a parameter-independent Lyapunov matrix (10).

**Lemma 5** Suppose that \( \Omega_0 = \{0\} \) holds. For a given positive number \( \gamma_{\infty} \), suppose that there exist positive definite matrices \( X_c \in \mathbb{S}^n \) and \( Z_c \in \mathbb{S}^n \), and parametrically affine matrices \( A_c(\theta) \in \mathbb{R}^{n \times n}, B_c(\theta) \in \mathbb{R}^{n \times n}, C_c(\theta) \in \mathbb{R}^{n \times n}, \) and \( D_c(\theta) \in \mathbb{R}^{n \times n} \), such that (28) and (29) hold. Then, the controller \( C(\theta) \) whose state-space matrices are given as in (30) makes the closed-loop system \( G_c(\theta) \) exponentially stable and satisfies (5) for all admissible trajectories \( \theta \).

**Lemma 6** Suppose that the measurement errors in the measured scheduling parameter are constantly zeros, i.e. \( \Omega_0 = \{0\} \) holds. For a given positive number \( \gamma_{\infty} \), if the condition of Theorem 2 is satisfied then the condition of Lemma 5 is always satisfied. Furthermore, the converse also holds.

The proof is omitted as it is very similar to that of Lemma 4. In summary, when no measurement errors in the measured scheduling parameters are supposed, our methods can design GS controllers which achieve the same performance as conventional design methods; that is, our proposed methods recover the conventional design methods.

4. Example

We consider the same example as in [16]. The state-space matrices of the LPV system (1) are given as follows with a scheduling parameter \( \theta \) with \( |\theta| \leq 1 \):

\[
\begin{pmatrix}
A(\theta) & B(\theta) & C(\theta) \\
B(\theta) & C(\theta) D(\theta) & D(\theta)
\end{pmatrix} =
\begin{pmatrix}
-0.89 & -0.896 & 1.010^{-1} - 0.119 \\
-142.6 & -178.25 & 0.4 - 130.8 \\
C(\theta) & D(\theta) & 1.010^{-1}
\end{pmatrix}.
\]

We design GS \( H_2 \) controllers for this system using Theorem 1 and GS \( H_{\infty} \) controllers using Theorem 2. The measurement error \( \delta \) in the measured scheduling parameter is supposed to be bounded by \( \xi \), i.e. \( |\delta| \leq \xi \). In Theorems 1 and 2, the line search
for $\varepsilon$ was conducted with 400 logarithmically linear-gridded points in $[10^{-10}, 10^{10}]$.

For reference, Figs. 1 and 2 show the relationships between $\varepsilon$ and $\gamma_2$ using (14) and (15) respectively. Similarly, Figs. 3 and 4 show the relationships between $\varepsilon$ and $\gamma_{\infty}$ using (14) and (15) respectively.

The design results are summarized in Table 1. The optimal $\gamma_2$ and $\gamma_{\infty}$ designed using Lemmas 3 and 5 are obtained as $0.434$ and $0.153$ respectively. These values are the same as those obtained by our methods with $\xi = 0$. This indicates that Lemmas 4 and 6 hold.

We next examine the $H_2$ and $H_{\infty}$ performance of the closed-loop system using QS. The results are shown in Table 2, where “$\infty$” means that the closed-loop system does not have QS. We confirm the following: (1) GS controllers designed using conventional methods have some robustness against measurement error in the measured scheduling parameter, e.g. until $\xi \leq 0.1$ holds, but when $\xi \geq 0.2$ holds they cannot stabilize the closed-loop system; and (2) GS controllers designed using our methods are in some cases more conservative than those designed using conventional methods when $\xi \leq 0.1$ holds, but our methods can design stabilizing GS controllers even if the measurement error is large, e.g. $\xi \geq 0.2$. This illustrates the advantage of our methods over conventional ones.

### Table 1 Optimal $\gamma_2$ and $\gamma_{\infty}$ for GS controllers.

<table>
<thead>
<tr>
<th>$\xi$</th>
<th>$\gamma_2$</th>
<th>$\gamma_{\infty}$</th>
<th>$\gamma_2$</th>
<th>$\gamma_{\infty}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(14)</td>
<td>(15)</td>
<td>(14)</td>
<td>(15)</td>
<td>(14)</td>
</tr>
<tr>
<td>0</td>
<td>0.434</td>
<td>0.434</td>
<td>0.434</td>
<td>0.153</td>
</tr>
<tr>
<td>0.01</td>
<td>0.493</td>
<td>0.459</td>
<td></td>
<td>0.170</td>
</tr>
<tr>
<td>0.05</td>
<td>0.787</td>
<td>0.586</td>
<td></td>
<td>0.231</td>
</tr>
<tr>
<td>0.1</td>
<td>1.427</td>
<td>0.848</td>
<td></td>
<td>0.310</td>
</tr>
<tr>
<td>0.2</td>
<td>3.626</td>
<td>1.669</td>
<td></td>
<td>0.488</td>
</tr>
<tr>
<td>0.5</td>
<td>17.305</td>
<td>5.177</td>
<td></td>
<td>1.147</td>
</tr>
<tr>
<td>1.0</td>
<td>64.441</td>
<td>13.941</td>
<td></td>
<td>2.530</td>
</tr>
<tr>
<td>2.0</td>
<td>212.19</td>
<td>36.918</td>
<td></td>
<td>5.976</td>
</tr>
</tbody>
</table>

### Table 2 Guaranteed $\gamma_2$ and $\gamma_{\infty}$ for $G_{c}(\theta, \theta + \delta)$ using the GS controllers in Table 1.

<table>
<thead>
<tr>
<th>$\xi$</th>
<th>$\gamma_2$</th>
<th>$\gamma_{\infty}$</th>
<th>$\gamma_2$</th>
<th>$\gamma_{\infty}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(14)</td>
<td>(15)</td>
<td>(14)</td>
<td>(15)</td>
<td>(14)</td>
</tr>
<tr>
<td>0</td>
<td>0.434</td>
<td>0.434</td>
<td>0.434</td>
<td>0.153</td>
</tr>
<tr>
<td>0.01</td>
<td>0.459</td>
<td>0.444</td>
<td>0.444</td>
<td>0.162</td>
</tr>
<tr>
<td>0.05</td>
<td>0.565</td>
<td>0.522</td>
<td>0.493</td>
<td>0.191</td>
</tr>
<tr>
<td>0.1</td>
<td>0.825</td>
<td>0.675</td>
<td>0.584</td>
<td>0.227</td>
</tr>
<tr>
<td>0.2</td>
<td>1.927</td>
<td>1.191</td>
<td>$\infty$</td>
<td>0.317</td>
</tr>
<tr>
<td>0.5</td>
<td>8.785</td>
<td>2.215</td>
<td>$\infty$</td>
<td>0.668</td>
</tr>
<tr>
<td>1.0</td>
<td>30.224</td>
<td>8.111</td>
<td>$\infty$</td>
<td>1.238</td>
</tr>
<tr>
<td>2.0</td>
<td>120.33</td>
<td>21.277</td>
<td>$\infty$</td>
<td>2.475</td>
</tr>
</tbody>
</table>

### 5. Conclusions

This paper tackled the design problems of Gain-Scheduled (GS) $H_2$ and $H_{\infty}$ output-feedback controllers for parametrically affine Linear Parameter-Varying (LPV) systems under the condition that only inexact scheduling parameters are available. Using parameter-independent Lyapunov functions, we proposed design methods for our addressed problems in terms of parametrically affine Linear Matrix Inequalities (LMIs) with single line search parameters. When no measurement errors exist in the measured scheduling parameters, our methods recover the conventional design.

Fig. 1  $\varepsilon$ vs. $\gamma_2$ using (14).

Fig. 2  $\varepsilon$ vs. $\gamma_2$ using (15).

Fig. 3  $\varepsilon$ vs. $\gamma_{\infty}$ using (14).

Fig. 4  $\varepsilon$ vs. $\gamma_{\infty}$ using (15).
methods with an appropriate choice of line search parameters. A simple numerical example borrowed from the literature supported our results.

Acknowledgments

The author really appreciates for the invaluable comments of anonymous reviewers.

This work was supported by the Ministry of Education, Culture, Sports, Science and Technology of Japan under Grant-in-Aid for Young Scientists (B) No. 20760287.

References


Appendix A Structured \(X_{cl}\)

We make the following claim.

Lemma 7 For a given parameter-dependent matrix \(N(\theta) \in S^{n_{\theta}}\), if there exist a positive definite matrix \(X_{cl} \in S^{n_{\theta}}\) and a controller \(C(\theta + \delta)\) such that (6) and (7) hold, then there always exist a positive definite matrix \(X_{cl}\) as in (10) such that (6) and (7) hold for some controller \(C(\theta + \delta)\).

Proof Suppose that the matrix \(X_{cl}\) and the state-space matrices of \(C(\theta + \delta)\) in (2), which satisfy (6) and (7), are respectively given as

\[
X_{cl} = \begin{bmatrix} X_1 X_2 \\ X_2^T X_1 \end{bmatrix} \in S^{n_{\theta}},
\]

\(X_1 \in \mathbb{R}^{n_{\theta} \times n_{\theta}}\), and

\[
A_\theta(\theta + \delta),
B_\theta(\theta + \delta), C_\theta(\theta + \delta) \text{ and } \theta.
\]

As \(X_{cl}\) is strictly positive, it can be assumed that \(X_{cl}\) is nonsingular without loss of generality. Applying a congruence transformation with \(\text{diag}(I_{n_{\theta}} X_1 X_2^T, I_{n_{\theta}})\) to (6) leads to the following inequality:

\[
\begin{bmatrix}
A(\theta) & B_\theta C_\theta(\theta + \delta) \\
B_\theta(\theta + \delta) C_\theta & A_\theta + \bar{A}(\theta + \delta) X_1 X_2^T X_1 \\
C_\theta(\theta + \delta) D_{12} C_\theta &= A_\theta + \bar{A}(\theta + \delta) X_1 X_2^T X_1 \\
& & - I_{n_{\theta}}
\end{bmatrix} < 0,
\]

where \(X = X_1 X_2 X_1^T, \bar{A}(\theta + \delta) = X_1 X_2 A_\theta(\theta + \delta) X_2 X_1^T, B_\theta(\theta + \delta) = X_1 X_2 B_\theta(\theta + \delta),\) and \(C_\theta(\theta + \delta) = C_\theta(\theta + \delta) X_2 X_1^T X_1^T.

Similarly, applying a congruence transformation with \(\text{diag}(I_{n_{\theta}}, I_{n_{\theta}}, X_1 X_2^T)\) to (7) leads to the following:

\[
\begin{bmatrix}
N(\theta) & * \\
B_\theta(\theta + \delta) & D_{12}
\end{bmatrix} \begin{bmatrix}
X_1 X_2^T \\
X_1 X_2^T
\end{bmatrix} > 0.
\]

Thus, \(X_{cl}\) in (10) with \(Y = \bar{X}\) and \(C(\theta + \delta)\) with \(A_\theta(\theta + \delta), B_\theta(\theta + \delta), C_\theta(\theta + \delta)\) and \(D_{12}(\theta + \delta)\) being respectively set as \(\bar{A}(\theta + \delta), \bar{B}(\theta + \delta), \bar{C}(\theta + \delta)\) and \(\theta\) satisfy (6) and (7) for the same \(N(\theta)\). Thus, our claim is proved.

Similarly, we also make the following claim.

Lemma 8 For a given positive number \(\gamma_{\infty}\), if there exist a positive definite matrix \(X_{cl} \in S^{n_{\theta}}\) and a controller \(C(\theta + \delta)\) such that (9) holds, then there always exist a positive definite matrix \(X_{cl}\) as in (10) such that (9) holds for some controller \(C(\theta + \delta)\).

As the proof is very similar to that of Lemma 7, it is omitted.
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