Exact Linearization of Two-Input Affine Systems via the Dynamic Extension Based on the Relative Degree Structure

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Abstract: This paper discusses the exact linearization problem of two-input affine systems via the dynamic extension based on the relative degree structure. A necessary and sufficient condition for two-input systems to be exactly linearizable via 1-degree dynamic extension is derived, and the input transformations of two-input systems with dynamic extension are classified into two forms. Finally, a method to derive the input transformation such that the transformed system is exactly linearizable via the dynamic extension is proposed. The proposed method is applied to a mechanical system.

Key Words: dynamic extension, exact linearization, relative degree.

1. Introduction

Exact linearization using static feedback transformation which is the combination of input transformation and coordinate transformation has been studied based on the differential geometry. A necessary and sufficient condition for exact linearization was summarized in [1]. This linearizing method is useful since there is no approximation. However, few systems can be transformed into linear system exactly via the static feedback transformation only. Hence, many researchers seek the way to expand the class of exactly linearizable systems. For example, orbital transformation and dynamic feedback were reported with some examples, see [2]–[5], and [6]. Charlet et al. [7],[8] showed a necessary condition for exact linearization via restricted dynamic feedback. However, there exists a big gap between this condition and a necessary and sufficient condition for the exact linearizability via the dynamic feedback. Guay et al. [4] revealed a necessary and sufficient condition for exact linearization via the dynamic feedback in terms of differential one form. This condition requires the existence of the dynamic feedback and generators that satisfy certain conditions. Even though the necessary and sufficient conditions were reported, it has not been reported how to derive such dynamic feedback and generators. It is still problematic how to check whether the given system is exactly linearizable via the dynamic feedback and how to construct transformations including the dynamic feedback. In order to discuss the linearizability via the dynamic feedback, some papers focused on the simple class of the dynamic feedback, which is called the dynamic extension. Sluis et al. showed that a bound on the extension degree which is needed to linearize a system by the dynamic extension, see [3]. However, the problem of the exact linearization via the dynamic extension is still unsolved. In this paper, we focus on the dynamic extension.

This paper discusses the exact linearization problem of two-input affine systems via the dynamic extension, and proposes a method to derive the specific transformations which linearize the system via dynamic extension. These discussions are based on the relative degree structure that is introduced in [9].

This paper is organized as follows. In the second section, some notations and definitions including the relative degree structure are prepared. Section 3 shows the condition for exact linearization via the 1-degree dynamic extension. In Section 4, the input transformation for the original system is discussed and classified based on the equivalence under the feedback transformation for extended system. After that the method to derive the transformations which linearize the system via dynamic extension is proposed in Section 5. In order to explain the proposed method, the inverted pendulum system with horizontal and vertical inputs is introduced, and the transformations that realize the exact linearization are derived in Section 6.

2. Preliminary

Consider the following inputs affine nonlinear system

$$\dot{x} = f(x) + g_1(x)u_1 + \cdots + g_n(x)u_n,$$

where $f(x), g_1(x), \ldots, g_n(x)$ are smooth vector fields defined on the state space $\mathbb{R}^n$, and $x$ is local coordinate, and $u_1, \ldots, u_n \in \mathbb{R}$. $\mathbb{R}^n$ denotes the tangent bundle of $\mathbb{R}^n$. We use the symbol $\Sigma$ to indicate the system (1). The set of input vector fields is denoted by $\mathcal{G}(x) = \{g_1(x), \ldots, g_n(x)\}$. $\mathcal{L}_{f(x)}h(x)$ denotes the Lie derivative of a scalar function $h(x)$ along with a vector field $f(x)$, and the function $\mathcal{L}_{f(x)}h(x)$ satisfies the recursion $\mathcal{L}_f h(x) = [f(x), h(x)]$. $\{f(x), g(x)\}$ represents the Lie bracket of two vector fields $f(x)$ and $g(x)$, and Lie bracket satisfies the Jacobi identity:

$$[[f_1(x), f_2(x), f_3(x)] + [f_2(x), f_3(x), f_1(x)] + [f_3(x), f_1(x), f_2(x)]] = 0,$$

where $f_1(x), f_2(x)$ and $f_3(x)$ are vector fields, $sp\{f_1(x), \ldots, f_3(x)\}$ denotes the smooth distribution spanned by smooth vector fields $f_1(x), \ldots, f_3(x)$. A distribution is said to be involutive if it is closed under Lie bracket. The smallest

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involutive distribution which includes the distribution \( \mathcal{D} \) and called involutive closure of \( \mathcal{D} \) and referred to as \( \bar{\mathcal{D}} \) or \( \mathfrak{d}\mathcal{D} \). \( \dim \mathcal{D} \) denotes the dimension of the distribution \( \mathcal{D} \). We suppose that all distributions defined below are smooth and regular, that is, their dimensions are constant over \( \mathbb{R}^n \). The vector field \( f_1(x) \) is said to be congruent to the vector field \( f_2(x) \) modulo \( \mathcal{D} \) if there exists \( g(x) \in \mathcal{D} \) such that \( f_1(x) = f_2(x) + g(x) \), and this congruence is denoted by \( f_1(x) \equiv f_2(x) \mod \mathcal{D} \). If \( f_1(x) \in \mathcal{D} \), then we say \( f_1(x) \equiv 0 \mod \mathcal{D} \). The distributions associated with the system (1) are defined as follows:

\[
\begin{align*}
\mathcal{G}_i(x) &= \text{sp}\{\mathcal{G}(x)\}, \\
\mathcal{G}_i'(x) &= \text{sp}\{\text{ad}_{f_1(x)}^{-1}\mathcal{G}(x), \mathcal{G}_i^{-1}(x)\}, \\
& \quad i \geq 2, (3)
\end{align*}
\]

where \( \text{ad}_{f_1(x)}^{-1}\mathcal{G}(x) = \text{sp}\{\text{ad}_{f_1(x)}^{-1}\mathcal{G}(x) | \mathcal{G}(x) \in \mathcal{G}(x)\} \), and \( \text{ad}_{f_1(x)}\mathcal{G}(x) \) is the successive Lie bracket defined by \( \text{ad}_{f_1(x)}\mathcal{G}(x) = \text{ad}_{f_1(x)}[\text{ad}_{f_1(x)}\mathcal{G}(x)] \) and \( \text{ad}_{f_1(x)}\mathcal{G}(x) = [f_1(x), \mathcal{G}(x)] \). The equation (3) implies that these distributions make nested structure \( \mathcal{G} \subseteq \mathcal{G}^{-1} \) and \( \mathcal{G}' \subseteq \mathcal{G}'^{-1} \). Hereinafter, we use both notations: \( \mathcal{G}^1, \ldots, \mathcal{G}^r \) and \( \sum_{i=1}^n \mathcal{G}(x) \frac{\partial}{\partial x} \) to represent a vector field \( \mathcal{G}(x) \). We sometimes omit the coordinate representation \( (x) \) for simplicity.

Two systems \( \Sigma_1 \) and \( \Sigma_2 \) are said to be feedback equivalent if there exist coordinate transformation and input transformation which transforms the system \( \Sigma_1 \) into \( \Sigma_2 \). If a system is feedback equivalent to a linear system, then the system is said to be exactly linearizable. The condition of exact linearization via static feedback is known as follows.

Theorem 1 [1] The system (1) is exactly linearizable with static feedback transformation if and only if the distributions (3) are regular and satisfy the following conditions:

\[
\mathcal{G}_i = \mathcal{G}_i', \quad i = 1, 2, \ldots, (4)
\]

\[
\exists k < \infty \quad \text{s.t.} \quad \dim \mathcal{G}_i = n. (5)
\]

Note that the exact linearizability is not depend on the feedback transformation since the distributions (3) are not depend on the feedback transformation as shown in [1].

2.1 Relative Degree Structure [9]

The notion of the system structure based on the relative degree is defined as follows.

Definition 1 [9] The relative degree structure of the system with \( n \) states and \( m \) inputs is defined as the pair of two sets of indices.

\[
(r_i^0, \ldots, r_i^r) = [k_i^0, \ldots, k_m^r],
\]

where each index is defined as follows.

\[
r_i^0 = r \quad \text{s.t.} \quad \sum_{j=1}^{r} \sigma^{j-1} < n - i + 1 \leq \sum_{j=1}^{r} \sigma^{j},
\]

\[
k_i^j = \{ | \{ k \geq 1 | k \in \mathbb{N} \} \}
\]

where \( | \{ \cdot \} | \) denotes the cardinality of the set \( \{ \cdot \} \) and \( \mathbb{N} \) denotes the natural number \( 1, 2, \ldots \). The indices \( \sigma^j \) and \( \gamma^j \) are

\[
\sigma^j = \dim \mathcal{G}^j - \dim \mathcal{G}^{j-1}, \quad \sigma^0 = 0,
\]

\[
\gamma^j = \dim \mathcal{G}^j - \dim \mathcal{G}^{j-1}.
\]

Each index is relative degree which characterizes the system, and in this paper, we assume these indices are finite for simple discussion. The indices \( [r_i^0, \ldots, r_i^r] \) are the relative degrees of the functions which can construct the coordinate transformation and have the largest relative degree. The other \( m \) indices denote the controllability indices of the largest linear subsystem generated by static feedback transformation, and these indices \( k_i^0, \ldots, k_m^r \) are derived in [10]. If a given system is exactly linearizable, then \( \sum_{i=1}^{m} k_i^r = n \). If a system is not exactly linearizable, then \( \sum_{i=1}^{m} k_i^r < n \). Each set of indices is ranked in descending order. Note that the distributions (3) are feedback invariant, that is, they are invariant under the feedback transformation, and then the relative degree structure is also feedback invariant. To list up the information which is indicated by the relative degree structure, the following example is considered.

Example 1 Consider a system which has the relative degree structure \( (2, 2, 2, 1, 1) - (2, 2) \) on a domain \( \mathbb{R}^6 \). Following properties are deduced by the relative degree structure.

- There exists coordinate \( h_1, \ldots, h_6 \) such that the system has the following form with proper input transformation.

\[
\begin{align*}
\frac{d}{dt} \begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \\ h_5 \\ h_6 \end{bmatrix} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + u_1 + u_2.
\end{align*}
\]

- \( [2, 2] \) indicates that \( \sigma^1 = 2, \sigma^2 = 4 \), \( \dim \mathcal{G}^1 = 2 \) and \( \dim \mathcal{G}^2 = 6 \).

- \( \sigma^2 = \gamma^2 = 2 \) means that \( \mathcal{G}^2 \) has dimension 4, and \( \{g_1, g_2, \text{ad} g_1, \text{ad} g_2\} \) is independent.

For the detailed description of the relative degree structure, see [9]. Hereinafter, we consider two-input systems.

2.2 Dynamic Extension [2],[8]

In general, the dynamic feedback has the form

\[
\begin{align*}
\dot{w} &= A(x, w) + B(x, w)w' \\
u &= \alpha(x, w) + \beta(x, w)v,
\end{align*}
\]

where \( w \in \mathbb{R}^d \) is extended state, and \( v \in \mathbb{R}^2 \) is new input. Even though the necessary and sufficient condition for exact linearization via dynamic feedback is derived in [4], it is not shown that how to check the existence of the dynamic feedback which satisfies the condition. It is also open problem that how to derive the dynamic feedback which satisfies the condition. In this paper, we consider a class of the dynamic feedback, which is defined by

\[
w_i = w_{j}^{(e)} = v_i, \quad i = 1, 2
\]
where \( e_i \) is extended degree of input \( u_i \), and \( w_i^{(e_i)} = \frac{\partial \gamma_i}{\partial e_i} \), and \( v_i \) is new input associated with \( u_i \). We call this class of the dynamic feedback \( \mu \)-degree dynamic extension, where \( \mu = e_1 + e_2 \). In the case that \( e_1, e_2 > 0 \), the system controlled by (8), which is called the extended system, becomes

\[
\Sigma : \dot{z} = F(z) + G_1v_1 + G_2v_2 = F(z) + Gv
\]

where the extended state \( w \) is \( [w_1, \ldots, w_1^{(e_1-1)}, w_2, \ldots, w_2^{(e_1-1)}]^T \), \( A \) and \( B \) are constant matrix and describe the dynamics (8), and \( z = (x, w)^T \in \mathbb{R}^{n+2} \) is a local coordinate of the extended system, and \( F, G_1, G_2 \) are vector fields of the extended system, and \( f(z) = f(x) + g_1(x)w_1 + g_2(x)w_2 \) and \( G_1, G_2 \) are constant vector fields. Note that if \( e_i = 0 \) for \( i = 1, 2 \), then \( G_i \) is equal with \( g_i \) and \( f(z) = f(x) + \sum_{i=0}^2 g_i(x)w_i \). The distribution of the extended system \( \mathcal{G} \) is defined in a similar way as in (3).

Through the discussion of Pfaffian system associated with the system (1), Sluis et al. [3] showed that the extension degree needs at most \( 2n - 2 \) for a extended system to be linearizable if the system (1) is linearizable via the dynamic extension. It is also shown that if the system (1) is dynamic feedback linearizable with dynamic extension only, then at least one extension degree can be set to zero. This fact is also proven via the discussion about the distribution associated with the system (1), see [11]. Hereinafter, we set the extension degrees to \( e_1 = \mu > 0, e_2 = 0 \). Then the \( \mu \)-degree dynamic extension (8) is rewritten as follows:

\[
\begin{bmatrix}
  u_1 \\
  u_2 \\
  w_1^{(\mu)}
\end{bmatrix} =
\begin{bmatrix}
  w_1 \\
  0 \\
  1
\end{bmatrix} +
\begin{bmatrix}
  0 \\
  1
\end{bmatrix}v_2.
\]

The problems of how to check the linearizability via dynamic extension and how to construct the transformations are still unsolved. We consider these problems in the following text.

At the end of this section, we define the exact linearizability via the dynamic extension.

**Definition 2** A system (1) is said to be exactly linearizable via the dynamic extension if the system (1) together with the dynamic extension (8) is exactly linearizable.

**Remark 1** Since the dynamic extension changes the dimension of the system, the dynamic extension also changes the relative degree structure of the system. Then the extended system still has a possibility to be exactly linearizable even if the original system is not exactly linearizable.

3. **Modification of Structural Distortion**

In this section, we focus on the 1-degree dynamic extension and derive the condition for the extended system to be exactly linearizable, and the condition is described by the vector fields of the original system.

Before we discuss the condition of the exact linearization via the 1-degree dynamic extension, we define the structural distortion for further discussion.

**Definition 3** (Structural distortion of the system) The system with \( \{r_1, \ldots, r_n\} - \{k_1, \ldots, k_m\} \) is said to have the structural distortion if \( \sum_{i=1}^m k_i < n \), and the distortion degree of the system is \( n - \sum_{i=1}^m k_i \) and referred to as \( d \).

If the distortion degree \( d \) is reduced by using \( \mu \)-degree dynamic extension, we say that the structural distortion of a system is modified by \( \mu \)-degree dynamic extension. If the distortion degree becomes 0 by the dynamic extension, then the system is exactly linearizable via the dynamic extension. In this paper, we consider the condition to modify the structural distortion. The structural distortion is caused by the defect of the involutiveness. For example, a two-link underactuated robot which is known as acrobat, has the 1 degree distortion, since the relative degree structure is \( (3, 3, 2, 1) - [3] \), and this distortion comes from the defect of the involutiveness at \( \mathcal{G}^3 \). The defect of the involutiveness of distribution \( \mathcal{G} \) is referred to as the \( i \)th defect degree and defined by the relative degree structure as follows:

\[
d^i = \sigma^i - \gamma^i
\]

where the indices \( \sigma^i \) and \( \gamma^i \) are defined in Definition 1. The sum of \( i \)th defect degree is equal to distortion degree, that is, \( \sum_{i=1}^m d^i = d \). In the next, we prepare some lemmas to discuss the condition to modify the structural distortion of two-input systems by 1-degree dynamic extension.

Next lemma states inclusion relation between \( \mu \)-degree dynamic extension and the sequence of 1-degree dynamic extension.

**Lemma 1** \( \mu \)-degree dynamic extension (10) for two inputs system can be decomposed to the sequence of 1-degree dynamic extension.

**Proof** \( \mu \)-degree dynamic extension displayed in (10) is decomposed as follows:

\[
u_1 = w_1, \quad \tilde{w}_i = u_i+1, \quad i \geq 0,
\]

where \( w_1 \) is the \( i \)th extended state, \( u_i+1 \) is the virtual input of the \( i \)th 1-degree dynamic extension, \( u_i+1 = u_i \), and \( u_1 = v_1 \). Lemma 1 means that the class of the exactly linearizable system via the \( \mu \)-degree dynamic extension is included in the class of the exactly linearizable system via the \( \mu \) times sequence of the 1-degree dynamic extension. In the rest of this section, we consider the condition to modify the structural distortion via the 1-degree dynamic extension for input \( u_1 \), that is, the extended state \( w = w_1 \).

The extended system is rewritten as follows:

\[
\frac{dx}{dt} = F + G_1v_1 + G_2v_2,
\]

where \( \tilde{f} = f + g_1w \). The distributions of the extended system are

\[
\mathcal{G}_1 = \text{sp} \{G_1, G_2\} = \text{sp} \left\{ \frac{\partial}{\partial w}, g_2 \right\},
\]

\[
\mathcal{G}_2 = \text{sp} \{\text{ad}_w G_1, \text{ad}_w G_2, \mathcal{G}_1\} = \text{sp} \{g_1, \text{ad}_w g_2, \mathcal{G}_1\},
\]

\[
\vdots
\]

\[
\mathcal{G}_m = \text{sp} \{\text{ad}_w^{m-2} G_1, \text{ad}_w^{m-2} G_2, \mathcal{G}_{m-1}\} = \text{sp} \{\text{ad}_w^{m-3} g_1, \text{ad}_w^{m-3} g_2, \mathcal{G}_{m-1}\}.
\]

Following two lemmas state the properties of the extended system.
Lemma 2 \[ f, \tilde{G}_e^k \subset \mathcal{G}_e^{k+1}. \]

**Proof:** In the case of \( i = 1 \), \( [f, \tilde{G}_e^1] = \text{sp} \{ \text{ad}_f G_1 \} \) is in \( \mathcal{G}_e^2 \) since \( \text{ad}_f G_2 = \text{ad}_f G_2 - w[f, G_1] \in \mathcal{G}_e^2 \). Assume that a number \( k \geq 1 \) satisfies \( [f, \tilde{G}_e^k] \subset \mathcal{G}_e^{k+1} \). Let \( g \in \mathcal{G}_e^{k+1} \). If \( g = \text{ad}_f G_i \) (\( i = 1 \) or \( 2 \), \( k \geq j \geq 0 \)), then \( g_i \) and \( [f, g] \) are in \( \mathcal{G}_e^{k+2} \), moreover,

\[ [f, g] = [f, [f, f_i]] = [f_2, f_1] \equiv [f, [f, f_i]], \]

by Jacobi identity (2). The equation (13) is applied recursively until \( f_i, f_2 \) are represented by \( \text{ad}_f G_i \) (\( i = 1 \) or \( 2 \), \( j \geq 0 \)) respectively. After that we apply (12). Consequently, we get \( [f, g] \in \mathcal{G}_e^{k+2} \).

**Lemma 3** There exist following congruences among the vector fields of the original system and the extended system.

\[ \text{ad}_f G_1 \equiv \text{ad}_f G_1 \mod \mathcal{G}_e^{k+1}, \text{ if } i = 1, \ldots, k_1, \]

\[ \text{ad}_f G_2 \equiv \text{ad}_f G_2 \mod \mathcal{G}_e^{k+1} \text{ if } i = 1, \ldots, k_2. \]

**Proof:** We use induction.

\[ \text{ad}_f G_1 = [f + g_1 w, g_1] = \text{ad}_f G_1, \]

\[ \text{ad}_f G_2 = [f + g_2 w, g_2] = \text{ad}_f G_2 - [g_2, g_1] w, \]

\[ \text{ad}_f^2 G_1 = [f + g_1 w, \text{ad}_f G_1] = \text{ad}_f^2 G_1 + [g_1, \text{ad}_f G_1] w \]

\[ \equiv \text{ad}_f^2 G_1 \mod \mathcal{G}_e^2, \]

\[ \text{ad}_f^2 G_2 = [f + g_2 w, \text{ad}_f G_2] = [f + g_2 w, \text{ad}_f G_2 - [g_2, g_1] w] \]

\[ = \text{ad}_f^2 G_2 + [g_1, \text{ad}_f G_2] w - [f, [g_2, g_1] w] - [g_1, [g_2, g_1]] w^2. \]

Using Jacobi identity (2), we get

\[ [f, [g_2, g_1]] = [g_2, [f, g_1]]. \]

Thus \( \text{ad}_f^2 G_i \equiv \text{ad}_f^2 G_i \equiv \text{ad}_f^2 G_i \mod \mathcal{G}_e^2 \).

Assume the equations (14) and (15) work when \( i = k \). Then

\[ \text{ad}_f^{k+1} G_1 = [f + g_1 w, \text{ad}_f^{k+1} G_1] \]

\[ \equiv \text{ad}_f^{k+1} G_1 \mod \mathcal{G}_e^{k+2} \]

where the second congruence comes from Lemma 2.

\[ \text{ad}_f^{k+1} G_2 = [f + g_2 w, \text{ad}_f^{k+1} G_2] \]

\[ \equiv \text{ad}_f^{k+1} G_2 \mod \mathcal{G}_e^{k+2} \]

where last congruence comes from Jacobi identity. \( \blacksquare \)

Now we state the condition for modifying the structural distortion.

**Theorem 2** By 1-degree dynamic extension for input \( u_1 (10) \), the structural distortion of the two inputs system with relative degree structure \((r_1, \ldots, r_n) - [k_1, k_2] \) is modified at least 1 degree if and only if the following two conditions are satisfied.

1. \( \text{ad}_f^2 G_2 \equiv \text{ad}_f^2 G_2 \equiv \text{ad}_f^2 G_2 \mod \mathcal{G}_e^{k+1} \)

2. One of the following relations is satisfied

\[ \text{sp} \{ [g_2, \text{ad}_f^{k-1} G_1], [\text{ad}_f^{k-1} G_1, g_2], [\text{ad}_f^{k-1} G_1, g_2], [\text{ad}_f^{k-1} G_1, g_2], \} \]

where \( G_1 \in \mathcal{G}_e^{k+1} \) and \( k_1 \geq k_2 \).

**Proof:** If we assume that the relative degree structure of the extended system (11) is \((r_1', \ldots, r_n') - [k_1', k_2'] \). Since the original system has the relative degree structure \((r_1, \ldots, r_n) - [k_1, k_2] \), there exist two cases as follows:

Case 1. \( \text{sp} \{ [g_1, \ldots, \text{ad}_f^{k-1} G_1], [\text{ad}_f^{k-1} G_1, g_2], [\text{ad}_f^{k-1} G_1, g_2], [\text{ad}_f^{k-1} G_1, g_2], \} \) (16)

Next, we consider the conditions to modify the distortion in each case.

**Case 1.** In this case, the tangent space of the extended system (11) is spanned by \( \mathcal{G}_e^{k+1} \) and \( k_1' \leq k_1 \). The assumption of this case implies \( \text{ad}_f^{k+1} G_1 \in \text{sp} \{ \text{ad}_f^{k+1} G_2, \tilde{G}_e^{k+1} \} \). Then Lemma 3 says

\[ \text{ad}_f^{k+1} G_1 = \text{ad}_f^{k+1} G_1 \in \text{sp} \{ \text{ad}_f^{k+1} G_2, \tilde{G}_e^{k+1} \} \]

\[ \subset \text{sp} \{ \text{ad}_f^{k+1} G_2, \tilde{G}_e^{k+1} \} \subset \mathcal{G}_e^{k+2} \].

It implies that

\[ \mathcal{G}_e^{k+2} = \text{sp} \{ \text{ad}_f^{k+1} G_2, \text{ad}_f^{k+1} G_1, \tilde{G}_e^{k+1} \} = \text{sp} \{ \text{ad}_f^{k+1} G_2, \tilde{G}_e^{k+1} \} \]

This means that \( k_1' \leq k_1 \). Thus the distortion degree of the extended system is at least \( n + 1 - (k_1 + k_2 + 1) = n - (k_1 + k_2) \), and no degree is modified by the dynamic extension.

**Case 2.** In this case, the tangent space of the extended system is spanned by \( \mathcal{G}_e^{k+1} \), and \( k_1' \leq k_1 + 1 \). Assume \( k_1' < k_1 + 1 \). The assumption of Case 2 implies that \( \text{ad}_f^{k+1} G_1 \in \tilde{G}_e^{k+1} \), and \( k_1' < k_1 + 1 \) means \( \text{ad}_f^{k+1} G_1 \in \text{sp} \{ \text{ad}_f^{k+1} G_2, \tilde{G}_e^{k+1} \} \) since \( \text{ad}_f^{k+1} G_1 \in \mathcal{G}_e^{k+1} \). This means that \( \text{ad}_f^{k+1} G_1 \not\in \mathcal{G}_e^{k+2} \), and this case is equal to Case 1. Thus no degree is modified. Thus we consider only the case \( k_1' = k_1 + 1 \).

The distortion of the system is modified if \( k_1' \) is larger than \( k_2 \). That is, under the assumption, all we have to check is

\[ \text{ad}_f^{k+1} G_2 \equiv 0 \mod \text{sp} \{ \text{ad}_f^{k+1} G_2, \tilde{G}_e^{k+1} \}. \]

**Lemma 3** means

\[ \text{ad}_f^{k+2} G_2 = \text{ad}_f^{k+2} G_2 \equiv \text{ad}_f^{k+2} G_2 \equiv \text{ad}_f^{k+2} G_2 \mod \mathcal{G}_e^{k+2} \]

The assumption implies that \( \text{ad}_f^{k+2} G_2 \equiv \text{sp} \{ \text{ad}_f^{k+1} G_1, \tilde{G}_e^{k+1} \} \)

(17) is rewritten as follows:

\[ \text{ad}_f^{k+2} G_2 \equiv \text{ad}_f^{k+2} G_2 \equiv \text{ad}_f^{k+2} G_2 \mod \mathcal{G}_e^{k+2} \]

(16)
where $m_{k_2} = \dim \mathcal{G}^{k_2}$ and $\{\partial_{g_k}^{k_2}\}_{k1=1}^{k2}$ are smooth scalar functions of $x$ and $w$. If $k_1 = k_2$, then $a_0 = 0$ since $\text{ad}^{2}_{g}g_1$ and $\text{ad}^{2}_{g_2}g_2 \in \mathcal{G}^{k_2}$. If $k_1 > k_2$ and $a_0$ is not zero, then the system is also belong to Case 1, and as mentioned in Case 1, no degree is modified by the dynamic extension. Thus we consider only $a_0 = 0$ for $k_1 \geq k_2$. The assumption of Case 2 and $a_0 = 0$ correspond to the condition 1 in Theorem 2.

In order to satisfy the condition (16), at least, one of the functions $\{\alpha_1, \ldots, \alpha_m\}$ must be nonzero. Notice that, from (17), if $\{g_2, \text{ad}^{k_2-1}_{g}g_1\} \notin \text{sp} \{\text{ad}^{k_2}_{g}G_1, \mathcal{G}^{k_2}\}$, then the condition (16) is satisfied, and it corresponds to the first part of the condition 2 in Theorem 2. The rest of the condition 2 guarantees that one of the functions $\{\alpha_1, \ldots, \alpha_m\}$ is nonzero, and the extended system satisfies (16).

**Remark 2** Using the distribution of the extended system, the condition 2 of Theorem 2 is simply expressed as follows:

$$\{g_2, \text{ad}^{k_2+1}_{g}g_1\} \notin \text{sp} \{\text{ad}^{k_2}_{g}G_1, \mathcal{G}^{k_2}\}, \quad \forall \theta \in \mathcal{G}^{k_2} \text{ such that } \text{ad}^{k_2}_{g_2}g_2 \equiv [g, \text{ad}^{k_2}_{g}g_1] \mod \text{sp} \{\text{ad}^{k_2}_{g}G_1, \mathcal{G}^{k_2}\}.$$ 

This replacement is based on the fact that $\text{sp} \{\mathcal{G}^{k_2-1}, \mathcal{G}^{k_2}\}$ is equal to $\text{sp} \{\text{ad}^{k_2}_{g}G_1, \mathcal{G}^{k_2}\}$ except for $\frac{\partial}{\partial w}$ and $\frac{\partial}{\partial \eta}$ has no mean for this condition.

**Remark 3** If a system satisfies the conditions of Theorem 2 by means of some input transformation for the original system, then the distortion of the transformed system is modified by 1-degree dynamic extension (10). As discussed above, the condition 1 of Theorem 2 means that the tangent space is spanned by $\text{sp} \{g_1, \ldots, \text{ad}^{k_2-1}_{g}g_1, g_2, \ldots, \text{ad}^{k_2-1}_{g_2}g_2\}$. That is to say, the extended input $u_1$ is associated with the index $k_1$. This implies that we should extend the input associated with $k_1$ to modify a distortion by 1-degree dynamic extension.

**Corollary 1** If $\mathcal{G}^{k_2}$ is involutive, then the distortion of the system is not modified via 1-degree dynamic extension (10).

**Proof:** If $\mathcal{G}^{k_2}$ is involutive, then the condition 2 of Theorem 2 is not satisfied.

In other words, the condition 2 of Theorem 2 says that if the $k_2$th defect degree is 0, then the distortion of the system is not modified by 1-degree dynamic extension (10).

**Corollary 2** If the distortion of the system can be modified via 1-degree dynamic extension (10), then $\dim \mathcal{G}^{k_2} > \dim \mathcal{G}^{k_2}$.

**Proof:** For the system whose distortion can be modified via 1-degree dynamic extension (10), there is a relation between the distributions $\mathcal{G}^{k_2}$ and $\mathcal{G}^{k_2}$ as follows:

$$\text{dim} \text{inv} \{\text{sp} \{\mathcal{G}^{k_2}, \frac{\partial}{\partial G}\} \} = \text{dim} \text{sp} \{\mathcal{G}^{k_2}, \text{ad}^{k_2+1}_{g}g_1\}.$$ 

It is trivial that $\text{dim} \text{inv} \{\text{sp} \{\mathcal{G}^{k_2}, \frac{\partial}{\partial \eta}\} \} = \dim \mathcal{G}^{k_2} + 1$. Since the distortion is modified by 1-degree dynamic extension (10), Remark 3 implies $\text{ad}^{k_2}_{g}g_1 \notin \mathcal{G}^{k_2}$. This means $\text{dim} \text{inv} \{\text{sp} \{\mathcal{G}^{k_2}, \text{ad}^{k_2+1}_{g}g_1\} \} = \dim \mathcal{G}^{k_2} + 1$. Therefore, $\dim \mathcal{G}^{k_2} \geq \dim \mathcal{G}^{k_2}$.

If $\dim \mathcal{G}^{k_2} = \dim \mathcal{G}^{k_2}$, then $\text{dim} \text{inv} \{\text{sp} \{\mathcal{G}^{k_2}, \text{ad}^{k_2+1}_{g}g_1\} \} = \dim \mathcal{G}^{k_2} + 1$, and $\text{inv} \{\mathcal{G}^{k_2}, \text{ad}^{k_2+1}_{g}g_1\} = \text{sp} \{\text{ad}^{k_2}_{g}G_1, \mathcal{G}^{k_2}\} = \text{sp} \{\text{ad}^{k_2}_{g}G_1, \mathcal{G}^{k_2}\}$, where the second equality comes from Lemma 3. Moreover, $\text{inv} \{\text{sp} \{\mathcal{G}^{k_2}, \mathcal{G}^{k_2}\} \} = \text{inv} \{\text{sp} \{\text{ad}^{k_2}_{g}G_1, \mathcal{G}^{k_2}\} \} = \text{sp} \{\text{ad}^{k_2}_{g}G_1, \mathcal{G}^{k_2}\}$.

Then the system can not satisfy the condition 2 of Theorem 2. It contradicts the assumption, and thus $\dim \mathcal{G}^{k_2} > \dim \mathcal{G}^{k_2}$.

In the next theorem, we formulate the condition of the vector fields of the original system, in order for the distortion to be modified more than 1 degree via the 1-degree dynamic extension (10).

**Theorem 3** By 1-degree dynamic extension for input $u_1$ (10), the structural distortion of the two inputs system with relative degree structure $(r_1, \ldots, r_m)$ is modified by $k > 1$ degree if and only if the following two conditions are satisfied.

1. $\text{ad}^{k_2}_{g}g_2 \in \mathcal{G}^{k_2}$.

2. There exists positive number $k > 1$ which is the largest number satisfying $\{g_2, \text{ad}^{k_2}_{g}g_2\} \notin \text{sp} \{\mathcal{G}^{k_2}, \text{ad}^{k_2}_{g}g_2\}$.

(18)

**Proof:** Similar to the proof of Theorem 2, the condition 1 implies that the vector field $\text{ad}^{k_2}_{g}g_2$ is described as follows:

$$\text{ad}^{k_2}_{g}g_2 = \sum_{i=1}^{m_{k_2}} f_i, \quad \text{where } \mathcal{G}^{k_2} = \text{sp} \{f_1, \ldots, f_{m_{k_2}}\}.$$ 

Then $\{f, \text{ad}^{k_2}_{g_2}g_2\} = \sum_{i=1}^{m_{k_2}} \{f, f_i\}$. If $f = \text{ad}^{k_2}_{g}g_2$ ($j = 0, \ldots, k_2 - 1, k = 1$ or 2), then $\{f, f_j\} \in \mathcal{G}^{k_2+1}$.

Otherwise we apply the Jacobi identity (2) recursively as shown in the proof of Lemma 2, and then $\{f, f_j\}$ is also in $\mathcal{G}^{k_2+1}$. Thus $\text{ad}^{k_2}_{g_2}g_2 \in \mathcal{G}^{k_2+1}$. Moreover, by similar discussion with the proof of Lemma 3, $\text{ad}^{k_2+1}_{g}g_2 \equiv -[g_2, \text{ad}^{k_2}_{g}g_1] w \mod \mathcal{G}^{k_2+1}$. By induction, $\text{ad}^{k_2+1}_{g}g_2 \equiv -[g_2, \text{ad}^{k_2+1}_{g}g_1] w \mod \mathcal{G}^{k_2+1}$. Then $\text{ad}^{k_2+1}_{g}g_2 \equiv -[g_2, \text{ad}^{k_2+1}_{g}g_1] w \mod \mathcal{G}^{k_2+1}$. By Lemma 3, $\text{ad}^{k_2+1}_{g}g_2 \equiv \text{ad}^{k_2+1}_{g_2}g_1 \mod \mathcal{G}^{k_2+1}$. Note that Remark 3 implies that $\text{ad}^{k_2}_{g_2}g_2 \notin \text{sp} \{\text{ad}^{k_2}_{g_1}, \mathcal{G}^{k_2}\}$, $i = 1, \ldots, k$ is necessary condition for the structural distortion to be modified $k$ degree in 1-degree dynamic extension (10). It means that the system satisfies (18) if the structural distortion is modified by $k$ degree. As discussed in Theorem 2, the condition 1 is necessary for modification of the structural distortion. Thus the conditions 1 and 2 of Theorem 3 are satisfied if the structural distortion is modified by $k$ degree, and the necessity of Theorem 3 is proven.
The sufficiency is proven as follows. The condition (18) means that \( \text{ad}_{f}^{k_2}\bar{G}_1 \not\parallel \text{sp}\{\text{ad}_{f}^{k_2}G_1, \bar{G}_e^{k_2}\} \). Thus \( \dim \bar{G}_e^{k} > \dim \bar{G}_e^{k-1} = 2 \). To complete the proof, we consider following claim.

**Claim**: If there exists \( k > 1 \) satisfying (18), then the relation (18) is satisfied for all \( i \in \{1, \ldots, k\} \).

**Proof of Claim**: Assume that (18) is not satisfied by \( i = k - 1 \). By the discussion in Remark 2, this assumption is rewritten as follows:

\[
\left[ g_2, \text{ad}^{k_2}G_1 \right] \in \text{sp}\{\text{ad}^{k_2}G_1, \bar{G}_e^{k_2}\}
\]

where \( \text{ad}^{k_2}G_1 = \text{ad}^{k_2}g_1 \equiv \text{ad}g_1 \mod \bar{G}_e^{k+1} \) by Lemma 3. Then

\[
\left[ f, \left[ g_2, \text{ad}^{k_2}G_1 \right] \right] \in \text{sp}\{\text{ad}^{k_2}G_1, \bar{G}_e^{k+1}\},
\]

where we use Lemma 2. Using Jacobi identity,

\[
\left[ g_2, \left[ \text{ad}^{k_2}G_1 \right] \right] \in \text{sp}\{\text{ad}^{k_2}G_1, \bar{G}_e^{k+1}\}.
\]

This contradicts the assumption of the claim. Thus equation (18) is satisfied by \( i = k - 1 \). By induction, the claim is proven.

This claim guarantees that the conditions 1 and 2 cover the conditions of Theorem 2, and the distortion of the system can be modified at least 1 degree. Moreover \( \dim \bar{G}_e^{k+1} = \dim \bar{G}_e^{k+1} = 2 \) (1) \( i = 1, \ldots, k \) since the claim guarantees

\[
\text{ad}_{f}^{k_2}G_1 \not\parallel \text{sp}\{\text{ad}_{f}^{k_2}G_1, \bar{G}_e^{k_2}\}, \quad 1 \leq i \leq k.
\]

This means that the distortion degree of the extended system is \( n + 1 - (k_1 + k_2 + k + 1) = n - (k_1 + k_2 + k) \). Thus the distortion is modified by \( k \) degree.

If the distortion degree \( d \) is equal to \( k \), then the system is exactly linearizable via 1-degree dynamic extension. Following example shows the 2 degree modification by means of 1-degree dynamic extension.

**Example 2** Consider the two inputs system

\[
\dot{x} = f(x) + g_1u_1 + g_2(x)u_2
\]

where

\[
f(x) = \left[ \begin{array}{c} x_2 \\ x_3 \\ 0 \\ -x_1x_6 + x_3x_5 + x_3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array} \right]^T
\]

\[
g_1 = \left[ \begin{array}{ccccccc} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & x_3 \\ \end{array} \right]^T
\]

\[
g_2(x) = \left[ \begin{array}{ccccccc} 0 & 0 & 0 & 0 & 1 & x_3 \\ \end{array} \right]^T
\]

The relative degree structure of this system is \((3, 2, 2, 2, 1, 1) - [3, 1]\). The distortion degree is 2, and the defects of the involutiveness exist in \( \bar{G}_1 \) and \( \bar{G}_2 \). This system satisfies the conditions of Theorem 2 since 1st defect degree is 1, \( \text{ad}g_2 \in \bar{G}_1 \) and \( g_2, g_1 \not\parallel \bar{G}_2 \). Moreover this system also satisfies the conditions of Theorem 3, that is, \( \bar{g}_2, \text{ad}g_1, \not\parallel \bar{G}_2 \). This means that the distortion degree of this system is modified 2 degree by 1-degree dynamic extension. Indeed, by 1-degree dynamic extension (10), the system is transformed into

\[
\frac{d}{dt} \begin{bmatrix} x \\ w \end{bmatrix} = \begin{bmatrix} f(x) + g_1w \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v_1 + \begin{bmatrix} g_2(x) \\ 0 \end{bmatrix} v_2.
\]

The relative degree structure of the extended system is \((4, 3, 3, 2, 1, 1) - [4, 3]\). Thus the extended system has the relative degree structure which is equivalent to the one of linear system and is exactly linearizable.

As mentioned in Remark 1, the dynamic extension might change the relative degree structure, and modify the structural distortion. The input transformation with the dynamic extension also has the ability to change the relative degree structure, and in the next section, the input transformation is considered.

### 4. Classification of Input Transformation

In Section 3, we only consider the extension of the input \( u_1 \). In this section, the input transformation is considered. The following example indicates the importance of the input transformation.

**Example 3** Consider the following system

\[
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = f + g_1u_1 + g_2u_2.
\]

This system has the relative degree structure \((2, 1, 1, 1) - [2, 1]\) and \( \text{ad}g_1, \text{ad}g_2 \not\parallel \bar{G}_2 \). Theorem 2 shows that the structural distortion of this system is not modified via (10). Now consider the input transformation

\[
\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.
\]

After this input transformation, this system satisfies the conditions of Theorem 2, and then the structural distortion is modified via (10). Moreover the transformed system is exact linearizable via dynamic extension (10).

This example shows that the input transformation with dynamic extension has the ability to modify the structural distortion even if the structural distortion with dynamic extension only is not modified. In the following text, the input transformation (20) is discussed to select a proper input to be extended.

\[
u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} a_1(x) \\ a_2(x) \end{bmatrix} + \begin{bmatrix} b_{11}(x) \\ b_{21}(x) \end{bmatrix} v_1 + \begin{bmatrix} b_{12}(x) \\ b_{22}(x) \end{bmatrix} v_2
\]

where \( b(x) \) is nonsingular matrix. As shown in above example, the restriction which is set in Section 3 is relaxed by the input transformation (20), that is, the combination of the input transformation (20) and the dynamic extension (10) relaxes the restriction on the selection of the input which is extended. As a result, the combined transformation is

\[
u = \begin{bmatrix} a_1(x) \\ a_2(x) \end{bmatrix} + \begin{bmatrix} b_{11}(x) \\ b_{21}(x) \end{bmatrix} w_1 + \begin{bmatrix} b_{12}(x) \\ b_{22}(x) \end{bmatrix} v_2,
\]

where \( \omega_1 = v_1 \).

Note that some input transformations with the form (20) generate the equivalent extended systems with respect to the feedback equivalence. Next theorem classifies the input transformation based on the equivalence under the feedback transformation of the extended system.
Theorem 4 The static state feedback for two inputs system with the dynamic extension (10) is classified into following two forms:

\[ IT1 : \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} a(x) & b(x) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u'_1 \\ u'_2 \end{bmatrix}, \]

\[ IT2 : \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 & a(x) \\ b(x) & c(x) \end{bmatrix} \begin{bmatrix} u'_1 \\ u'_2 \end{bmatrix}, \]

where \( b(x) \neq 0. \)

Proof: Consider the general input transformation (20). There exist following two cases.

Case 1. \( b_{22}(x) = 0 \) : The extended system is

\[ \dot{z} = \left[ f + a_1 g_1 + a_2 g_2 + (b_1 g_1 + b_2 g_2) w_1 \right] A w + \begin{bmatrix} 0 \\ B \end{bmatrix} v_1 + \begin{bmatrix} b_2 g_2 \\ 0 \end{bmatrix} v_2, \]

where \( A \) and \( B \) are constant matrix and describe the dynamics (21b), and \( w = [w_1, \ldots, w_{\mu-1}]^T. \)

The input transformation for extended system

\[ \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{b_{22}}(a_2 + b_2 w_1) \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{b_{22}} \end{bmatrix} \begin{bmatrix} v'_1 \\ v'_2 \end{bmatrix} \]

transforms the extended system into

\[ \dot{z} = \left[ f + a_1 g_1 + b_11 g_1 w_1 \right] A w + \begin{bmatrix} 0 \\ B \end{bmatrix} v_1 + \begin{bmatrix} g_2 \\ 0 \end{bmatrix} v_2. \]

This transformed system is equivalent to the extended system with IT1, where \( a(x) = a_1(x) \) and \( b(x) = b_11(x). \)

Case 2. \( b_{22}(x) \neq 0 \) : The input transformation for the extended system

\[ \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{b_{22}}(a_1 + b_11 w_1) \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{b_{22}} \end{bmatrix} \begin{bmatrix} v'_1 \\ v'_2 \end{bmatrix} \]

transforms the extended system into

\[ \dot{z} = \left[ \tilde{f} + a_1 g_1 + b_11 g_1 w_1 \right] A w + \begin{bmatrix} 0 \\ B \end{bmatrix} v_1 + \begin{bmatrix} g_2 \\ 0 \end{bmatrix} v_2, \]

where \( \tilde{f} = f + (a_2 - a_1 \frac{b_2}{b_{22}}) g_2 + (b_21 - b_11 \frac{b_2}{b_{22}}) g_2 w_1. \)

This transformed system is equivalent to the extended system with input transformation IT2, where \( a(x) = a_2(x) - a_1 \frac{b_2(x)}{b_{22}(x)}, \)

\( b(x) = b_2(x) - b_{11}(x) \frac{b_2}{b_{22}} \) and \( c(x) = b_{22}(x). \)

Theorem 4 means that the input transformation (20) and dynamic extension (10) are not commutable and that the set of all input transformation for two inputs system with dynamic extensions can be considered as the set of input transformations expressed by (IT1) or (IT2). Owing to this classification, in the following part, we consider at most 3 functions: \( a(x), b(x), \) and \( c(x) \) to construct the input transformation.

5. Algorithm to Derive the Input Transformation

In this section, the method to derive the linearizing transformations is proposed. First of all, the outline is shown, and after that, we explain the detail.

Step 1. Calculate the relative degree structure by Definition 1.
Step 2. Select an extension degree \( \mu. \)

Step 3. Derive the necessary and sufficient condition for the system to be exactly linearizable via \( \mu \)-degree dynamic extension (10), and describe it by the vector fields of the original system.
Step 4. Derive the input transformation for the original system to satisfy the conditions derived in Step 3.
Step 5. Get the extended system by applying the input transformation and the dynamic extension.
Step 6. Transform the extended system into linear system.

Then we give some explanations for each step.

About Step 2: The theorems derived in Section 3 give a useful information about the degree of extension. In general, smaller degree system is easy to handle. Namely, the smaller extension degree is better. If \( \mathcal{G}^2 \) is not involutive, then the distortion of the system may be modified by 1-degree dynamic extension. By Corollary 1, if \( \mathcal{G}^2 \) is involutive, then the distortion is not modified by 1-degree dynamic extension. In this case, we only use the extension to arrange the defects of involutiveness. As mentioned in Remark 1, the dynamic extension changes the relative degree structure of the system, and then the defects of involutiveness also are changed. After that, the distortion of the extended system might be modified by the 1-degree dynamic extension. As seen above, we can get the perspective on the choice of the extension degree using Theorem 2 and Theorem 3.

About Step 3: Assume that the extension degree is \( \mu. \)

Then the extended system is

\[ \frac{dx}{dt} \begin{bmatrix} x \\ w \end{bmatrix} = \begin{bmatrix} f(x) + g_1(x) w_1 \\ A w \end{bmatrix} + \begin{bmatrix} 0 \\ B \end{bmatrix} v_1 + \begin{bmatrix} g_2(x) \\ 0 \end{bmatrix} v_2, \]

where \( w = [w_1, \ldots, w_{\mu-1}]^T. \) Applying Theorem 1, the necessary and sufficient condition for the extended system to be exactly linearizable is derived as the condition of the distributions of the extended system as follows:

\[ \mathcal{G}^i = \mathcal{G}^i \quad \forall i \in \mathbb{N} \]

\[ \exists k < \infty \text{ s.t. } \dim \mathcal{G}^k = n + \mu \]

Based on the relative degree structure of the original system, this condition is transformed into the condition of the distributions of the original system. For example, we derive the necessary and sufficient condition for the 6 states 2 inputs system with \( (2,2,2,2,1,1) \rightarrow (2,2) \) as follows.

Theorem 5 The system with \( (2,2,2,2,1,1) \rightarrow (2,2) \) is exactly linearizable via 2-degree dynamic extension (10) if and only if the system vector fields satisfy the following conditions.

1. \( [g_1, g_2] \in \mathfrak{sp} \{g_2\}. \)
2. \( \mathfrak{sp} \{g_2, ad_{g_2} f\} \) is involutive.
3. \( \mathfrak{sp} \{g_1, g_2, ad_{g_2} g_2, [g_1, ad_{g_2} g_2], [g_1, ad_{g_2} g_2] \} \) is involutive and has 4 dimension.
4. \( \mathfrak{sp} \{g_1, g_2, ad_{g_2} f, g_2, [g_1, ad_{g_2} g_2], [g_1, ad_{g_2} g_2] \} \) has 6 dimension.
The system distributions of the extended system are
\[
\mathcal{G}_2 = \text{sp}\left\{g_2, \frac{\partial}{\partial \theta_1}\right\}, \quad \mathcal{G}_3 = \text{sp}\left\{\text{ad}_{g_2} \frac{\partial}{\partial \theta_1}, \mathcal{G}_3\right\},
\]
\[
\mathcal{G}_4 = \text{sp}\left\{\text{ad}_2 \mathcal{G}_2, \mathcal{G}_3\right\}, \quad \mathcal{G}_5 = \text{sp}\left\{\text{ad}_2 \mathcal{G}_2, \mathcal{G}_4\right\}.
\]

Since \(\mathcal{G}_2 \subset \mathcal{G}_4\), \(\mathcal{G}_3\) spans the tangent space at each point on \(\mathbb{R}^n\).

By Theorem 1, the necessary and sufficient condition for exact linearization is \(\mathcal{G}_2 = \mathcal{G}_i\) for \(i = 1, 2, 3\) and \(\dim \mathcal{G}_4 = 8\). Then if the system with \((2, 2, 2, 1, 1)\) is exactly linearizable by 2-degree dynamic extension, then the relative degree structure of the extended system is \((4, 4, 3, 3, 2, 1, 1)\).

\(\mathcal{G}_2\) is a 2-dimensional involutive distribution trivially. \(\dim \mathcal{G}_3 = 4\) since \(\{g_2, \text{ad}_2 g_2\}\) is independent. The distribution \(\text{sp}\{G_1, G_2, \text{ad}_2 G_1\} = \text{sp}\{g_1, g_1, g_1, g_1\}\) is a 3-dimensional involutive distribution, where \(F_1, G_1, G_2\) are vector fields of the extended system. Then \(\mathcal{G}_3\) is involutive if and only if

\[
\begin{align*}
[G_1, \text{ad}_2 G_2] &= \left[\frac{\partial}{\partial \theta_1}, \text{ad}_2 g_2\right] = 0, \\
[2, G_2, \text{ad}_2 G_3] &= \left[g_2, \text{ad}_2 g_2 + [g_1, g_1] w_1\right], \\
[\text{ad}_2 G_1, \text{ad}_2 G_2] &= \left[g_1, g_1\right].
\end{align*}
\]

are in \(\mathcal{G}_5\). If \(\{g_2, [g_1, g_2]\}\) is independent, then \(\text{sp}\{g_2, [g_1, g_2]\} = \text{sp}\{g_1, g_2\}\). This means that \(\{g_2, \text{ad}_2 g_2, \text{ad}_2 g_3, [g_1, g_1]\}\) is independent, and then \(\mathcal{G}_3\) is involutive. Thus the condition 1 is needed. As a result, \(\mathcal{G}_3 = \text{sp}\{g_2, \text{ad}_2 g_2, \frac{\partial}{\partial \theta_1}\}\), and the condition 2 is needed for the involutiveness of \(\mathcal{G}_3\).

Next, let us consider the condition that \(\mathcal{G}_4\) is a 6-dimensional involutive distribution. Vector fields \(\text{ad}_2 g_2 + [g_1, \text{ad}_2 g_1] w_1\) and \(g_1\) appear in \(\mathcal{G}_4\). Thus \(\mathcal{G}_2\) is a 6-dimensional involutive distribution if and only if \(\text{sp}\{g_1, g_1, \text{ad}_2 g_2, \text{ad}_2 g_3, [g_1, g_1]\}\) is involutive and 4-dimensional. By the involutiveness of \(\mathcal{G}_3\), the vector fields \([g_1, \text{ad}_2 g_2]\) and \([g_1, g_1, \text{ad}_2 g_3]\) are in \(\mathcal{G}_5\). Thus \(\text{ad}_2 G_2 \equiv \text{ad}_2 g_2 + [f_1, g_1, \text{ad}_2 g_1] w_1 \mod \mathcal{G}_5\). The last condition is needed for \(\dim \mathcal{G}_4 = 8\).

In this step, the necessary and sufficient condition is derived based on the restriction of the dynamic extension described in (10), that is, the extended input is \(u_1\). This restriction of the extension is important in order to derive the necessary and sufficient condition. On account of this restriction, the condition (22) of the distributions of the extended system is transformed into the condition of the distributions of the original system. In the next step, the input transformation is introduced to relax the restriction on the selection of the input to be extended.

**About Step 4:**

To enhance the dynamic extension, we use the input transformation as mentioned in Section 4. For two-input systems, Theorem 4 says that general input transformations are classified into two forms. As a result, only three functions \(a(x), b(x), c(x)\) are supposed to be designed to construct the input transformation.

The existence of an input transformation depends on a system. However we can check the existence of an input transformation by constructing the concrete input transformation based on the conditions derived in step 3. If there exists an input transformation which changes the vector fields of the original system to satisfy the conditions derived in step 3, then the system is exactly linearizable via the dynamic extension. Otherwise, the system is not exactly linearizable via \(\mu\)-degree dynamic extension.

**About Steps 1, 5, and 6:** These steps are simple calculation based on the definitions and algorithm.

In the next section, the procedure proposed in this section is demonstrated.

### 6. Deriving Example of Exact Linearization via Dynamic Extension: Inverted Pendulum System Utilizing Vertical and Horizontal Inputs \[12\]

Consider the simplified system of the inverted pendulum system utilizing vertical and horizontal inputs as shown in Fig. 1. This system has the strong nonlinearity and can be transformed into bilinear system by using feedback transformation as shown in \([12]\). In this section, we linearize this system by using the method proposed in Section 5. The state space realization of this system is

\[
\begin{align*}
\dot{x} &= f(x) + g_1(x) u_1 + g_2(x) u_2, \\
x &= [x_H \ x_V \ \dot{x}_H \ \dot{x}_V \ \dot{y} \ \theta], \\
&= [x_H \ x_V \ \dot{x}_H \ \dot{x}_V \ \dot{y} \ \theta], \\
g_1 &= [0 \ 0 \ 0 \ 0 \ \theta], \\
g_2 &= [0 \ 0 \ 0 \ 0 \ \theta],
\end{align*}
\]

where \(h = (\dot{x}_H \cos \theta - \dot{x}_V \sin \theta) + \theta, \quad p = \frac{m}{1 + m}, \quad q = \frac{m}{1 + m}, \quad r = \rho \left(\dot{x}_H \cos \theta - \dot{x}_V \sin \theta\right) + \theta, \quad \beta = \frac{m L}{M + m \rho \sin^2 \theta}.\)

For a simple calculation, this state space realization has already been transformed into input-output normal form presented in \([1]\). The coordinate function \(h\) is selected so that its relative degree is 2. All variables conform with Fig. 1, that is, \(m, M, l, I\) are physical parameters, \(g\) is gravity constant, \(x_H, x_V, \theta\) denote generalized coordinates, and \(F_H, F_V\) are respectively vertical and horizontal inputs. We apply the linearizing method proposed in Section 5 to this system.

**Step 1.** In order to derive the relative degree structure, we calculate the distributions defined in (3) and their involutive closure as follows:

\[
\begin{align*}
\mathcal{G}_1 &= \mathcal{G}_1 = \text{sp}\{g_1, g_2\}, \\
\mathcal{G}_2 &= \text{sp}\{\text{ad}_2 g_1, \text{ad}_2 g_2, \mathcal{G}_1\}, \\
\mathcal{G}_3 &= \text{sp}\{g_1, \text{ad}_2 g_2, [\text{ad}_2 g_1, \text{ad}_2 g_2]\}, \\
\mathcal{G}_4 &= \text{sp}\{g_1, \text{ad}_2 g_2, [\text{ad}_2 g_1, \text{ad}_2 g_2]\}, \\
\mathcal{G}_5 &= \text{sp}\{g_1, \text{ad}_2 g_2, [\text{ad}_2 g_1, \text{ad}_2 g_2]\},
\end{align*}
\]
Then $\gamma^1 = 2, \gamma^2 = 2$ and $\sigma^1 = 2, \sigma^2 = 4$. Thus the relative degree structure is $\mathfrak{g}(2) = (2, 2, 2, 1, 1) - [2, 2]$. 

**Step 2.** By the relative degree structure, 2nd defect degree is 2. Thus the system has possibilities of modification by 1-degree dynamic extension. However the equation (20) is not satisfied by $k = 2$ since the distribution at right hand side of (18) spans full space. Then Theorem 3 says that the system is not exactly linearizable via 1-degree dynamic extension. Thus the extension degree have to be larger than 2.

**Step 3.** The necessary and sufficient condition for the system with (22) to be exactly linearizable via 2-degree dynamic extension is given in Theorem 5.

**Step 4.** The system (23) does not satisfy the condition 2 of Theorem 5, and the system with the input transformation IT1 also fails the condition 2. We seek the input transformation IT2 so that the transformed system satisfies the conditions in Theorem 5.

The transformed vector fields are

$$f' = f + ag_2, \quad g'_1 = bg_2, \quad g'_2 = g_1 + cg_2.$$ 

Before the conditions in Theorem 5 are discussed, we list the property of the vector fields of the original system (23). By calculating the Lie bracket, we get the following equations:

$$[g_1, g_2] = 0, \quad (24)$$

$$[g_1, \text{ad}_f g_2] = [g_2, \text{ad}_f g_1], \quad (25)$$

$$[g_1, \text{ad}_f g_2] = T[g_1, \text{ad}_f g_1], \quad T = \frac{1 - 2\sin^2 x_0}{2\cos x_0 \sin x_0}, \quad (26)$$

$$[g_2, \text{ad}_f g_2] = -[g_1, \text{ad}_f g_1]. \quad (27)$$

By (24), we get

$$\text{ad}_f g'_2 = \text{ad}_f g_1 + \text{ad}_f g_2 + \left(\mathcal{L}_{f'} c = \mathcal{L}_{g'_1} a\right) g_2.$$ 

Using (25), (26), and (27),

$$[g'_2, \text{ad}_f g'_2] = (1 + 2cT - c^2)[g_1, \text{ad}_f g_1] + \left(\mathcal{L}_{g'_1} c = \mathcal{L}_{g'_2} a\right) \text{ad}_f g_2 \quad \text{and} \quad g'_2$$

$$+ \left(\mathcal{L}_{g'_1} c = \mathcal{L}_{g'_2} a\right) \text{ad}_f g_2 - \left(\mathcal{L}_{g'_2} a\right) \mathcal{L}_{g'_1} c \right) g_2. \quad (28)$$

By simple calculation, the vector fields, $[g_1, \text{ad}_f g_1], \text{ad}_f g_2, \text{ad}_f g_1, g_1$ and $g_2$ are independent. Thus to satisfy $[g'_2, \text{ad}_f g'_2] \in [g'_2, \text{ad}_f g'_2]$, the coefficients of $[g_1, \text{ad}_f g_1], \text{ad}_f g_2$ and $g_2$ in (28) have to be 0. By the equation,

$$1 + 2cT - c^2 = 0,$$

the function $c$ is derived as follows:

$$c = T + \sqrt{T^2 + 1} = \frac{\cos x_6}{\sin x_6} \quad \text{or} \quad -\tan x_6.$$ 

In the case that $c = \frac{\cos x_6}{\sin x_6}$, there exist $a$ and $b$ such that the transformed vector fields satisfy the condition 1 and 2 of Theorem 5. However these vector fields do not satisfy the condition 3. Then we consider the case $c = -\tan x_6$.

In this case, we get

$$\mathcal{L}_{g'_1} c = 0, \quad \mathcal{L}_{\text{ad}_f g'_1} c = 0.$$ 

Then the coefficient of the second term in (28) is 0. The coefficient of the third term is 0 if

$$\mathcal{L}_{g'_1} c = \mathcal{L}_{f'} c = \frac{\rho x_4 \cos x_6 + x_5}{\sin^2 x_6}.$$ 

The condition for $a$ is

$$\frac{\partial a}{\partial x_3} + a \frac{\partial a}{\partial x_4} = -\frac{\rho x_4 \cos x_6 + x_5}{\sin^2 x_6}.$$ 

Thus in order for the coefficient of $g'_1$ in (28) to be 0, $a$ is

$$a = \rho \frac{x_3^2 + x_4^2}{\cos x_6} - \frac{2x_3x_4}{\cos^2 x_6} + a',$$ 

$$a' = (a_1 x_3 + a_2 x_4 + a_3)(x_3 \tan x_6 + x_4 + a_4),$$

where $a_1, a_2, a_3, a_4$ are functions of $x_1, x_2, x_3, x_6$. Moreover to satisfy the condition 1 of Theorem 5, $b$ is derived as follows:

$$b = b_1(x_3 \tan x_6 + x_4 + b_2),$$

where $b_1, b_2$ are functions of $x_1, x_2, x_3, x_6$, and $b \neq 0$. Applying the input transformation IT2 with the functions $a(x), b(x), c(x)$ derived above, the vector fields of the transformed system satisfy the condition 1 and 2 of Theorem 5. Moreover the vector fields of the transformed system also satisfy the condition 3 and 4 regardless of the selection of free functions $a_1, a_2, a_3, a_4, b_1, b_2$. All conditions of Theorem 5 are satisfied, and thus the system is exactly linearizable via the dynamic extension (21) with the input transformation IT2. The summary of this step is as follows.

The inverted pendulum system with vertical and horizontal inputs is exactly linearizable via 2-degree dynamic extension (21) with the following input transformation.

$$u_1 = \left[\begin{array}{c} u_1 \\ u_2 \end{array}\right] = \left[\begin{array}{c} 0 \\ a(x) \end{array}\right] + \left[\begin{array}{c} 0 \\ b(x) \end{array}\right] \frac{1}{c(x)} \left[\begin{array}{c} u'_1 \\ u'_2 \end{array}\right], \quad (29)$$

$$a(x) = \rho \frac{x_3^2 + x_4^2}{\cos x_6} - \frac{2x_3x_4}{\cos^2 x_6} + a',$$ 

$$a' = (a_1 x_3 + a_2 x_4 + a_3)(x_3 \tan x_6 + x_4 + a_4),$$

$$b(x) = b_1(x_3 \tan x_6 + x_4 + b_2), \quad b(x) \neq 0,$$

$$c(x) = -\tan x_6,$$

where $a_1, a_2, a_3, a_4, b_1, b_2$ are functions of $x_1, x_2, x_3, x_6$.

**Steps 5 and 6.** We select the free functions $a_1, a_2, a_3, a_4, b_1, b_2$ as follows:
\[
\begin{align*}
  a(x) &= \rho \left( \frac{x_1^2 + x_2^2}{\cos x_6} - \frac{2x_1x_3}{\cos^2 x_6} + g' \right), \\
  a' &= \rho (2x_3 \tan x_6 - 1)(x_3 \tan x_6 + x_4) + \frac{x_2^2}{\rho \cos x_6} + g \tan x_6, \\
  b(x) &= \frac{1}{\cos^2 x_6}, \quad c(x) = -\tan x_6.
\end{align*}
\]

The system which is transformed by the input transformation and 2-degree dynamic extension is denoted as follows:

\[
\begin{align*}
  \frac{d}{dt} \begin{bmatrix} x \\ w_1 \end{bmatrix} &= \begin{bmatrix} f + (a + bw_1)g_2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} g_1 + cg_2 \\ 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \\
  &= F + G_1v_1 + G_2v_2, \quad (30)
\end{align*}
\]

After the input transformation and the 2-degree dynamic extension, the following transformations convert the extended system into linear system with \((4, 4, 3, 2, 2, 1, 1) - [4, 4].
\]

\[
\begin{align*}
  Z &= \begin{bmatrix} z_{hh}, z_{hv}, z_{hv}, z_{hh}, z_{hv}, z_{hv}, z_{hh}, z_{hv}, z_{hv}\end{bmatrix}^T, \\
  z_H &= x_1 + \frac{\sin x_6}{\rho}, \\
  z_V &= x_3 + \frac{\cos x_6}{\rho} - \frac{1}{\rho}, \\
  \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= -\left(L_G L_f^T Z\right)^{-1} L_f^T Z + \left(L_G L_f^T Z\right)^{-1} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \\
  L_G L_f^T Z &= \begin{bmatrix} L_{g1} L_f^T z_{hh} & L_{g1} L_f^T z_{hv} & L_{g2} L_f^T z_{hh} & L_{g2} L_f^T z_{hv} \\ L_{g1} L_f^T z_{hv} & L_{g1} L_f^T z_{hv} & L_{g2} L_f^T z_{hv} & L_{g2} L_f^T z_{hv} \end{bmatrix}, \\
  L_f^T Z &= \begin{bmatrix} L_f^T z_{hh} \\ L_f^T z_{hv} \end{bmatrix}.
\end{align*}
\]

7. Conclusion

In this paper, the exact linearization for two-input affine systems via the dynamic extension was discussed, and there were two main contributions. First, the method to derive the transformations which linearize the system with the dynamic extension was proposed. This method consists of two main steps. In the first main step, the condition for exact linearization of the extended system was interpreted into the condition of the original system in order to derive the necessary and sufficient condition for the system to be exactly linearizable via \(\mu\)-degree dynamic extension. This interpretation was achieved based on the relative degree structure of the original system and the restriction on the selection of input to be extended. In the second main step, the restriction on the selection of the input to be extended was relaxed by constructing the input transformation. As an example, the proposed method was applied to the two-input mechanical system, and the linearizing transformations were derived.

The second contribution of this paper was to introduce the distortion of the system using relative degree structure. By the analysis of the distortion, the necessary and sufficient condition for reducing the distortion degree of the system with 1-degree dynamic extension was revealed.

References


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