Optimal Linear Quadratic Regulators for Control of Nonlinear Mechanical Systems with Redundant Degrees-of-Freedom

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Abstract: An optimal regulator problem for endpoint position control of a robot arm with (or without) redundancy in its total degrees-of-freedom (DOF) is solved by combining Riemannian geometry with nonlinear control theory. Given a target point, within the task-space, that the arm endpoint should reach, a task-space position feedback with joint damping is shown to asymptotically stabilize reaching movements even if the number of DOF of the arm is greater than the dimension of the task space and thereby the inverse kinematics is ill-posed. Usually the speed of convergence without incurring further energy consumption, an optimal control design for minimizing a performance index composed of an integral of joint dissipation energy plus a linear quadratic form of the task-space control input and output is introduced. It is then shown that the Hamilton-Jacobi-Bellman equation derived from the principle of optimality is solvable in control variables and the Hamilton-Jacobi equation itself has an explicit solution. Although the state of the original dynamics (the Euler-Lagrange equation) with DOF-redundancy contains uncontrollable and unobservable manifolds, the dynamics satisfies a nonlinear version of the Kalman-Yakubovich-Popov lemma and the task-space input-output passivity. An inverse problem of optimal regulator design for robotic arms under the effect of gravity is also tackled by combining Riemannian geometry with passivity-based control theory.

Key Words: optimal regulator, nonlinear mechanical systems, redundant systems, Hamilton-Jacobi equation, KYP-lemma.

1. Introduction

Ever since the birth of optimal regulators in the year of 1960 discovered by R. E. Kalman [1], linear quadratic Gaussian analysis (LQG-analysis) has played over a half century a central role in design of controllers for not only linear dynamical systems but also nonlinear mechanical systems. It is worth emphasizing that most of the effective feedback controllers have been investigated via existence of nonnegative definite solutions to the Riccati quadratic matrix equation [2] or Kalman-Yakubovich-Popov conditions (see the details in [3]). This research direction has been enriched through extending the notions of passivity and dissipativity to nonlinear dynamical systems owing to Willems [4],[5] and as well through extending the conditions of a matrix equality or inequality form [6],[7] for positive realness and strictly positive realness to nonlinear systems initiated by Anderson and Moylan. An interesting problem of inverse optimal regulation initially posed by Kalman [8] in the case of linear time-invariant systems was also extended to a class of nonlinear dynamical systems by Moylan and Anderson [7],[9]. In a recent book by Brogliato et al. [3], the details of such extensions from linear systems to nonlinear systems can be found.

Given a robot arm composed of a serial connection of rigid links through rotational joints with one-DOF, its motion is subject to the Euler-Lagrange equation. If the center of the first joint, denoted by $J_0$, is fixed at the origin $O$ in Euclidean space $E^3$ in which the inertial frame coordinates $x = (X, Y, Z)^T$ is introduced (see Fig. 1), the pair considering of the joint angle vector $q = (q_1, \cdots, q_6)^T$ and the joint angular velocity vector $\dot{q} = (\dot{q}_1, \cdots, \dot{q}_6)^T$ can be employed as the state of the robot. It is well known [10] that the motion of the robot arm like the one shown in Fig. 1 is governed by the Euler-Lagrange equation described by

$$G(q)\ddot{q} + \left\{\frac{1}{2} G(q) + S(q, \dot{q})\right\}\dot{q} + C_0 \dot{q} + g(q) = u$$

(1)

where $G(q) = (g_{ij}(q))$ expresses the $n \times n$ inertia matrix, $u$ is an $n$-dimensional control input with the physical dimension of...
torque, $g(q) = \partial P(q)/\partial q$ with a scalar function $P(q)$ is called the gravity potential, $S(q, \dot{q})$ is a skew-symmetric matrix $S = (S_{ij})$ defined by

$$S_{ij} = \frac{1}{2} \left( \frac{\partial}{\partial q_i} \left( \sum_{k=1}^n \dot{q}_k g_{jk} \right) - \frac{\partial}{\partial q_j} \left( \sum_{k=1}^n \dot{q}_k g_{ik} \right) \right),$$

and $G(q)$ means $dG(q)/dt = \sum_{i=1}^n (\partial G/\partial q_i) \dot{q}_i$. In 1981 [11], despite of the complexity of nonlinear terms with uncertain parameters in (1), a simple set-point position control scheme described by

$$u = -C_v \ddot{q} + g(q_d) - ADqb$$

was proposed, which is called a PD control with damping shaping. Here, $q_d$ denotes a given target position described in joint angle coordinates in $\mathbb{R}^n$ (joint space or configuration space), $\Delta q = q - q_d$ stands for the position error vector, $A$ and $C_v$ denote a positive definite constant matrix, and $-C_v \ddot{q}$ stands for a joint angular velocity feedback called “damping shaping”. The closed-loop dynamics of motion of the arm is obtained by substituting (3) into (1) as follows:

$$G(q) \ddot{q} + \left( \frac{1}{2} \dot{G}(q) + S(q, \dot{q}) \right) \dot{q} + (C_v + C) \dot{q} + g(q) - g(q_d) + A\Delta q = 0$$

In the previous paper [11], it is shown that there exists an attractor region in $\mathbb{R}^n \times \mathbb{R}^n$ (called the phase space) centering the equilibrium state ($q = q_a$, $\dot{q} = 0$) that any solution trajectory of equation (4) starting from any initial state inside the attractor converges asymptotically to the equilibrium state as time tends to infinity. If gain tuning for damping shaping (choice for $C_v$ in (4)) and gravity compensation (choice for $A$) is required, it is convenient to consider an extra control $v$ for the system of (4), that is, consider the system

$$G(q) \ddot{q} + \left( \frac{1}{2} \dot{G}(q) + S(q, \dot{q}) + C \right) \dot{q} + g(q) - g(q_d) + A\Delta q = v$$

where $C = C_v + C_1$. By multiplying (5) by the inverse of $G(q)$ from the left, and defining $p = \dot{q}$, it is possible to rewrite (5) into the general state-space form of nonlinear dynamics by defining the state $x = (q^T, p^T)^T$ in the following way

$$\dot{x} = f(x) + g(v), \quad y = h(x)$$

where $y$ signifies an output vector with some dimension.

2. Preliminary Remarks on Passivity Analysis

In order to study optimal regulator problems for nonlinear robotic systems, it is necessary to introduce a nonlinear version of the Kalman-Yakubovich-Popov lemma concerning the system of (6). Assume that $\text{dim}(v) = \text{dim}(y)$.

KYP-lemma (see [3])

The following two statements are equivalent:

(a) There exist $V(x) \geq 0$ ($V(0) = 0$), $S(x) \geq 0$ such that

$$V(x(t)) - V(x(0)) = \int_0^t y^T(\tau)w(\tau)d\tau - \int_0^t S(x(\tau))d\tau$$

(b) There exists a $C^1$-nonnegative function $V(x)$ with $V(0) = 0$ such that

$$\frac{\partial V}{\partial x^T} f(x) = -S(x), \quad \frac{\partial V}{\partial x^T} g(x) = h^T(x)$$

It is said that the system is strictly passive for $S(x) > 0$ and passive for $S(x) \geq 0$. In general, it is not an easy task to find such scalar functions $V(x)$ (that is called a storage function) and $S(x)$ that satisfy (7) or (8) in a direct calculation of function forms of $f$, $g$, and $h$ in (6). Fortunately in the case of such a robotic system as shown in (5), taking an inner product of equation (5) and angular velocity vector $\dot{q}$ as the output $y$ yields

$$y^T(t)v(t) = \frac{d}{dt} V(x(t)) + S(x(t))$$

where

$$V(x) = \frac{1}{2} \dot{q}^T G(q) \dot{q} + P(q) - P(q_d) + \Delta q^T g(q_d)$$

$$+ \frac{1}{2} \Delta q^T A\Delta q$$

$$S(x) = \dot{q}^T Cq + \dot{q}^T C \dot{p}$$

Since $V(x)$ is positive definite in a neighborhood of $x_d = (q_d^T, 0)^T$, and $S(x) \geq 0$, the robotic system of (5) is passive in the vicinity of $x_d$. This fact was first found in [12] by noticing the skew symmetric property of $S(q, \dot{q})$ in (1). It is worth remarking that, although $V(x) > 0$, the nonnegative definiteness of $S(x)$ does not directly imply the asymptotic stability of $x_d$ because $S(x) = \dot{p}^T C \dot{p}$ is not positive definite with respect to the state vector $x_d$. Nevertheless, it is shown [10] that, when $v = 0$ in (5), $V(x(t))$ along the solution trajectory to (4) converges to zero as $t \to \infty$ with an exponentially decaying speed.

In the case of endpoint position control for a robotic arm shown in Fig. 1, suppose that a target endpoint position of the arm is given as $x_d = (x_d, y_d, z_d)$ in Euclidean space $E^3$. It is well known that, for a given robot posture $q$, the robot endpoint in $E^3$ is determined by a function $x(q)$ of $C^\infty$-class that is called the forward kinematics. Unfortunately, for given $x_d$ in $E^3$, there is an infinite number of robot postures that satisfy $x(q) = x_d$ due to the redundancy in DOF and hence it is said [13],[14] that the inverse kinematics is ill-posed. Fortunately, if a task space position feedback with full joint damping and feedforward gravity compensation described by

$$u = g(q) - C_v \ddot{q} - J^T(q) \Delta \dot{x} + J^T(q) \dot{w}$$

is employed, then the closed-loop dynamics described below satisfies the passivity, where substituting (12) into (1) yields

$$G(q) \ddot{q} + \left( \frac{1}{2} \dot{G}(q) + S(q, \dot{q}) + C \right) \dot{q} + g(q) - g(q_d) + A\Delta q = J^T(q) \dot{w}$$

Here in (12) $\Delta x = x(q) - x_d$, $J(q) = \partial x(q)/\partial q$, $\dot{w}$ denotes an extra output signal in Euclidean task space $E^3$, and the output is regarded as $y = \dot{x} = J(q) \dot{q}$ (or $J(q) \dot{p}$) [15],[16]. Note that $J(q)$ is a $3 \times n$ matrix, called the Jacobian matrix, that is non-square. Since there is a $(n - 3)$-dimensional null-space of $J(q)$, the system of (13) is neither completely controllable nor completely observable. It is shown in [15] that taking an inner product of (13) and $\dot{q}(\epsilon = p)$ yields the same formula (7) with

$$V(x) = \frac{1}{2} \dot{q}^T G(q) \dot{q} + \frac{k}{2} \|\Delta x\|^2, \quad S(x) = \dot{q}^T C \dot{p}$$

where $\|\Delta x\|$ denotes the Euclidean norm in $E^3$. Note that $V(x)$ is not positive definite in state $x(\epsilon = (q^T, p^T)^T)$, but it is nonnegative.
definite in some sense. Notwithstanding these hurdles, the previous paper [17] shows that there is an attractor region in $R^n \times R^n$ such that, for any initial posture $x(0)$ in the region, any solution trajectory of (13) (in this case, $v = 0$) converges asymptotically to $x_d$ with an exponentially decaying speed. This shows that the system of (13) is stabilizable in some sense. Motivated by this observation, there arises an optimal regulator problem for the system of (13) to find a better control $v$ that achieves the best gain tuning so as to minimize the input-output quadratic performance index

$$I[x_0; v(t), t \in (0, \infty)] = \lim_{t \to \infty} \int_0^t \left( S(x) + \frac{c}{2} ||v(t)||^2 + \frac{c^{-1}}{2} ||v(t)||^2 \right) \, dt$$

$$V[x_0; (0, \infty)] = \min_v I[x_0; v, t \in (0, \infty)]$$

where $c$ is an appropriate positive constant, $x_0$ denotes an initial state, and $S(x)$ signifies joint energy dissipation.

Moylan and Anderson [7],[9] solved a similar type of optimal regulator problems for a restricted class of nonlinear systems described by (6) with a condition that $g(x)u(t) = G(u(t))$ with a constant $n \times m$-matrix $G$. Furthermore, they assumed that the system is completely controllable and completely observable, by which they could use the standard Hamilton-Jacobi method together with the KYP-lemma.

This paper tackles the optimal regulator problem for a class of redundant nonlinear systems described by (13) that is neither controllable nor observable (but stabilizable and detectable in some sense) by showing solvability of the related Hamilton-Jacobi equation. In section 3, stability of an equilibrium manifold is defined by regarding the set of all postures of a given robot as a Riemannian manifold $[M, g_{ij}]$ where $G(q) = (g_{ij}(q))$ denotes the inertia matrix. Then, by using the basic property of Riemannian connection, it is shown that the related Hamilton-Jacobi equation is solvable. Moreover, a general type of optimal regulator problems with performance index

$$I[x_0; v(t), t \in (0, t_1)] = E(x(t_1))$$

$$+ \int_0^{t_1} \left( S(x(t)) + \frac{c}{2} ||v(t)||^2 + \frac{c^{-1}}{2} ||v(t)||^2 \right) \, dt$$

$$V[x_0; (0, t_1)] = \min_{v} I[x_0; v, t \in (0, t_1)]$$

is solved by using the KYP-lemma, where $E(x)$ signifies a non-negative function of $C^\infty$-class. In section 5, a more general case is treated, when the gravity term is compensated adaptively by using a regressor-based estimator.

3. Stability on a Manifold

It is worth noting that, in case of multi-joint reaching movements with DOF-redundancy, some durable oscillatory motion of joints may arise while the endpoint position control is almost achieved, as pointed out by Seraji [18]. In order to distinguish between metric of the endpoint in $E^3$ and another metric to be defined among robot postures (corresponding to a joint space), it is necessary to introduce a Riemannian manifold with such a required metric. For a given robot arm with $n$-DOF as a serial connection of $n$ rigid links through each rotational joint, the set of its all postures denoted by $M$ can be regarded as a Riemannian manifold $[M, g_{ij}]$ with Riemannian metric $g_{ij}(q)$ (see [17]).

The Riemannian distance connecting two postures $q^1$ and $q^2$ in $M$ is defined by

$$d(q^1, q^2) = \inf_q \int_0^T \sqrt{\sum_{i,j} g_{ij}(q(t)) \dot{q}_i(t) \dot{q}_j(t)} \, dt$$

where the infimum denoted by “$\inf$” is taken over all motions $q(t), t \in [0, T]$, connecting $q(0) = q^1$ and $q(T) = q^2$. For a given target endpoint $x_d = (X_d, Y_d, Z_d)$ in $E^3$, if the robot’s DOF is greater than 3 (the dimension of $E^3$), then there is an infinite number of postures whose endpoint coincide with $x(q) = x_d$ where $x(q)$ signifies a 3-dimensional vector-valued function of $C^\infty$-class called the forward kinematics in robotics. All the postures satisfying $x(q) = x_d$ constitute an $(n - 3)$-dimensional submanifold denoted by $N_p$, where $P$ signifies the target point $P = x_d$ in $E^3$. Suppose one posture $q^*$ on $N_p$ at which the Jacobian matrix $J(q)(= \partial x(q)/\partial q)$ is non-degenerate (see Fig. 2). In the previous papers [16],[17], it is shown that, if $C$ and $k$ in (13) are adequately given, there exists a Riemannian ball with radius $r_0$ defined as $B(q^*; r_0) = \{q | d(q, q^*) < r_0\}$ such that, for any initial posture $q(0)$ in $B(q^*; r_0)$ with $q(0) = 0$, the solution trajectory of equation (13) with $v = 0$ remains in some Riemannian ball $B(q^*; r_1)$ with $r_1 > 0$ and the arm endpoint converges to $x_d$ asymptotically with exponentially decaying speed as $t \to \infty$, while the arm posture tends to some posture $q^*$ on $N_p$ (see Fig. 2). During motion of the robot arm, it is important to assure that $J(q)$ is non-degenerate.

4. Solvability of the HJ-Equation

It is convenient to rewrite (13) in the state space form

$$\begin{cases} \dot{q} = p \\ \dot{p} = -G^{-1}\left( \frac{1}{2} \dot{G} + S + C \right) p + J^T(k \Delta x - v) \end{cases}$$

in which description of variables $q$ and $\dot{q} (= p)$ inside $G$, $\dot{G}$, $S$, and $J$ is omitted. The expression of (20) can be regarded as a nonlinear system of equation (6) with state $x = (q^T, p^T)^T$ and the output nonlinear form

$$y = x(q) = J(q)q = J(q)p$$

To apply the principle of optimality contributed by Bellman [19] for the optimization problem of (17) and
where \( \frac{\partial V}{\partial t} = V_t = 0 \), \( V_p = G(q)p \), \( V_q = kJ^T(q)\Delta x + \frac{\partial K}{\partial q} \) \( \) (30) 
where \( K \) denotes the kinetic energy of the system defined by

\[
K = \frac{1}{2} q^T G(q) \dot{q} = \frac{1}{2} p^T G(q)p
\]

Then, substituting (30) into (28) yields

\[
\begin{align*}
H(q, p, V_q, V_p; t) &= p^T C p + \frac{c}{2} \|Jp\|^2 - \frac{c}{2} p^T J^T Jp
\end{align*}
\]

\[
\begin{align*}
&+ V^T G^{-1} \left( \frac{1}{2} G + S + C \right) p + kJ^T \Delta x
\end{align*}
\]

\[
= p^T \frac{\partial K}{\partial q} - \frac{1}{2} p^T Gp = 0 = -V_t
\]

It should be remarked that the second equality follows from the skew symmetry of matrix \( S \) and the third from the definitions of \( G \) and \( K \) described below (2) and in (31). The optimal control signal \( v^* \) is obtained using (30) and (26) in such a way that

\[
v^*(t) = -cJG^{-1}(Gp) = -c\dot{x}(t)
\] (33)

This means that the optimal control for the optimization problem of (17) and (18) is given by a negative velocity feedback in task space \( E^3 \).

It is important to note that the scalar function defined by (29) is also a solution to the HJ-equation of (27) for the semi-infinite-time horizon optimization problem of (15) and (16), since \( E(t_1) \) or \( V(q, p; t_1) \) tends to vanish as \( t_1 \to \infty \) along the solution trajectory of the closed-loop differential equation: \( \dot{v} = -cJG^{-1}(Gp) = -c\dot{x}(t) \) (33)

that follows from substitution of (33) into (20) by putting \( v = \dot{x} = -\dot{c} \). Note that the initial condition for (34) is set as \( q(0) = q^0 \) such that \( x(q^0) = x^0 \) and \( q(0) = p(0) = 0 \). It is also worth remarking that the optimal negative velocity feedback with constant gain \( c \) in task space is established not only for the case of infinite-time horizon \([0, \infty)\) but also for any finite time horizon \([0, t_1]\) as far as the function \( E(x(t)) \) in (17) is defined by (25). In other words, if \( E(x(t)) \) in the performance index of (17) is set as (25) then the value function \( V[x; (0, t_1)] \) in the sense of (17) and (18) is coincident with \( E(x(t)) \), the value of \( E(x(t)) \) defined in (25) at \( t = 0 \). Finally it should be remarked that the third equality in (32) follows from the property of Levi-Civita connection of the Riemannian metric (see [20]), if (1) is expressed via the Euler equation with Christoffel's symbol (see [21]).

5. Proof of Optimality via KYP-Lemma

The optimality of task-space velocity feedback \( v^*(t) \) given by (33) for the performance index of (17) and (18) or that of (15) and (16) can be proved explicitly via the derivation of the KYP-lemma for the system of (13) in the form of the Lagrange equation or (20) in the state-space form. In fact, taking the inner product of \( p(= \dot{q}) \) and (13) by referring to (21) yields

\[
\int_0^\tau y^T(\tau)w(\tau)d\tau = E(x(t)) - E(x(0))
\]

\[
+ \int_0^\tau p^T(\tau)Cp(\tau)d\tau
\]

(35)
where $E(x(t))$ denotes the value of the total energy at $t$ as defined by (25). Since $E(x(t)) \geq 0$ for all $t \geq 0$, (35) shows the passivity of the system described by (13) or (20). However, the system is not strictly passive since the function $S(x) = p^TCp$ in the KYP-lemma is not positive definite in $x$. Moreover, the system with input $v$ and output $y = x$ is neither completely controllable nor completely observable. In fact, the storage function $V(x)$ in the KYP-lemma is given by $V(x) = E(x)$ in this case and $E(x)$ is not positive definite with respect to the state $x = (q^T, p^T)^T$. Hence, Moylan's treatment of the optimality proof [9] cannot be directly applied to the concerned system. Notwithstanding this ill-posed situation, the zero-input dynam-
ition $V(\tau) = \int_0^\tau Y(q(\tau))\dot{q}(\tau)d\tau$ of (1) as
terminant $\Delta \theta = \hat{\theta}(0) - \int_0^1 \Gamma^{-1} Y^T(q(\tau))\dot{q}(\tau)d\tau$ (41)
where $\Gamma$ denotes an appropriate $m \times m$ positive definite constant matrix and $\hat{\theta}(0)$ expresses an initially estimated vector. Instead of the direct compensation of $q$, it is possible to use the estimator $\hat{q}(\theta) = Y(q(\theta))$ and construct the controller for the dynamics of (1) as
$$
\dot{q} = Y(q)\theta
$$
(40)
Then, substituting (42) into (1) yields
$$
G(q)\dot{q} + \left\{ \frac{1}{2} \dot{G} + S + C \right\} \dot{q} + J^T(c^T \dot{x} + k\Delta x) = 0
$$
(39)
that follows from substitution of (33) into (13).

Conversely, the system of (39) can be regarded as an optimal regulator system with the open-loop structure of (13) under the performance index of (15) and (16) or (17) and (18). This is a nonlinear extension of Kalman’s converse theorem concerning a linear quadratic optimal regulator problem [8]. Note that the total dissipation energy of (39) (the integral of the right hand side of (38) over $t \in [0, \infty)$) coincides that of the original system (13) with $v = 0$.

6. Optimal Regulator with Adaptive Gravity Compensation

When the gravity term $g(q)$ in (1) cannot be directly compensated owing to the uncertainty of constant parameters ap-
pearing linearly in the gravity potential $P(q)$ or its gradient $g(q) = \partial P/\partial q$, it is possible to use a model-based adaptive scheme originally introduced by Slotine and Li [22] in robot control. In particular, it is possible to use a far simpler adaptive scheme proposed by Kelly [23], that is restricted to estimating only the gravity by noticing the fact that $g(q)$ can be expressed by a set of dynamic constant parameters denoted by an $m$-dimensional vector $\theta = (\theta_1, \ldots, \theta_m)^T$ together with an $n \times m$-matrix $Y(q)$ composed of sinusoidal functions of measurable joint angles in such a way that
$$
g(q) = Y(q)\theta
$$
(40)
Then, an estimator of $\theta$ denoted by $\hat{\theta}$ can be constructed in the following causal form
$$
\dot{\hat{\theta}}(t) = \hat{\theta}(0) - \int_0^1 \Gamma^{-1} Y^T(q(\tau))\dot{q}(\tau)d\tau
$$
(41)
where $\Gamma$ denotes an appropriate $m \times m$ positive definite constant matrix and $\hat{\theta}(0)$ expresses an initially estimated vector. Instead of the direct compensation of $q$, it is possible to use the estimator $\hat{q}(\theta) = Y(q(\theta))$ and construct the controller for the dynamics of (1) as
$$
u = Y(q)\hat{\theta} = C_1 \hat{q} - J^T(q(k\Delta x + v)
$$
(42)
Then, substituting (42) into (1) yields
$$
G(q)\dot{q} + \left\{ \frac{1}{2} \dot{G} + S + C \right\} \dot{q} + J^T(c^T \dot{x} + k\Delta x) = Y(q)\Delta \theta
$$
(43)
where $\Delta \theta = \dot{\theta} - \theta$ and $C = C_0 + C_1$. Since the inner product of $\dot{q}$ and the last term of the left-hand side of (43) reduces to
$$
-\dot{q}^T Y(q)\Delta \theta = \int_0^1 \frac{d}{dt}(G(\theta))^T \Delta \theta = \frac{d}{dt} \left( \frac{1}{2} \Delta \theta^T \Gamma \Delta \theta \right)
$$
(44)
it follows from the inner product of $\dot{q}$ and (43) that
$$
\int_0^1 x^T(r)\omega(r)dr = \dot{E}(t) - E(0) + \int_0^1 x^T(r)\dot{C}\dot{q}(r)dr
$$
(45)
where
$$
\dot{E}(t) = E(x(t)) + \frac{1}{2} \Delta \theta^T(t) \Gamma \Delta \theta(t)
$$
(46)
and $E(x(t))$ is defined in (25). Equation (45) shows the passivity of the system described by (43) concerning input $v$ and output $y = \dot{x}$. Hence, it is possible to consider the same optimal regulator problem by replacing (17) with
$$
I \left[ \dot{q}^T, \dot{v}(t), (0, t_1) \right] = \dot{E}(t)
$$
$$+ \int_0^t \left( \frac{c}{2} \left| v^T(r) \right|^2 + \frac{c}{2} \left| \dot{v}(r) \right|^2 \right)dr
$$
(47)
Then, the corresponding Hamilton-Jacobi equation follows as
$$
-\dot{V}(q, p; t) = H(q, p, V_q, V_p; t)
$$
(48)
where
$$
\dot{V} = V(q, p; t) + \frac{1}{2} \Delta \theta^T(t) \Gamma \Delta \theta(t)
$$
(49)
$$
H = H(q, p, V_q, V_p; t) + V_q^T G^{-1}(q) Y(q) \Delta \theta
$$
(50)
where $V$ and $H$ are given in (29) and (28) respectively. Note that $V_t = 0$ but $V_q$ becomes as follows:

$$\frac{\partial V}{\partial t} = V_t + \frac{\partial}{\partial \theta} \left\{ \frac{1}{2} \Delta \theta^T \Gamma \Delta \theta \right\} = -p^T \dot{Y}(q) \Delta \theta \quad (51)$$

Since $V_t = \bar{V}_t$, and $V_q = \hat{V}_q$, $\hat{V}(q, p, t)$ satisfies the Hamilton-Jacobi equation expressed by (48). Thus, it is possible to conclude that the optimal regulator can be constructed by the same task-space velocity feedback described in (33), provided that the zero-input dynamics of (43) with $\nu = 0$ is exponentially asymptotically stable in a Riemannian ball $B(q^*; r_1)$.

7. Conclusions

Optimal regulator problems for multi-joint endpoint reaching movements of a robot arm with redundant degrees-of-freedom are studied under the situation that the input-output pair, constructed in the task space (Euclidean space $E^2$ or $E^3$), satisfies neither controllability nor observability. It is shown that if the zero-input dynamics of the robot is exponentially asymptotically stable on an equilibrium manifold, then a nonlinear version of the Kalman-Yakubovich-Popov condition is valid for this case, even if the system has an uncontrollable manifold. Furthermore, it is shown that the corresponding Hamilton-Jacobi equation is solvable and that its solution leads to a linear velocity feedback, in the task space, that optimizes a quadratic performance index.

Finally, it is important to remark that a similar optimal regulator design in the case of $y = x + \alpha x$, that is, the output is a linear combination of $x$ and $\dot{x}$ in the task space, should be investigated in a more sophisticated mathematical way. It should be also pointed out that the convergence proof of a solution to the zero-input dynamics of (43) by using the adaptive gravity estimation is not yet given from the theoretical viewpoint but ascertained in some cases by numerical simulation [24].

References


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He received the B.S. degree from Kyoto University in 1959 and the Dr. Eng. degree from the University of Tokyo in 1967. He was an Associate Professor of Osaka University during 1968–1973 and a Professor during 1973–1990. In 1988 he joined the Faculty of Engineering of the University of Tokyo as a Professor and in 1997 retired. Since 1997 he has been with Ritsumeikan University as a Professor. He is an IEEE Fellow (1983), an IEICE Fellow (2000), an RSJ Fellow (2003), a JSME Fellow (2005), and an IEEE Life Fellow (2007). He was awarded the Royal Medal with Purple Ribbon from the Japanese Government in 2000, the IEEE 3rd Millennium Memorial Medal in 2000, the Pioneer Award from the IEEE Robotics and Automation Society in 2006, and the 2007 Rufus Oldenburger Medal from the ASME.