Adaptive Generalized State Tracking Control on Descriptor Form

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Abstract: In this paper, we propose an adaptive generalized state tracking control on the descriptor form which is more general and natural than the state space form. The generalized state tracking problem covers not only state tracking but also output tracking problems. The proposed method guarantees asymptotic stability of the tracking error even if unknown parameters which may cause instability of controlled systems exist. The proposed adaptive adjusting laws are designed by solving a generalized Lyapunov equation. Some conditions assumed in the proposed method can be checked directly from the descriptor form.

Key Words: descriptor form, adaptive control, tracking control, generalized Lyapunov equation, multivariable systems.

1. Introduction

The descriptor form which is more general and natural than the state space form has rich representation capability based on the following features [1]–[4]: 1) modeling the impulsive behavior or differential operator, and 2) preserving construction and variables of object systems. In particular, the feature 2) can make modeling of controlled systems and dealing with their uncertainties simpler, which works more effectively in the cases where the controlled systems are larger and more complex. These advantages of the descriptor form yielded many researches which include pole assignment [5],[6], optimal regulator [7]–[9], H∞ control [10]–[12], system theory [13]–[15], and so on.

Meanwhile, it is well known that adaptive control with an adjustable controller is more useful for controlled systems which have wide range of uncertainties than the control with a fixed controller [16]. However adaptive control theories based on the descriptor form are not developed enough against other control theories for fixed controllers like above. A few researches of them are an adaptive observer [17], adaptive stabilizing control [18],[19], and adaptive regularization control [20]. However, adaptive tracking control on the descriptor form is not established yet as far as the authors know.

In this paper, we propose an adaptive generalized state tracking control on the descriptor form with unknown parameters. The generalized state tracking problem covers not only state tracking but also output tracking problems. The proposed method guarantees asymptotic stability of the tracking error even if unknown parameters which may cause instability of the controlled systems exist. The proposed adaptive laws are designed by solving a generalized Lyapunov equation [13]. Some conditions assumed in the proposed method can be checked directly from the descriptor form.

This paper is organized as follows. Section 2 gives some preliminary information on the descriptor form. Section 3 formulates the adaptive generalized state tracking control problem. Section 4 proposes the adaptive controller for the problem using a generalized Lyapunov equation. Section 5 checks how the proposed method can be expressed in the case where the controlled system is described as a state space form. Section 6 shows numerical examples of the proposed method. Section 7 gives conclusions.

2. Preliminaries

Consider a dynamic equation of the descriptor form

\[ E \dot{x}(t) = Ax(t) + Bu(t), \quad x(0_) = x_0, \]

where \( x(t) \in \mathbb{R}^n \) is the generalized state vector, \( u(t) \in \mathbb{R}^m \) is the input vector, \( A, E \in \mathbb{R}^{n \times n} \) and \( B \in \mathbb{R}^{n \times m} \) are constant matrices, and \( \text{rank}(E) = r (\leq n) \). Note that \( x_0 \) is an arbitrary initial state and \( x(0_) := \lim_{t \to 0} x(t) \).

Here the following definitions are introduced.

Definition 1 [3],[9]

i) \((E,A)\) is regular if \( \det(sE - A) \neq 0, \quad s \in \mathbb{C}, \) which is necessary and sufficient condition for (1) to have a unique solution \( x(t), t \geq 0 \) by a smooth input \( u(t), t \geq 0 \) and an arbitrary initial state \( x_0 \).

ii) Finite eigenvalues of \( Av = \lambda E v \) are referred as finite dynamic modes of \((E,A)\). Infinite eigenvalues of \( Av = \lambda E v \), which are zero eigenvalues of \( Ev = \rho Av (\rho = 0) \), are referred as non-dynamic modes and categorized into two kinds. One is grade-one infinite eigenmode \( v^1 \neq 0 \) that satisfy \( Ev^1 = 0 \), which are called as static modes. Another is grade-k \( (\geq 2) \) infinite eigenmode \( v^j \) corresponding to generalized eigenvectors \( v^j, j = 2, \ldots, k \), defined by \( Ev^j = Av^{j-1} \), which is \( E \) is \( 0 \) satisfying \( Ev^j = 0 \), which are called as impulse modes.

iii) \((E,A)\) is stable if all finite dynamic modes are in the open left-half complex plane.

The following facts are concerning these definitions.

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Lemma 1 [3] \((E, A)\) is regular and has no impulse modes if and only if
\[
\deg \det(sE - A) = \text{rank} E = r.
\] (2)

Lemma 2 [3] If \((E, A)\) is regular and has no impulse modes, there exist nonsingular matrices \(L, R \in \mathbb{R}^{n \times n}\) satisfying
\[
LER = \begin{bmatrix} I_n & 0 \\ 0 & I_{r=1} \end{bmatrix}, \quad LR = \begin{bmatrix} A_1 & 0 \\ 0 & I_{r=1} \end{bmatrix}
\] (3)
whose \(A_1 \in \mathbb{R}^{r \times r}\).

Lemma 3 [13] If \((E, A)\) is regular, has no impulse modes and is stable, then for each \(Q > 0 \in \mathbb{R}^{m \times m}\) there exists \(P > 0 \in \mathbb{R}^{n \times n}\) solution of the generalized Lyapunov equation:
\[
E^TPA + A^TPE + E^TQE = 0.
\] (4)
Furthermore, \(E^TP \geq 0\) is unique for each \(Q > 0\).

Lemma 4 [3] Let \(u(t) \equiv 0\) in (1). If \((E, A)\) is regular, has no impulse modes and is stable, then the solution \(x(t), t \geq 0\) of (1) is \(\lim_{t \to \infty} x(t) = 0\) for any \(x_0\).

3. Problem Statement

We consider a controlled system of the descriptor form
\[
E \dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0,
\] (5)
where \(x(t) \in \mathbb{R}^n\) is the generalized state, \(u(t) \in \mathbb{R}^m\) is the input, \(A, E \in \mathbb{R}^{n \times n}\) and \(B \in \mathbb{R}^{n \times m}\) are unknown constant system matrices. Let \(E\) allow to have singularity, i.e. \(\text{rank} E = r(\leq n)\). Note that \(x_0\) is an arbitrary initial state and \(x(0) := \lim_{t \to 0^+} x(t)\).

The objective of our study is to design the control input \(u(t)\) such that the generalized state \(x(t)\) asymptotically tracks a reference signal \(x_{0d}(t)\) given from the reference model
\[
E_{sa} \dot{x}_{sa}(t) = A_{sa} x_{sa}(t) + B_{sa} r(t), \quad x_{sa}(0) = x_{sa0}
\] (6)
where \(A_{sa}, E_{sa} \in \mathbb{R}^{n \times n}\) and \(B_{sa} \in \mathbb{R}^{n \times m}\) are known constant matrices, \(\text{rank} E_{sa} = r(\leq n)\), and the reference input \(r(t) \in \mathbb{R}^m\) is bounded, piecewise continuous and sufficiently rich, namely such that the generalized state tracking error
\[
e(t) = x(t) - x_{sa}(t)
\] (7)
is \(\lim_{t \to \infty} e(t) = 0\).

Remark 1 In case that (5) has an output vector \(y(t) = Cx(t) + Du(t)\) and (6) has a target signal \(y_{sa}(t) = C_{sa} x_{sa}(t) + D_{sa} r_{sa}(t)\), output tracking problems to let \(y(t)\) follow \(y_{sa}(t)\) is formulated as generalized state tracking problems like above by defining extended general variables \(x^{T} y^{T}\) and \(x_{sa}^{T} y_{sa}^{T}\) obviously.

This novel problem aims at the following significance:

1. The output (static variables) and/or state (dynamic variables) tracking can be achieved by the same approach even if unknown parameters which may cause instability exist.
2. The controller can be directly designed from the descriptor form that can preserve construction of object systems and is easier to model and more convenient to deal with unknown parameters than the state space form.

Here we suppose the following conditions.

Assumption 1 \((E_{sa}, A_{sa})\) is regular and has no impulse modes.

This is the standard condition that a unique solution exists and no impulse phenomena occur.

Assumption 2 \((E_{sa}, A_{sa})\) is stable.

Assumption 3 There are unknown matrices \(K_3^{\ast} \in \mathbb{R}^{m \times n}\), \(K_2^{\ast} \in \mathbb{R}^{m \times n}\) and \(K_1^{\ast} \in \mathbb{R}^{m \times n}\) satisfying the following matching conditions and rank condition.
(a) \(E = (I_n + B_{sa} K_3^{\ast}) E_{sa}\), \(\text{det}(I_n + B_{sa} K_3^{\ast}) \neq 0\).
(b) \(A + BK_1^{\ast} = B_{sa}, \quad B = B_{sa} K_2^{\ast}, \quad \text{det} K_2^{\ast} \neq 0\).
(c) \(\rank \begin{bmatrix} E_{sa} \\ K_3^{\ast} K_1^{\ast} + K_2^{\ast} A_{sa} \end{bmatrix} = \rank E_{sa}\).

These mean matching conditions that the structure of controlled model is partially immersed to the structure of the reference model. These also include \(\rank E = \rank E_{sa} = r\). Here \(\rank E = r\) can be known using partial information of \(E\) which can be estimated in practical case and \(\rank E = \rank E_{sa}\) can be satisfied when the reference model is designed based on a practical approach that utilizes the descriptor form of the controlled system.

Assumption 4 There exists a known nonsingular matrix \(F \in \mathbb{R}^{n \times n}\) to satisfy \(MF > 0\) where \(M = (I_n + K_1^{\ast} B_{sa})^{-1} K_3^{\ast}\).

This means \(F\) makes the input channel positive. Note that \(F\) can be designed using partial information of \(M\) which can be estimated in practical case.

Now, from Assumption 3(a) and inverse matrix lemma [21], the identity
\[
(I_n + B_{sa} K_3^{\ast})^{-1} = I_n - B_{sa} (I_n + K_3^{\ast} B_{sa})^{-1} K_3^{\ast}
\] is established. From this equation and Assumption 3(a)–(b), (5) is transformed into
\[
E_{sa} \dot{x}(t) = A_{sa} x(t) + B_{sa} (I_n + K_3^{\ast} B_{sa})^{-1} K_3^{\ast} A_{sa} r(t)
- B_{sa} (I_n + K_3^{\ast} B_{sa})^{-1} K_3^{\ast} A_{sa} x(t)
+ B_{sa} (I_n + K_3^{\ast} B_{sa})^{-1} K_3^{\ast} B_{sa} \dot{u}(t).
\] (8)

Using the following relation
\[
I_n - (I_n + K_3^{\ast} B_{sa})^{-1} K_3^{\ast} B_{sa} = (I_n + K_1^{\ast} B_{sa})^{-1},
\]
we transform (8) into
\[
E_{sa} \dot{x}(t) = A_{sa} x(t) + B_{sa} (I_n + K_1^{\ast} B_{sa})^{-1}
\times [K_2^{\ast} u(t) - (K_3^{\ast} K_1^{\ast} + K_2^{\ast} A_{sa}) x(t)].
\] (9)

From Assumption 3(c), there exists matrix \(K_3^{\ast} \in \mathbb{R}^{n \times n}\) satisfying the equation
\[
K_3^{\ast} K_1^{\ast} + K_2^{\ast} A_{sa} = K_3^{\ast} E_{sa}.
\] (10)

Therefore (9) is rewritten as
\[
E_{sa} \dot{x}(t) = A_{sa} x(t) + B_{sa} (I_n + K_1^{\ast} B_{sa})^{-1}
\times [K_2^{\ast} u(t) - K_3^{\ast} E_{sa} x(t)].
\] (11)
Subtracting (6) from (11) and using (7), we obtain the following error equation:
\[ E_\varepsilon(t) = A_\varepsilon e(t) + B_\varepsilon (I_\varepsilon + K_2^* B_\alpha) \cdot K_2^* \times \left[ u(t) - K_1^{-1} K_2^* E_\varepsilon(t) - K_2^{-1} (I_\varepsilon + K_1^* B_\alpha) r(t) \right]. \] (12)

If \( K_1^*, K_2^* \text{ and } K_2^* \) were known, the control law for (12):
\[ u(t) = K_2^{-1} K_1^* E_\varepsilon(t) + K_2^{-1} (I_\varepsilon + K_1^* B_\alpha) r(t) \] (13)

would result in the closed-loop error equation
\[ E_\varepsilon(t) = A_\varepsilon e(t) \]
which \(\lim_{t \to \infty} e(t) = 0\) from Assumption 1, 2 and Lemma 4. However, since \( K_1^*, K_2^* \text{ and } K_2^* \) are actually unknown, we need to use the adaptive version of (13):
\[ u(t) = \hat{\Theta}_\varepsilon(t) E_\varepsilon(x(t)) + \hat{\Theta}_\varepsilon(t) r(t), \] (14)

which \( \hat{\Theta}_\varepsilon(t) \in \mathbb{R}^{m_\varepsilon \times n_\varepsilon} \) and \( \hat{\Theta}_\varepsilon(t) \in \mathbb{R}^{m_\varepsilon \times n_\varepsilon} \) is the estimates of the unknown coefficient matrices of \( x(t) \) and \( r(t) \) in (13).

In this paper, we propose an adaptive adjusting laws for \( \hat{\Theta}_\varepsilon(t) \) and \( \hat{\Theta}_\varepsilon(t) \) which ensure stability of the closed-loop error equation (12) and (14).

4. Proposed Adaptive Controller

To design an adaptive adjusting laws for \( \hat{\Theta}_\varepsilon(t) \) and \( \hat{\Theta}_\varepsilon(t) \) such that the origin of closed-loop error equation (12) and (14) is stable, we define parameter errors as
\[ \tilde{\Theta}_\varepsilon(t) = \hat{\Theta}_\varepsilon(t) - K_1^{-1} K_2^*, \]
\[ \tilde{\Theta}_\varepsilon(t) = \hat{\Theta}_\varepsilon(t) - K_2^{-1} (I_\varepsilon + K_1^* B_\alpha) \]
(15)

to transform the closed-loop equation (12) and (14) into
\[ E_\varepsilon(t) = A_\varepsilon e(t) + B_\varepsilon M [\tilde{\Theta}_\varepsilon(t) E_\varepsilon(x(t)) + \hat{\Theta}_\varepsilon(t) r(t)]. \] (16)

Now there exist the following nonsingular matrices \( L \) and \( R \)
from Assumption 1 and Lemma 2:
\[ LE_\varepsilon R = \begin{bmatrix} I_\varepsilon & 0 \\ 0 & 0 \end{bmatrix}, \quad LA_\varepsilon R = \begin{bmatrix} A_{\varepsilon 1} & 0 \\ 0 & L_{\varepsilon - r} \end{bmatrix}, \] (17)

where \( A_{\varepsilon 1} \in \mathbb{R}^{r \times n_\varepsilon} \) has stable eigenvalues from Assumption 2. Multiplying \( L \) by the left side of (16) and transforming the error vector \( e(t) \) into
\[ \begin{bmatrix} e_1(t) \\ e_2(t) \end{bmatrix} := R^{-1} e(t) \] (18)

with \( R \), we obtain
\[ \begin{bmatrix} e_1(t) \\ e_2(t) \end{bmatrix} = A_{\varepsilon 1} \begin{bmatrix} e_1(t) \\ e_2(t) \end{bmatrix} + \begin{bmatrix} B_{\varepsilon 1} \\ B_{\varepsilon 2} \end{bmatrix} [\tilde{\Theta}_\varepsilon(t) E_\varepsilon(x(t)) + \hat{\Theta}_\varepsilon(t) r(t)], \]

where
\[ \begin{bmatrix} B_{\varepsilon 1} \\ B_{\varepsilon 2} \end{bmatrix} = LB_{\varepsilon 1}, \quad B_{\varepsilon 2} \in \mathbb{R}^{r \times m_\varepsilon} \text{ and } B_{\varepsilon 2} \in \mathbb{R}^{(n-r) \times m_\varepsilon}. \]

Accordingly, the following relation is obtained.
\[ e_2(t) = -B_{\varepsilon 2} M [\tilde{\Theta}_\varepsilon(t) E_\varepsilon(x(t)) + \hat{\Theta}_\varepsilon(t) r(t)]. \] (19)

Meanwhile, we stand the following relation from (17):
\[ \begin{bmatrix} e_1(t) \\ e_2(t) \end{bmatrix} = \begin{bmatrix} I_\varepsilon & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} e_1(t) \\ e_2(t) \end{bmatrix} = L E_\varepsilon R \begin{bmatrix} e_1(t) \\ e_2(t) \end{bmatrix} = LE_\varepsilon e(t) \]
which gives
\[ e_1(t) = L_1 E_\varepsilon e(t) \] (20)
where \( L = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}, \quad L_1 \in \mathbb{R}^{r \times r} \) and \( L_2 \in \mathbb{R}^{(n-r) \times r} \). From (18), (19) and (20), \( e(t) \) is represented as
\[ e(t) = R \begin{bmatrix} e_1(t) \\ e_2(t) \end{bmatrix} = \begin{bmatrix} L_1 E_\varepsilon e(t) \\ -B_{\varepsilon 2} M [\tilde{\Theta}_\varepsilon(t) E_\varepsilon(x(t)) + \hat{\Theta}_\varepsilon(t) r(t)] \end{bmatrix} \] (21)

Therefore, if \( \lim_{t \to \infty} E_\varepsilon e(t) = 0, \lim_{t \to \infty} \hat{\Theta}_\varepsilon(t) = 0 \) and \( \lim_{t \to \infty} \tilde{\Theta}_\varepsilon(t) = 0 \) are proved, \( \lim_{t \to \infty} e(t) = 0 \) is accomplished.

Thus we consider the following positive function about \( E_\varepsilon(t), \hat{\Theta}_\varepsilon(t) \) and \( \tilde{\Theta}_\varepsilon(t) \):
\[ V(E_\varepsilon e(t), \hat{\Theta}_\varepsilon(t), \tilde{\Theta}_\varepsilon(t)) = e^T(t) E_{\varepsilon \varepsilon}^T P E_\varepsilon e(t) \]
\[ + 2tr(\hat{\Theta}_\varepsilon(t) M^T \hat{\Theta}_\varepsilon(t)) \]
(22)

where \( \text{tr} [\cdot] \) is trace of a square matrix. Note that \( M^T > 0 \) from Assumption 4, \( P \) is a solution of the generalized Lyapunov equation
\[ E_{\varepsilon \varepsilon}^T PA_{\varepsilon 1} + A_{\varepsilon 1}^T P E_{\varepsilon \varepsilon} + E_{\varepsilon \varepsilon}^T Q E_{\varepsilon \varepsilon} = 0 \] (23)

which \( Q \) is an arbitrary positive definite matrix, based on Lemma 3.

Differentiating \( V(\cdot, \cdot, \cdot) \) along the trajectory of (16), we get
\[ \dot{V}(\cdot, \cdot, \cdot) = 2e^T(t) E_{\varepsilon \varepsilon}^T P E_\varepsilon e(t) + 2tr[\hat{\Theta}_\varepsilon(t) M^T \hat{\Theta}_\varepsilon(t)] \]
\[ + 2tr[\hat{\Theta}_\varepsilon(t) M^T \hat{\Theta}_\varepsilon(t)] \]
\[ = -e^T(t) E_{\varepsilon \varepsilon}^T P E_\varepsilon e(t) \]
\[ + 2tr[\hat{\Theta}_\varepsilon(t) M^T \hat{\Theta}_\varepsilon(t) + I^{-1} B_{\varepsilon 2}^T P E_\varepsilon e(t) x(t)^T E_{\varepsilon \varepsilon}] \]
\[ + 2tr[\hat{\Theta}_\varepsilon(t) M^T \hat{\Theta}_\varepsilon(t) + I^{-1} B_{\varepsilon 2}^T P E_\varepsilon e(t) r(t)^T]. \] (24)

To let \( \dot{V}(\cdot, \cdot, \cdot) \leq 0, \) we choose adaptive laws of the form
\[ \hat{\Theta}_\varepsilon(t) = -I^{-1} B_{\varepsilon 2}^T P E_\varepsilon e(t) x(t)^T, \]
\[ \hat{\Theta}_\varepsilon(t) = -I^{-1} B_{\varepsilon 2}^T P E_\varepsilon e(t) r(t)^T, \]
(25)
(26)

which mean
\[ \hat{\Theta}_\varepsilon(t) = -I^{-1} B_{\varepsilon 2}^T P E_\varepsilon e(t) x(t)^T, \]
\[ \hat{\Theta}_\varepsilon(t) = -I^{-1} B_{\varepsilon 2}^T P E_\varepsilon e(t) r(t)^T \]
(27)
(28)

where \( \hat{\Theta}_\varepsilon(0) \) and \( \tilde{\Theta}_\varepsilon(0) \) are arbitrary values. So there is a positive scalar \( \alpha \), and \( V(\cdot, \cdot, \cdot) \) satisfies
\[ \dot{V}(\cdot, \cdot) = -e(t)^T E_u^T Q E_u e(t) \leq -\alpha \|E_u e(t)\|^2 \leq 0 \quad (29) \]

Therefore the origin of (16), (25) and (26) is uniformly stable, and \( E_u e(t), \dot{\Theta}(t) \) and \( \dot{\Theta}(t) \) are uniformly bounded, that is \( E_u e(t), \dot{\Theta}(t), \dot{\Theta}(t) \in L_{\infty} \). Therefore from (16), (21), (27) and (28), \( e(t), E_u e(t), \dot{\Theta}(t) \) and \( \dot{\Theta}(t) \) are bounded. Furthermore from (29), \( E_u e(t) \in L_2 \). Accordingly, since \( E_u e(t) \in L_2 \cap L_{\infty} \) and \( E_u e(t) \in L_{\infty} \), from Barbalat’s lemma, \( \lim_{t \to \infty} E_u e(t) = 0 \). Finally since \( r(t) \) is sufficiently rich, \( \lim_{t \to \infty} \dot{\Theta}(t) = 0 \) and \( \lim_{t \to \infty} \dot{\Theta}(t) = 0 \) are achieved [22], so that \( \lim_{t \to \infty} e(t) = 0 \) from (21).

Summarizing the above proof, we establish the following theorem.

**Theorem 1** Suppose that (5) and (6) satisfy Assumption 1 - Assumption 4. Executing the control input as

\[
\begin{align*}
\dot{x}(t) & = \dot{\Theta}(t)E_u x(t) + \tilde{\Theta}(t)r(t), \\
\dot{\Theta}(t) & = -\Gamma^{-1}B_u^T P E_u e(t) (E_u x(t))^T, \\
\dot{\tilde{\Theta}}(t) & = -\Gamma^{-1}B_u^T P E_u e(t) r(t)^T
\end{align*}
\]

on arbitrary \( x_0, x_u, \dot{\Theta}(0) \) and \( \tilde{\Theta}(0) \) makes \( \lim_{t \to \infty} e(t) = 0 \) and \( \dot{\Theta}(t) \) and \( \dot{\tilde{\Theta}}(t) \) be bounded. Here \( \Gamma \) is a nonsingular matrix satisfying \( \Gamma \Gamma^T > 0 \), and \( P \) is a positive definite matrix satisfying (23).

### 5. Discussion on State Space Form

The proposed method can be applied to the state space form because the descriptor form is generalized model of it. However we would not argue the proposed method has enough utility if the conditions of the proposed method got more severe than those of a conventional method. Hence it is important to show that the conditions include those of a representative adaptive state tracking control e.g.[23].

Now we analyze how the assumed conditions and the controller in Theorem 1 are rewritten in case that the controlled systems are represented as state space form i.e. \( E = E_u = I_n \).

Based on Definition 1, in this case, Assumption 1 is naturally satisfied and Assumption 2 is rewritten as follows.

**(Assumption 2')** \( A_u \) is stable.

On Assumption 3, the condition (a) is naturally satisfied as \( K_1^* = 0 \), (b) is not rewritten, and (c) is naturally satisfied from \( E_u = I_n \), namely Assumption 3 is rewritten as follows.

**(Assumption 3')** There are unknown \( K_1^*, K_2^* \) satisfying the following equations.

\[
A + BK_1^* = A_u, \quad B = B_u K_2^*, \quad \det K_2^* \neq 0.
\]

Assumption 4 is also rewritten as follows because \( K_1^* = 0 \).

**(Assumption 4')** There exists a nonsingular matrix \( \Gamma \) to satisfy \( \Gamma \Gamma^T > 0 \) whose \( M = K_2^* \).

Furthermore, the proposed controller is obviously rewritten as follows:

\[
\begin{align*}
\dot{x}(t) &= \dot{\Theta}(t) x(t) + \tilde{\Theta}(t) r(t), \\
\dot{\Theta}(t) &= -\Gamma^{-1} B_u^T P E_u e(t) (E_u x(t))^T, \\
\dot{\tilde{\Theta}}(t) &= -\Gamma^{-1} B_u^T P E_u e(t) r(t)^T
\end{align*}
\]

where \( \Gamma \) is a nonsingular matrix satisfying \( \Gamma \Gamma^T > 0 \) whose \( M = K_2^* \), and \( P \) is a positive definite matrix satisfying the usual Lyapunov equation:

\[
PA_u + A_u^T P + Q = 0.
\]

In addition, the sufficient richness of \( r(t) \) is not obviously required from the proof of Theorem 1 in this case.

From this discussion, we confirm that the above adaptive controller and the conditions are fully coincident with the conventional representative adaptive state tracking control e.g.[23].

### 6. Numerical Examples

#### 6.1 Example 1: Generalized State Tracking

Consider the system composed from coarse equations

\[
\Sigma : \begin{bmatrix} \dot{x}_1(t) + 2x_1(t) + 4x_3(t) = 0 \\
\dot{x}_2(t) - x_1(t) + 2x_2(t) - x_3(t) = 0 \\
\alpha \dot{x}_1(t) - x_2(t) - x_1(t) + \beta x_3(t) - x_3 - yu(t) = 0 
\end{bmatrix}
\]

where \( x_1 \) and \( x_3 \) are the dynamic variables, \( x_2 \) is the static variable, \( u(t) \) is the input, \( \alpha \) and \( \beta \) are unknown, and \( y > 0 \) is unknown.

The descriptor form of \( \Sigma \) is constructed easily by just placing the coarse equations as follows:

\[
\begin{bmatrix} 1 & 0 & 0 & x_1(t) \\
0 & 1 & 0 & x_2(t) \\
\alpha & -1 & 0 & x_3(t) 
\end{bmatrix} \begin{bmatrix} x_1(t) \\
\dot{x}_1(t) \\
\dot{x}_2(t) = 1 - \delta & 1 & \gamma 
\end{bmatrix} + 0 u(t) \quad (31)
\]

whose matrices have 3 unknown elements, and \( \text{rank} E = 2 \) is obvious although \( \alpha \) is unknown.

On the other hand, the state space form with the output of \( \Sigma \) is constructed by more calculations as follows:

\[
\begin{bmatrix} x_1(t) \\
\dot{x}_1(t) = -\alpha & 2 - 2 & \alpha/2 & -1 \end{bmatrix} \begin{bmatrix} x_2(t) \\
\dot{x}_2(t) \\
\gamma 
\end{bmatrix} + 0 u(t) \quad (31)
\]

whose matrices have 6 unknown elements.

Comparing the two forms shows that the descriptor form is superior to the state space one from the viewpoint of easier modeling and fewer unknown elements.

Now, let us apply Theorem 1 to the descriptor form (31) with the reference model:

\[
\begin{bmatrix} 1 & 0 & 0 & \dot{x}_{\text{ref}}(t) \\
0 & 1 & 0 & \dot{x}_{\text{ref}}(t) \\
0 & -1 & 0 & \dot{x}_{\text{ref}}(t) 
\end{bmatrix} \begin{bmatrix} x_{\text{ref}}(t) \\
\dot{x}_{\text{ref}}(t) \\
\dot{x}_{\text{ref}}(t) 
\end{bmatrix} + 0 \dot{r}(t).
\]

Firstly, check the assumptions of Theorem 1: Assumption 1 is satisfied because \( \text{deg} \text{det}(sE_{\text{ref}} - A_u) = \text{rank} E_{\text{ref}} = 2 \) according to Lemma 1; Assumption 2 is satisfied because the finite dynamic modes of \( E_{\text{ref}}, A_u \) are \(-1, -2\) based on Definition 1 (iii); Assumption 3 is satisfied because \( K_1^*, K_2^* \) which hold the conditions (a)–(c) exist as
states $x$, $Q$ equation (23) with tem with finite dynamic modes $-\gamma$ and $x_t$ and $(0$ reference signal while $0$. Here we set $\Gamma = 1$. Secondly, the positive solution of the generalized Lyapunov equation $\Sigma P - \varepsilon P\Sigma + \varepsilon \varepsilon = 0$ is satisfied by selecting $\varepsilon = 1$ and $\gamma > 0$ in this case. Here we set $\Gamma = 0.1$.

Secondly, the positive solution of the generalized Lyapunov $\Theta x_t = x_t$ and $\hat{\Theta} x_t$ are computed online based on (27) and $\Gamma = 0.1$ under the $\hat{\Theta} x_t$ and $\hat{\Theta} x_t$ are adjusted adaptively online.

The simulation is done under setting the initial dynamic states $x_0 = [1 0^T]$, $x_{m0} = [0 0]$, the initial gains $\hat{\Theta}(0) = (0, 0), \hat{\Theta}(0) = 0$, the reference input signal as $r(t) = \sin(t) + 0.5 \sin(0.5t)$ and the unknown parameters as $\alpha = 0.5$, $\beta = 3$ and $\gamma = 0.5$ which cause the instability of the controlled system with finite dynamic modes $-2.8508, 0.3508$.

Here $\hat{\Theta}(t)$ and $\hat{\Theta}(t)$ are computed online based on (27) and (28) with $P$ of (32) and $\Gamma = 0.1$ under the $\hat{\Theta}(0)$ and $\hat{\Theta}(0)$.

The simulation results are shown in Fig. 1 (a)-(f). From these results, it is observed that the generalized state converges to the reference signal while $\hat{\Theta}(t)$ and $\hat{\Theta}(t)$ are adjusted adaptively online.

### 6.2 Example 2: State Tracking

We consider the system expressed as the standard 2-order differential equations

$$\Sigma: \alpha \dot{p}(t) + \beta \ddot{p}(t) + \gamma p(t) = u(t)$$

where $p(t)$ is the dynamic variable and $u(t)$ is the input.

Here suppose the simple case that $\alpha > 0$ is unknown parameter and $\gamma$ and $\beta$ are known parameters as $\gamma = 1$ and $\beta = 1$.

The descriptor form of $\Sigma$ is easily made from (33) by setting $x_1(t) = p(t)$ and $x_2 = \dot{p}(t)$ as follows.

$$P = \begin{bmatrix} 1 & 0 \\ 0 & \alpha \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

where $\text{rank} E = 2$ is obvious from $\alpha > 0$.

Meanwhile, the state space form of $\Sigma$ is made by using the inverse of the parameter as follows:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1/\alpha & -1/\alpha \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/\alpha \end{bmatrix} u(t).$$

Comparing the two forms, we confirm that the descriptor form is easier to model and has 1 unknown element against the state form has 3 unknown elements.

Now, we apply the proposed method to (34) to let $x(t)$ track $x_{m0}(t)$ of the reference model:

$$\begin{bmatrix} x_{m1} \\ x_{m2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_{m1} \\ x_{m2} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r(t).$$

Firstly, check the assumptions for the proposed method: Assumption 1 is satisfied because $\text{deg} \ det(A_{m} - A_{m1}) = \text{rank} E_{m} = 2$ according to Lemma 1; Assumption 2 is satisfied because the finite dynamic modes of $[E_{m1}, A_{m1}]$ are $-0.5000 \pm 0.8660i$ based on Definition 1 (iii); Assumption 3 is satisfied because $K^*_1$, $K^*_2$ and $K^*_3$ which hold the conditions (a)-(c) exist as

$$K^*_1 = 1, \; K^*_2 = [0 \ 0], \; K^*_3 = [0 \ -1 + \alpha],$$

and Assumption 4 is satisfied by choosing $\Gamma > 0$ because $M = 1/\alpha, \alpha > 0$. Here we set $\Gamma = 0.02$.

Next, the positive solution of the generalized Lyapunov equation (23) with $Q = I_2$ is calculated as

$$P = \begin{bmatrix} 2.2500 & 0.1250 \\ 0.1250 & 0.1875 \end{bmatrix}.$$
(a) Trajectories of $x_1(t)$ and $x_2(t)$ when $u(t) = 0$.

(b) Trajectory of $u(t)$.

(c) Trajectories of $x_1(t)$ and $x_{M1}(t)$.

(d) Trajectories of $x_2(t)$ and $x_{M2}(t)$.

(e) Trajectories of $\hat{\Theta}_x(t)$.

(f) Trajectory of $\hat{\Theta}_r(t)$.

Fig. 2 Simulation results of example 2.

The simulation is executed under the initial states $x_0 = [1 \ 0]^T$, $x_{M0} = [0 \ 0]^T$, the initial gains $\hat{\Theta}_x(0) = [0 \ 0]^T$, $\hat{\Theta}_r(0) = 0$, the reference input signal as $r(t) = \sin(t) + \sin(0.5t)$ and the unknown parameters $\alpha = 10$ giving finite dynamic modes $-0.050 \pm 0.3122i$ which cause weak convergence property of the free response of controlled system as shown in Fig. 2 (a).

Here $\hat{\Theta}_x(t)$ and $\hat{\Theta}_r(t)$ are calculated online based on (27) and (28) with $P$ of (38) and $\Gamma = 0.02$ under the $\hat{\Theta}_x(0)$ and $\hat{\Theta}_r(0)$.

The simulation results are shown in Fig. 2 (b)–(f). From these results, we observe that each state converges to the reference signal as $\hat{\Theta}_x(t)$ and $\hat{\Theta}_r(t)$ are adjusted adaptively online.

7. Conclusions

In this paper, we have established the adaptive generalized tracking control based on the descriptor form. The proposed method can guarantee asymptotic stability of the tracking error even if unknown parameters which may cause instability of the controlled systems exist. Some conditions assumed in the proposed method can be checked directly from the descriptor form which is easier to model the controlled system and deal with their uncertainties than the state space form. The unique feature of the proposed method is using a positive definite solution of the generalized Lyapunov equation. The proposed method includes the well known conventional adaptive state tracking control based on the state space form. The effectiveness of the proposed method have been confirmed through two numerical examples.

The authors will consider the following future works:

1. To extend the proposed method to the case of $\text{rank} E \neq \text{rank} E_M$.

2. To extend the proposed method to nonlinear descriptor systems.

3. To apply the proposed method or its extended versions.

References


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