Estimation of Transition Times of a Hopping Machine Based on Wavelet Analysis

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Abstract: The transition time of a hopping machine (HM) as a piecewise linear (PWL) system is estimated by using a wavelet analysis method proposed in previous papers. The HM is a nonlinear system because its equation of motion has a nonlinear drag coefficient term owing to air resistance. The effectiveness of the proposed method for real (nonlinear) systems is verified through instrumental experiments performed on the HM. First, the effect of nonlinearity on the estimation of a transition time is examined. The HM has a discrete transition caused by the nonlinearity in a real system. In addition, two types of discrete transitions are detected from the output data: one caused by the change between discrete free-fall and spring-mass states and the other caused by inversion of the sign of the nonlinear term owing to a change in the velocity of the mass. These transition times are estimated using the proposed method. Finally, the proposed method is compared with a conventional method based on a clustering technique.

Key Words: piecewise linear system, transition time, wavelet analysis, hopping machine, nonlinearity.

1. Introduction

The transition time of a hopping machine (HM) as a piecewise linear (PWL) system is estimated in this study by using a method based on wavelet analysis [1],[2] in numerical and instrumental experiments. In addition, the proposed method is compared with the conventional method based on the clustering technique [3] to verify its effectiveness. The HM is a good example of a system that can be modeled as a PWL system in numerical simulations; however, in real situations, the systems are nonlinear because of air resistance. Therefore, this paper focuses on the discrete transition caused by the nonlinearity.

Hybrid systems have attracted attention because of their wide range of applicability [4]–[6]. Identification methods of PWL and piecewise affine (PWA) systems have been studied intensively [3],[7]–[12]. Conventional identification methods are based on clustering datasets (composed of output and input data) of a piecewise affine autoregressive exogenous (PWARX) model according to their location. In contrast, we proposed a new method for estimating the transition time from an output of a continuous-time model of the PWL (PWA) system via wavelet analysis [13]. It is difficult to apply conventional methods to hybrid systems in which there exists a discrete transition from one discrete state (submodel or mode) to itself. The proposed method can be applied to detect such discrete transitions.

Discrete transitions are caused by a switching of differential equations or a jump of a continuous state. These effects will appear in the output after several integration times. In addition, there are cases when a system undergoes multiple discrete transitions that occur in different orders of derivatives of its output. Our method can be applied to the analysis of the motion of such systems offline (e.g., for analysis of walking of a biped robot). The HM is a good example of a system with multiple discrete transitions because of the ease with which its equation of motion can be analyzed.

The HM we study is a PWL system with two discrete states—a free-fall state and a spring-mass state—if air resistance is ignored. However, when air resistance is considered, the HM becomes a nonlinear system because its equation of motion has a nonlinear term drag coefficient term owing to air resistance. Thus, in this study, the transition time of the HM is estimated under conditions in which air resistance exists. The effect of the nonlinearity on the precision of the estimation is investigated in numerical and instrumental experiments. In addition, two types of discrete transitions are detected from the output data: one caused by the change between discrete free-fall and spring-mass states and the other caused by inversion of the sign of the nonlinear term owing to a change in the velocity of the mass. These two types of discrete transitions occur in different orders of derivatives of the output. Therefore, detection of these discrete transitions requires the use of different functions (see Section 3).

In Section 2, the HM is summarized, and its equation of motion is obtained to analyze the discrete transition. In Section 3, the continuous wavelet transform and the method used to estimate the transition time through wavelet analysis are explained. Numerical experiments are presented in Section 4. Two types of transition times are estimated by changing the condition of air resistance, and the effect of the nonlinearity is examined without noise. In Section 5, the transition time of a real system is estimated in instrumental experiments. The effect of noise (in the instrumental experiments) on the estimation is verified. In Section 6, estimation results obtained using the proposed method are compared with those obtained from the conventional method with use of a K-means-like algorithm [3] and a
support vector machine (SVM). The advantage of the proposed method over the conventional method is discussed. Finally, the paper is concluded in Section 7.

2. Hopping Machine

In this section, the hopping machine is briefly summarized. In the presence of air resistance, the equations of motion for the HM and those of the Taylor expansion are obtained, and types of discrete transitions are then identified.

2.1 Formulation of the Hopping Machine

The HM is modeled using a hybrid automaton [14], as illustrated in Fig. 1. Consider the HM as the system with two discrete states—namely, a free-fall state and a spring-mass state—and two discrete transitions. The time when the discrete state changes and a change in the discrete state are called a transition time and a discrete transition, respectively. The transition time of the HM is the moment when the discrete state changes between free-fall and spring-mass states. Let \( y(t) \) denote the height of the mass and assume that the discrete transition occurs when \( y(t) \) reaches 0. The equation of motion for the HM is described as

\[
\dot{m}y(t) = \begin{cases} 
-\frac{mg}{m} & \text{if } y(t) > 0, \\
-\frac{mg - K_i y(t)}{m} & \text{otherwise},
\end{cases}
\]  

where \( m > 0 \) is mass, \( p > 0 \) is the acceleration due to gravity, and \( K_i > 0 \) is a spring constant. Thus, the HM is described as the autonomous PWL system,

\[
\dot{x}(t) = A_p x(t) \quad \text{for } p = 1, 2, \\
y(t) = C x(t), \quad x(t) = [y(t) \dot{y}(t) 1]^\top, \\
A_1 = \begin{bmatrix} 1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -g \\
0 & 0 & 0 
\end{bmatrix}, \quad A_2 = \begin{bmatrix} -K_i/m & 0 & -g \\
0 & 0 & 0 \\
0 & 0 & 0 
\end{bmatrix}, \\
C = [1 \ 0 \ 0], \\
p = \begin{cases} 
1 & \text{if } y(t) > 0 \text{ (free-fall)}, \\
2 & \text{otherwise (spring-mass)},
\end{cases}
\]

where \( x(t) \in \mathbb{R}^3 \) is a continuous state, \( y(t) \in \mathbb{R} \) is an output, \( A_p \in \mathbb{R}^{3 \times 3} \) is a system matrix corresponding to a discrete state \( p \), and \( C \in \mathbb{R}^{1 \times 3} \) is a common output matrix.

Assuming that the discrete transition occurs at \( t = 0 \) and that \( v_0 \) is the velocity of the mass at the transition time \( t = 0 \), the Taylor expansion of \( y(t) \) around the transition time \( t = 0 \) can be obtained as follows:

\[
y(t) = \begin{cases} 
-v_0 t - \frac{g t^2}{2} & \text{if } t < 0, \\
-v_0 t + \frac{K_i v_0}{6m} t^3 + \cdots & \text{otherwise}.
\end{cases}
\]  

The third derivative of the output is discontinuous at the transition time \( t = 0 \). This means that the HM is a system with a discrete transition of \( k = 3 \). Let the parameter \( k \) be defined as the order of the discrete transition. When the discrete transition of \( k \) occurs, its effect appears in the \( k \)th derivative of the output (see [1],[2]).

2.2 Air Resistance

When air resistance is considered, the HM becomes a nonlinear system. The equation of motion has a nonlinear drag coefficient \( c > 0 \) (defined as \( c = p c_d S / 2 \), \( p \) is air density, \( C_d \) is a drag coefficient, and \( S \) is a surface area of the mass):

\[
\ddot{y}(t) = \begin{cases}
-\frac{mg + sgn(\dot{y}(t))c(t)\dot{y}(t)}{m} & \text{if } \dot{y}(t) > 0, \\
-\frac{mg + sgn(\dot{y}(t))c(t)\dot{y}(t)}{m} - K_i y(t) & \text{otherwise},
\end{cases}
\]

\[
sgn(\dot{y}(t)) = \begin{cases}
-1 & \text{if } \dot{y}(t) > 0, \\
+1 & \text{otherwise}.
\end{cases}
\]  

According to Eq. (9), the sign of the \( c \) term changes depending on the direction of the velocity \( y(t) \), so the HM has four discrete states (Fig. 2). The discrete state changes between spring-mass1 and spring-mass2 when the mass cannot jump with the spring. The HM is described as the extension of Eqs. (2)–(6):

\[
\dot{x}(t) = \bar{A}_p \dot{x}(t) \quad \text{for } \bar{p} = 1, 2, 3, 4, \\
y(t) = \bar{C} x(t), \quad \bar{C} x(t) = [y(t) \dot{y}(t)]^\top, \\
\bar{A}_p = \begin{bmatrix} A_p & 0 \\
0 & C \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \quad \bar{p} = \begin{cases} 
1 & \text{if } y(t) > 0 \text{ and } \dot{y}(t) > 0 \text{ (free-fall1)}, \\
2 & \text{if } y(t) > 0 \text{ and } \dot{y}(t) \leq 0 \text{ (free-fall2)}, \\
3 & \text{if } y(t) \leq 0 \text{ and } \dot{y}(t) \leq 0 \text{ (spring-mass1)}, \\
4 & \text{if } y(t) \leq 0 \text{ and } \dot{y}(t) > 0 \text{ (spring-mass2)},
\end{cases}
\]

\[
p = \begin{cases} 
1 & \text{if } \bar{p} = 1 \text{ or } 2, \\
2 & \text{if } \bar{p} = 3 \text{ or } 4.
\end{cases}
\]

If the discrete transition occurs at \( t = 0 \), the output in each discrete state can be calculated from the Taylor expansion of the output around the transition time \( t = 0 \) and its multiderivatives:

\[
y(t) = \begin{cases}
-v_0 \frac{dt}{dt} - \frac{c dt}{12m} \frac{dt}{dt} + \cdots & \text{if } \text{free-fall1}, \\
-v_0 \frac{dt}{dt} + \frac{c dt}{12m} \frac{dt}{dt} + \cdots & \text{if } \text{free-fall2}, \\
-v_0 \frac{dt}{dt} + \frac{K_i v_0}{6m} \frac{dt}{dt} + \left(\frac{c dt}{12m} + \frac{c K_i}{36m}\right) \frac{dt}{dt} + \cdots & \text{if } \text{spring-mass1}, \\
-v_0 \frac{dt}{dt} + \frac{K_i v_0}{6m} \frac{dt}{dt} + \left(\frac{c dt}{12m} + \frac{c K_i}{36m}\right) \frac{dt}{dt} + \cdots & \text{if } \text{spring-mass2}.
\end{cases}
\]  

The effect of the change in the function in Eq. (9) appears in
the $i^t$ term, i.e., the fourth derivative of the output $y(t)$. The
HM is the system with a discrete transition of $k = 4$. Therefore,
there exist discrete transitions of $k = 3$ and $k = 4$ in the HM.
These transition times will be estimated by using the proposed
method in this paper.

3. Estimation Method

In this section, the wavelet transform and the method to esti-
mate the transition time are explained.

3.1 Wavelet Transform

The continuous wavelet transform is defined as

$$ (W_\psi(y)(b, a) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{a}} \psi^* \left( \frac{t-b}{a} \right) y(t) dt, \quad (16) $$

where $*$ denotes a complex conjugate, and $a \in R$ and $b \in R$ are
a scale and a translation parameter, respectively. In this study,
$a > 0$ is assumed. The Mexican hat wavelet,

$$ \psi(t) = (1 - 2^2t^2)e^{-t^2}, \quad (17) $$
is adopted as a mother wavelet function.

The transition time is denoted by $\tau$. For the sake of simplic-
ity, the discrete transition occurs at $\tau = 0$; i.e., it will be denoted
by $b$ instead of $b - \tau$.

3.2 Construction of the Function $Z_k(b, a)$

To detect the discrete transition that appears in the $i^t$ terms,
a function to negate the effect of $t$, $t^2$, $t^3$, and $t^4$ terms is
constructed. A function $Z_k(b, a)$, which is a linear combination
of $(W_\psi(y)(b, a))^j$ for some $j \in S, j = 0, 1, \ldots, k$, is constructed
depending on the order of the discrete transition $k$ (see [1],[2]).
In this study, $Z_3(b, a)$ and $Z_4(b, a)$ are obtained to extract the $t^3$
and $t^4$ terms in Eq. (15).

When $k = 3$, we define $Z_3(b, a)$ as follows:

$$ Z_3(b, a) = \left| \frac{2(W_\psi(y)(b, a) + (W_\psi(y)(b + a, a)
+ (W_\psi(y)(b - a, a))}{2} \right|, \quad (18) $$

When $k = 4$, we define $Z_4(b, a)$ as follows:

$$ Z_4(b, a) = \left| \frac{2(W_\psi(y)(b + a, a) + 2(W_\psi(y)(b - a, a)
- (W_\psi(y)(b + 2a, a) + (W_\psi(y)(b - 2a, a))}{2} \right|, \quad (19) $$

The functions $Z_3(b, a)$ and $Z_4(b, a)$ are proportional to approximately $a^{3.5}$ and $a^{4.5}$, respectively, at the transition time $b = 0$.

To estimate the transition time, define $s_3(b)$ as the slope of the
log $Z_3(b, a)$ against log $a$ for each time $b$. (Note that the com-
mon logarithm is used in this study.) An estimated transition
time will be defined as the time $b$ when $s_3(b)$ becomes approx-
imately $k + 0.5$ or a local minimum. If the scale parameter $a$ is
large, $s_3(b)$ becomes a local minimum at the transition time.

Remark 1 It is important for the estimation to determine the appro-
riate range of the scale parameter $a$. Precision of the esti-
mation is affected by white Gaussian noise (see [1]), and the
intensity of the noise effect depends on the range of the scale
parameter $a$. If the lower limit of the range is too small, the esti-
mation is affected by the noise intensively. In contrast, when the
range is large, the effect of noise is less but the precision of the
estimation is lost. We have previously investigated a method for
determining the appropriate range in the present research [15].
The height of the cylinder was set hopping by a commercial apparatus of the HM. A cylindrical brass mass (0.61 kg in weight and 40 mm in diameter) was set inside an acrylic pipe (with an inside diameter of 42 mm and a length of 0.50 m). The height of the mass was measured with a laser displacement sensor (LDS; model CD33-250NV, Optex FA), with an effective range of approximately 0.25 m. The grounding of the spring was detected using an electric switch composed of the spring and a couple of aluminum plates; i.e., the discrete state of the HM was the spring-mass state while electric current runs through the electrical switch. The time-series output data from the LDS and the electrical switch were recorded as voltage with a USB data logger (model NI USB-6009, National Instruments). The sampling period was 1/1000 s.

An air hole (at a height of 10 mm) was made between the acrylic pipe and the aluminum plates to reduce air resistance. The drag coefficient with the air hole was estimated as approximately 0.21 kg/m by comparing the output with that of a numerical experiment.

### Table 1 Estimated transition times in the numerical experiments.

<table>
<thead>
<tr>
<th>k = 3</th>
<th>( \tau_0 ): true values of ( y(t) ) (c = 0.0 kg/m) (s)</th>
<th>( \hat{\tau}_0 ): estimated values (s)</th>
<th>( \tau_1 ): true values of ( y(t) ) (c = 0.5 kg/m) (s)</th>
<th>( \hat{\tau}_1 ): estimated values (s)</th>
<th>( \tau_2 ): true values of ( y(t) ) (c = 1.0 kg/m) (s)</th>
<th>( \hat{\tau}_2 ): estimated values (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \tau_0 )</td>
<td>0.225</td>
<td>0.425</td>
<td>0.877</td>
<td>1.077</td>
<td>1.529</td>
<td>1.729</td>
</tr>
<tr>
<td>( \hat{\tau}_0 )</td>
<td>0.222</td>
<td>0.425</td>
<td>0.876</td>
<td>1.077</td>
<td>1.528</td>
<td>1.729</td>
</tr>
<tr>
<td>( \tau_1 )</td>
<td>0.233</td>
<td>0.438</td>
<td>0.763</td>
<td>0.978</td>
<td>1.232</td>
<td>1.456</td>
</tr>
<tr>
<td>( \hat{\tau}_1 )</td>
<td>0.232</td>
<td>0.439</td>
<td>0.762</td>
<td>0.978</td>
<td>1.231</td>
<td>1.454</td>
</tr>
<tr>
<td>( \tau_2 )</td>
<td>0.241</td>
<td>0.452</td>
<td>0.704</td>
<td>0.932</td>
<td>1.103</td>
<td>1.350</td>
</tr>
<tr>
<td>( \hat{\tau}_2 )</td>
<td>0.239</td>
<td>0.452</td>
<td>0.702</td>
<td>0.931</td>
<td>1.103</td>
<td>1.349</td>
</tr>
</tbody>
</table>

### 5. Instrumental Experiments

In this section, the transition time of the HM is estimated in instrumental experiments to verify the effectiveness of the proposed method for a real system. Figure 9 shows the experimental apparatus of the HM. A cylindrical brass mass (0.61 kg in weight and 40 mm in diameter) was set hopping by a compression coil spring (with a spring constant of 200 N/m and a length of 0.2 m) inside an acrylic pipe (with an inside diameter of 42 mm and a length of 0.50 m). The height of the mass was measured with a laser displacement sensor (LDS; model CD33-250NV, Optex FA), with an effective range of 0.25 ± 0.15 m.
merical simulation. In the case without the air hole, the drag coefficient was estimated as approximately 0.81 kg/m by using the same method. The effect of the intensity of the nonlinearity is investigated under these two conditions of the drag coefficient.

The outputs and the states of the electrical continuity between aluminum plates are shown in Fig. 10, where $y_3(t)$ and $y_4(t)$ are the outputs for $c \sim 0.21$ kg/m and $c \sim 0.81$ kg/m, respectively. The output $y_3(t)$ has sparser data because the mass went beyond the effective range of the LDS. The attenuation of $y_4(t)$ was larger than that of $y_3(t)$ because of air resistance. The wavelet transform of the output is shown in Fig. 11. Figures 12 and 13 show $s_3(b)$ and $s_4(b)$ at $\log a \in [-2.5, -0.5]$, respectively. The estimated transition times are listed in Table 2. The transition times of both $k = 3$ and $k = 4$ could be estimated, although the outputs included noise from vibration of the spring, collision between the mass and the acrylic pipe, and chattering of the electric switch. The estimation errors are larger than those of the numerical experiments because of the noise. However, no large difference exists among the results under the different conditions as well as among the numerical experiments. In addition, in spite of the sparser data, transition times around the sparser data could be estimated from $y_3(t)$. Figures 14 and 15 show plots of $\log Z_3(b, a)$ and $\log Z_4(b, a)$ against $\log a$ and those linear approximations for each transition time. It can be observed that the linear approximations were all proportional to $\log a$.

### 6. Comparison with Conventional Method

In this section, an estimation result obtained from the proposed method is compared with that obtained from the conventional method based on a clustering technique to verify the effectiveness of the proposed method. In the conventional identification method of the PWARX model, each dataset (composed of output and input data) is assigned to a submodel according to its location. Feature vectors composed of local parameter vectors obtained from local data (vectors of regressors and output data) and centroids of the local data are classified into submodels by using a clustering technique. A variation of the batch K-means algorithm (modified K-means) proposed in [3] is the representative method based on a clustering technique. After feature vectors are classified, a hyperplane separating pairs of regions (to which the dataset belongs) identified by the submodels must be obtained. An SVM [16] with a linear kernel is the most general method used to determine the hyperplane. Thus, modified K-means and SVM are used for the classification of the dataset in this study.

Transition times of $k = 3$ are estimated in both numerically.
is a nonlinear system with discrete transitions of $k = 3$ (vertical dotted lines). (i) $c \sim 0.21$ kg/m. (ii) $c \sim 0.81$ kg/m.

Fig. 13 Slopes $s_3(b)$ at $\log a \in [-2.5, -0.5]$ and the transition times of $k = 4$ (vertical dashed lines). (i) $c \sim 0.21$ kg/m. (ii) $c \sim 0.81$ kg/m.

Graphical and instrumental experiments. The transition time is the time $t$ when the submodel to which each dataset (the output $y(t)$) belongs changes between free-fall and spring-mass states. The vector of regressors is defined as $[y_d(n-1) \ y_d(n-2)]^\top$ (where $y_d(n)$ is a sampled value of $y(t)$ at $t = nh$ for each $n \in [1, \ldots, N]$, $h$ is the sampling period, and $N$ is a length of the data). The datasets are classified into two submodels with modified K-means 100 times. (Initializations of classifications are changed at random.) Means of 100 times are defined as estimated transition times. Figure 16 shows a comparison between results obtained using the proposed method and those obtained from the conventional method. The transition times estimated using the proposed method are more precise than those estimated using the conventional method (Table 3). The largest error exists in the second estimated transition times of (ii) ($t = 0.326$ s) because of the vibration of the spring after it jumps intensively, which is detected. The transition times estimated using the conventional method are divided into several patterns depending on initial assignments of the datasets to the submodels. In contrast, the transition time estimated using the proposed method is unique if the scale parameter is determined.

7. Conclusion

The transition time of an HM has been estimated using the proposed wavelet analysis method to examine the effect of nonlinearity and verify its effectiveness for real systems. The HM is a nonlinear system with discrete transitions of $k = 3$ and $k = 4$ operating under the condition in which air resistance exists. Both types of transition times can be estimated from the drag coefficients in the experiments. The nonlinearity caused by air resistance does not affect the precision of the estimation. These results suggest that the transition time of the nonlinear system can be estimated if a function $Z_k(b, a)$ that negates the effect of the $t, t^2, \ldots, t^{k-1}$ terms can be constructed. Furthermore, the advantage of the proposed method over the conventional method based on the clustering technique is shown by

Fig. 14 Functions $Z_3(b, a)$ at $\log a \in [-3.0, 0.0]$ in increments of 0.05 and those linear approximations at $\log a \in [-2.5, -0.5]$ for $b = r_3, r$. (i) $c \sim 0.21$ kg/m. (ii) $c \sim 0.81$ kg/m.

Fig. 15 Functions $Z_3(b, a)$ at $\log a \in [-3.0, 0.0]$ in increments of 0.05 and those linear approximations at $\log a \in [-2.5, -0.5]$ for $b = r_3, r$. (i) $c \sim 0.21$ kg/m. (ii) $c \sim 0.81$ kg/m.

Fig. 16 Transition times estimated using the proposed method (filled circles) and using modified K-means and SVM ($\times$: each result; $\diamond$: means), the output (continuous line), and the transition times of $k = 3$ (vertical dotted lines). (i) Numerical experiment with $c = 0.81$ kg/m. (ii) Instrumental experiment with $c = 0.81$ kg/m.
Table 3 Estimated transition times of $k=3$ in the experiments.

<table>
<thead>
<tr>
<th></th>
<th>Numerical experiment with $c=0.81$ kg/m</th>
<th>Instrumental experiment with $c=0.81$ kg/m</th>
</tr>
</thead>
<tbody>
<tr>
<td>true values of $t_1(t)$ (s)</td>
<td>0.238 0.447 0.723 0.946 1.143 1.380 1.528 1.781 1.891</td>
<td>0.097 0.292 0.605 0.804 1.033 1.233 1.412 1.641 1.750</td>
</tr>
<tr>
<td>estimated values (the proposed method) (s)</td>
<td>0.237 0.447 0.719 0.943 1.144 1.379 1.528 1.777 1.891</td>
<td>0.118 0.326 0.601 0.810 1.033 1.243 1.429 1.618 1.794</td>
</tr>
<tr>
<td>standard deviations (s)</td>
<td>0.005 0.005 0.005 0.006 0.008 0.009 0.010 0.013 0.014</td>
<td>0.017 0.021 0.023 0.030 0.034 0.0  0  0  0  0</td>
</tr>
</tbody>
</table>

References


