Characterization of Finite Frequency Properties for \( n \)-Dimensional Behaviors Using Quadratic Differential Forms

Chiaki KOJIMA * and Shinji HARA *

Abstract: Many of practical design specifications are provided by finite frequency properties described by inequalities over restricted finite frequency intervals. In this paper, the authors consider a characterization of the finite frequency domain inequalities (FFDs) for \( n \)-dimensional systems from a view point of a dissipation theory using quadratic differential forms (QDFs), which are useful algebraic tools for the dissipation theory based on the behavioral approach. The QDFs allow us to derive a clear characterization of the FFDs using some inequality analogous to dissipation inequality with a compensation rate and an inequality of an integral of the supply rate with a matrix integral quadratic constraint as a main result. This characterization leads to a physical interpretation in terms of the dissipativity for subbehavior with some rate constraints. The authors also show how to resolve a difficulty on the expression of a compensation rate peculiar to \( n \)-dimensional systems. The results of this paper can be regarded as a finite frequency version for the characterizations of frequency properties over the entire frequency domain due to Pillai and Willems (2002).

Key Words: finite frequency properties, dissipativity, quadratic differential forms, behavioral approach

1. Introduction

Many of practical design specifications are provided by sets of finite frequency properties which are expressed as inequalities over restricted finite frequency intervals. The properties play important role for dynamical system design including control and signal processing. In \( n \)-dimensional systems \([1]–[4] \), the finite frequency properties appear in many context such as filter design \([5]–[7] \), image processing \([8],[9] \), and etc. including Fornasini-Marchesini \([10],[11] \) and Roessor \([9] \) (discrete-time) state-space systems.

Dissipativity is one of the most important properties which captures a dynamical system from the viewpoint of energy and power interactions with its external environment \([12]–[14] \). It is well-known that the dissipativity can be equivalently transformed to the matrix inequality over the imaginary axis \([12] \). Hence, it may be important to articulate the relationship between finite frequency properties and dissipativity. This claim can also be validated by the fact that a stability condition for a feedback system is given in terms of integrals over entire frequencies, called integral quadratic constraint \([15] \).

A quadratic differential form (QDF) is a useful algebraic tool in dissipation theory based on the behavioral approach \([16],[17] \), because it has a one-to-one correspondence to a two-variable polynomial matrix. The behavioral approach is a theoretic framework which does not assume an input-output relationship, a particular representation and causality in advance. Since an \( n \)-dimensional system has an infinite-dimensional state space and no causality in the space coordinates, we can naturally analyze and design an \( n \)-dimensional system based on the approach. Using QDFs, Willems and Trentelman \([18] \) proved that a dissipativity of a behavior is equivalent to a certain frequency domain inequalities on the entire frequency range. This equivalence is characterized by the dissipation inequality in terms of QDFs. This characterization was extended to \( n \)-dimensional systems by Pillai and Willems \([19] \). On the other hand, for a characterization of finite frequency properties, the authors of this paper clarified that the properties are equivalent to a dissipativity of some rate constrained subbehavior in time domain based on QDFs for one-dimensional systems \([20] \). A key point was to prove the existence of a compensation rate which appears in the inequality corresponding to the dissipation inequality for the finite frequency case. Since the most of physical systems are described by partial differential-algebraic equations at the beginning of analysis and synthesis, we have a great interest in how the finite frequency properties can be expressed from a theoretical viewpoint of dissipativity in \( n \)-dimensional systems. However, there has never been derived a characterization for the \( n \)-dimensional case. Hence, we conceived to derive a characterization of the properties from this viewpoint in \( n \)-dimensional systems.

In this paper, motivated by the observation in the above paragraphs, we will characterize the finite frequency properties of \( n \)-dimensional systems based on QDFs. As a main result, we derive a characterization of the properties using some inequality and an integral of a supplied rate. We also characterize the properties in terms of dissipativity of some subbehavior of the original behavior. These characterizations are obtained by generalizing the idea of \([20] \) for the one-dimensional system to the \( n \)-dimensional case. Although a nonnegative property of a compensation rate played an important role in \([20] \), we find that a straightforward extension of \([20] \) is not easy in \( n \)-dimensional systems, since an expression of a compensation rate satisfying the property is not clear in this case. Hence, we need a further theoretical consideration for the characterization in the general-

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ization. We show how this theoretical difficulty can be resolved in $n$-dimensional systems. The results of this paper allow us to understand the significance of the properties directly. Figure 1 illustrates a series of these results comparing with the previous works [18]–[20]. In this figure, the notations “1-D” and “n-D” mean one-dimensional and $n$-dimensional systems, respectively. The arrows represent the inclusive relations of the results. Especially, the contributions of this paper are illustrated with the black arrows.

Fig. 1 Relationship between this paper and the previous works.

The organization of this paper is as follows. In Section 2, we review some basic definitions and results on the behavioral system theory and QDFs. We give the problem formulation of finite frequency characterization for $n$-dimensional systems in Section 3. In Section 4, we derive a characterization of the finite frequency properties using QDFs as a main result. A numerical example demonstrates our characterization in Section 5.

We adopt the following notations in this paper.

The sets of $p \times q$ real and complex matrices are denoted by $\mathbb{R}^{p \times q}$ and $\mathbb{C}^{p \times q}$, respectively. We also denote $\mathbb{S}^{p \times q}$ and $\mathbb{H}^{p \times q}$ as the sets of $q \times q$ real symmetric and Hermitian matrices, respectively. The set of $p \times q$ real coefficient $n$-variable polynomial matrices are denoted by $\mathbb{P}^{p \times q}[\xi]$. The sets of $p \times q$ complex coefficient $n$- and 2n-variable polynomial matrices are denoted by $\mathbb{C}^{p \times q}[\xi]$ and $\mathbb{C}^{p \times q}[\zeta, \eta]$, respectively. We also denote $\mathbb{H}^{p \times q}[\zeta, \eta]$ as the set of Hermitian $2n$-variable polynomial matrices, i.e. $\Phi(\zeta, \eta) = \Phi(\bar{\zeta}, \bar{\eta})^*$ for any $\Phi \in \mathbb{H}^{p \times q}[\zeta, \eta]$, where $\bar{\zeta}$ and $\bar{\eta}$ denote the complex conjugates of $\zeta$ and $\eta$, respectively.

We denote $\mathbb{T}^n$ as the set of maps from $\mathbb{T}$ to $\mathbb{W}$. Define $\mathbb{C}^n(\mathbb{R}^n, \mathbb{V})$ as the set of infinitely differentiable functions from $\mathbb{R}^n$ to the vector space $\mathbb{V}$, and denote $\mathbb{D}^n(\mathbb{R}^n, \mathbb{V})$ as the subset of set $\mathbb{C}^n(\mathbb{R}^n, \mathbb{V})$ with compact support.

Finally, the row dimension of the matrix $A$ is denoted by $\text{rowdim}(A)$. We define the rank of polynomial matrix $R(\xi)$ and constant matrix $R(\lambda)$ are denoted by $\text{rank}R(\xi)$ and $\text{rank}R(\lambda)$, respectively. We denote the matrix $\begin{bmatrix} A_1^T & A_2^T & \cdots & A_q^T \end{bmatrix}$ by $\text{col}(A_1, A_2, \cdots, A_q)$. We also define $\text{He}(A) := \frac{1}{2}(A + A^*)$.

2. Preliminaries

In this section, we review the basic definitions and results from the behavioral system and dissipation theory for $n$-dimensional behaviors from the references [16],[19].

2.1 Linear Continuous-time Systems

In the behavioral system theory, a dynamical system is defined as a triple $\Sigma = (\mathbb{T}, \mathbb{W}, \mathbb{B})$, where $\mathbb{T}$ is the set of independent variables, and $\mathbb{W}$ is the signal space in which the trajectories take their values on. The behavior $\mathbb{B} \subseteq \mathbb{W}^T$ is the set of all possible trajectories.

In this paper, we consider an $n$-dimensional linear time-invariant continuous-time system $\Sigma = (\mathbb{R}^n, \mathbb{C}^q, \mathbb{B})$ with the independent variable $x := (x_1, \cdots, x_n) \in \mathbb{R}^n$. Such a $\Sigma$ is typically represented by the linear partial differential-algebraic equation expressed as

$$\sum_{i=0}^{l_1} \cdots \sum_{i_q=0}^{l_q} R_{i_1,\cdots,i_q}(\lambda) \frac{\partial^{i_1} \cdots \partial^{i_q} \phi_{x_1} \cdots \phi_{x_n}}{\partial \lambda_{1}^{i_1} \cdots \partial \lambda_{p}^{i_q}} w(x_1, \cdots, x_n) = 0,$$

where $R_{i_1,\cdots,i_q}(\lambda) \in \mathbb{C}^{p \times q}$ and $L_k \geq 0$. The variable $w \in \mathbb{C}^n(\mathbb{R}^n, \mathbb{C}^q)$ is called the manifest variable.

For a simplicity of the description, we introduce the multi-index notation [19],[21] by $i := (i_1, \cdots, i_k) \in \mathbb{Z}^n$ and $\xi := (\xi_1, \cdots, \xi_n)$, where $i_k (k = 1, \cdots, n)$ is a nonnegative integer. By using this notation, we define the $n$-variable polynomial matrix $R \in \mathbb{C}^{p \times q}[\xi]$ by

$$R(\xi) := \sum_{i=0}^{l_1} \cdots \sum_{i_q=0}^{l_q} R_{i_1,\cdots,i_q}(\lambda) \xi_{i_1} \cdots \xi_{i_q} = R(\xi_1, \cdots, \xi_n),$$

where $\xi_{i_k}$ is defined by $\xi_{i_k} := \xi_{i_1}^{i_1} \cdots \xi_{i_q}^{i_q}$ and $L \in \mathbb{Z}^n$ is given by $L = (L_1, \cdots, L_n)$. For the multi-index $i = (i_1, \cdots, i_k)$, we define the corresponding partial differential operator $\frac{\partial}{\partial x} \equiv \frac{\partial}{\partial x_1} \cdots \frac{\partial}{\partial x_n}$. Then, (1) is expressed as

$$\sum_{i=0}^{L_1} \cdots \sum_{i_q=0}^{L_q} R_{i_1,\cdots,i_q}(\lambda) \frac{\partial^{i_1} \cdots \partial^{i_q} \phi_{x_1} \cdots \phi_{x_n}}{\partial \lambda_{1}^{i_1} \cdots \partial \lambda_{p}^{i_q}} w = 0,$$

in short hand, where $\frac{\partial}{\partial x}$ denotes $\frac{\partial}{\partial x_1} \cdots \frac{\partial}{\partial x_n}$. Then, the behavior $\mathbb{B} \subseteq \mathbb{C}^n(\mathbb{R}^n, \mathbb{C}^q)$ is defined as the kernel of the operator $R(\frac{\partial}{\partial x})$ given by

$$\mathbb{B} = \left\{ w \in \mathbb{C}^n(\mathbb{R}^n, \mathbb{C}^q) \mid R(\frac{\partial}{\partial x})w = 0 \right\}.$$

For this reason, (3) is called the kernel representation of $\mathbb{B}$. The representation (3) is said to be the minimal representation of $\mathbb{B}$ if $\text{rowdim}R \leq \text{rowdim}R'$ holds for any other $R' \in \mathbb{R}^{p \times q}[\xi]$ which induces a kernel representation of $\mathbb{B}$.

The behavior $\mathbb{B} \subseteq \mathbb{C}^n(\mathbb{R}^n, \mathbb{C}^q)$ is called controllable if for any $w_1, w_2 \in \mathbb{B}$ and $X_1, X_2 \subset \mathbb{R}^n$ with disjoint closure, there exists $w \in \mathbb{B}$ such that $w|_{X_1} = w_1|_{X_1}$ and $w|_{X_2} = w_2|_{X_2}$, where $w|_{X}$ denotes the restriction of the trajectory $w \in \mathbb{C}^n(\mathbb{R}^n, \mathbb{C}^q)$ to the domain $X \subset \mathbb{R}^n$. The behavior $\mathbb{B}$ is controllable if and only if $\text{rank}R(A)$ is constant for all $A \in \mathbb{C}^n$ [16].

Whenever $\mathbb{B}$ is controllable, it can be described by an image representation

$$w = M \left( \frac{\partial}{\partial x} \right) \ell,$$

where $M \in \mathbb{R}^{p \times q}$ and the variable $\ell \in \mathbb{C}^n(\mathbb{R}^n, \mathbb{C}^q)$ is called the latent variable. Then, $\mathbb{B}$ is given by

$$\mathbb{B} = \{ w \in \mathbb{C}^n(\mathbb{R}^n, \mathbb{C}^q) \mid \exists \ell \in \mathbb{C}^n(\mathbb{R}^n, \mathbb{C}^q) \text{ s.t. (5)} \}.$$
When $\mathcal{B}$ is represented by an image representation, $\mathcal{B}$ is called observable if $w = \mathcal{M}(\mathcal{B}) \ell = 0$ implies $\ell = 0$. The representation (5) is observable if and only if the constant matrix $\mathcal{M}(\lambda)$ is of full column rank for all $\lambda \in \mathbb{C}^d$ [16].

As we have mentioned in the above, every controllable behavior admits an image representation. However, for $n$-dimensional behaviors, it should be noted that every controllable behavior does not necessarily have an observable image representation contrary to the one-dimensional case [16].

2.2 Quadratic Differential Forms

We review the definition and the basic results of QDFs [18],[19] for $n$-dimensional behaviors, which play a central role in this paper.

We consider a Hermitian $2n$-variable polynomial matrix in $\mathbb{C}^{n \times n}[\zeta, \eta]$. Similarly to Section 2.1, we use the multi-index notation [19],[21] by $i := (i_1, \ldots, i_n) \in \mathbb{Z}^n$, $j := (j_1, \ldots, j_n) \in \mathbb{Z}^n$ and $\zeta := (\zeta_1, \ldots, \zeta_n)$, $\eta := (\eta_1, \ldots, \eta_n)$, where $i_k$ and $j_k$ ($k = 1, \ldots, n$) are nonnegative integers. We also denote $\zeta^i := \zeta_1^{i_1} \cdots \zeta_n^{i_n}$ and $\eta^j := \eta_1^{j_1} \cdots \eta_n^{j_n}$. We can describe any matrix in $\mathbb{C}^{n \times n}[\zeta, \eta]$ as

$$\Phi(\zeta, \eta) = \sum_{i \in \mathbb{Z}^n} \sum_{j \in \mathbb{Z}^n} \Phi_{ij} \zeta^i \eta^j, \quad \Phi_{ij}, \Phi_{ij} \in \mathbb{C}^{n \times n},$$

where the above sum ranges over all nonnegative multi-indices $i \in \mathbb{Z}^n$, $j \in \mathbb{Z}^n$, and is assumed to be finite. The degree of $\Phi(\zeta, \eta)$ with respect to $\zeta_k$ and $\eta_k$ ($k = 1, \ldots, n$) are defined as $\deg_\zeta := \max_{i \in \mathbb{Z}^n} i_k$ and $\deg_\eta := \max_{j \in \mathbb{Z}^n} j_k$, respectively, where $I \subset \mathbb{Z}^n$ is the set defined by $I := \{(i, j) \in \mathbb{Z}^n \mid \Phi_{ij} \neq 0\}$. For $\Phi(\zeta, \eta)$ in (6), we define the mapping

$$\partial : \mathbb{C}^{n \times n}[\zeta, \eta] \to \mathbb{C}^{n \times n}[\zeta], \quad \partial(\Phi)(\zeta) := \Phi(-\zeta, \eta).$$

We define a quadratic differential form for $n$-dimensional behaviors in the following [19].

A quadratic differential form (QDF) is induced by the $2n$-variable polynomial matrix $\Phi(\zeta, \eta)$ in (6). The QDF is represented by

$$Q_\Phi : C^\infty(\mathbb{R}^n, \mathbb{C}^n) \to C^\infty(\mathbb{R}^n, \mathbb{R}),$$

$$Q_\Phi(\ell) := \sum_{k=0}^K \sum_{j=0}^J \left( \frac{d^k}{dx^k} \right) \Phi_{ij} \frac{d^j}{dx^j},$$

where $K := (K_1, \ldots, K_n) \in \mathbb{Z}^n$, $K_k := \deg_\zeta \Phi = \deg_\eta \Phi$ ($k = 1, \ldots, n$). There is a one-to-one correspondence between the QDF and the $2n$-variable polynomial matrix

$$\Phi(\zeta, \eta) = \sum_{k=0}^K \sum_{j=0}^J \Phi_{ij} \zeta^i \eta^j.$$

This means that $\zeta$ and $\eta$ correspond to the partial differentiations on $\ell^t$ and $\ell$, respectively.

Consider the $2n$-variable polynomial matrix $\Psi(\zeta, \eta) = \text{col}(\Psi_1(\zeta, \eta), \ldots, \Psi_n(\zeta, \eta), \Psi_1^1, \ldots, \Psi_n^1) \in H^{\text{sup}}[\zeta, \eta]$. This induces a vector of QDFs (VQDFs)

$$Q_\Psi : C^\infty(\mathbb{R}^n, \mathbb{C}^n) \to C^\infty(\mathbb{R}^n, \mathbb{R}),$$

$$Q_\Psi(\ell) := \text{col}(Q_{\Psi_1}(\ell), \ldots, Q_{\Psi_n}(\ell)).$$

The divergence of the VQDF $Q_\Psi(\ell)$ is defined by $\text{div}Q_\Psi(\ell) := \sum_{k=0}^K \sum_{j=0}^J \frac{d}{dx} Q_{\Psi_k}(\ell)$. This is also a QDF. Let $\nabla \Phi \in H^{\text{sup}}[\zeta, \eta]$ induce $\text{div}Q_\Psi(\ell)$, i.e. $\text{div}Q(\ell) = Q_\Phi(\ell)$. Then, it is given by $\nabla \Phi(\zeta, \eta) = \sum_{k=0}^K \sum_{j=0}^J \Phi_{ij} \zeta^i \eta^j$.

2.3 Dissipation Theory

In this section, we review the basic definitions and properties of dissipativity for $n$-dimensional behaviors using QDFs [19].

We assume that $\mathcal{B}$ in (4) is controllable in this section. Then, $\mathcal{B}$ has an observable image representation (5). Let $\Phi \in H^{\text{sup}}[\zeta, \eta]$ in (8) be given.

We give the definition of dissipativity of a behavior.

Definition 1 [19] Let $\Phi \in H^{\text{sup}}[\zeta, \eta]$ in (8) be given. Then, a behavior $\mathcal{B}$ is called dissipative with respect to the supply rate $Q_\Phi(w)$ if the inequality

$$\int_{\mathbb{R}^n} Q_\Phi(w)dx \geq 0$$

holds for all $w \in \mathcal{B} \cap D^{\infty}(\mathbb{R}^n, \mathbb{C}^n)$.

We may think of $Q_\Phi(w)$ as the power delivered to the behavior $\mathcal{B}$. The dissipativity implies that the net flow of energy into the system is nonnegative. This shows the system dissipates energy. Hence, due to this dissipation, the rate of increase of the energy stored inside of the system does not exceed the power supplied to it. This interaction between supply, storage, and dissipation is now formalized in Definition 2 and Proposition 1 below.

We give the definitions of storage function and dissipation rate.

Definition 2 [19] Assume that $\mathcal{B}$ is controllable. Let $\Phi \in H^{\text{sup}}[\zeta, \eta]$ be given.

(i) The VQDF $Q_\Psi(\ell)$ induced by

$$\Psi(\zeta, \eta) = \text{col}(\Psi_1(\zeta, \eta), \ldots, \Psi_n(\zeta, \eta)),$$

where $\Psi_1, \ldots, \Psi_n \in H^{\text{sup}}[\zeta, \eta]$, is called a storage function for $\mathcal{B}$ with respect to the supply rate $Q_\Phi(\ell)$ if

$$\text{div}Q_\Psi(\ell) \leq Q_\Phi(w)$$

holds for all $w \in \mathcal{B}$ with the image representation (5). We call (11) the dissipation inequality.

(ii) The QDF $Q_\Psi(\ell)$ induced by $\Delta \in H^{\text{sup}}[\zeta, \eta]$ is called a dissipation rate for $Q_\Phi(w)$ if $Q_\Psi(\ell) \geq 0$, $\forall w \in \mathcal{B}$ and

$$\int_{\mathbb{R}^n} Q_\Psi(w)dx = \int_{\mathbb{R}^n} Q_\Delta(\ell)dx, \forall w \in \mathcal{B} \cap D^{\infty}(\mathbb{R}^n, \mathbb{C}^n)$$

hold with the image representation (5).

There is a one-to-one relation between a storage function $Q_\Psi(\ell)$ and a dissipation rate $Q_\Delta(\ell)$ defined by

$$\text{div}Q_\Psi(\ell) = Q_\Phi(w) - Q_\Delta(\ell).$$

The equation (12) is called the dissipation equality.

The next proposition gives a characterization of dissipativity in terms of a storage function and a dissipation rate.

Proposition 1 [19] Let $\Phi \in H^{\text{sup}}[\zeta, \eta]$ be given. The following statements (i), (ii) and (iii) are equivalent.

(i) The behavior $\mathcal{B}$ is dissipative with respect to the supply rate $Q_\Phi(w)$. 

(ii) There exists a 2n-variable polynomial matrix $\Psi \in \mathbb{R}^{m \times m}$ in (10) satisfying the dissipation inequality (11) for all $\ell \in \mathcal{D}^n(\mathbb{R}^n,\mathbb{R}^m)$ with the image representation (5).

(iii) There exist 2n-variable polynomial matrices $\Psi \in \mathbb{R}^{m \times m}$ in (10) and $\Lambda \in \mathbb{R}^{m \times m}$ satisfying the dissipation inequality (12) and $Q_\Lambda(\ell) \geq 0$ for all $\ell \in \mathcal{D}^n(\mathbb{R}^n,\mathbb{R}^m)$ with the image representation (5).

Remark 1 As we have remarked in Section 2.1, the observability of (5) does not always hold for n-dimensional behaviors. This implies that the storage function does not necessarily become a function of manifest variable [19]. Hence, the uniqueness of the storage function does not hold, i.e. there will be many possible storage functions.

Remark 2 We give an interpretation of the inequality (24) in terms of flux [19] in this remark. This enables us to further clarify the above physical interpretation.

Suppose that the independent variable $x_1 = t$ represents the time variable and the remaining variables $x_2, \ldots, x_n$ are the space variables. Then, the dissipation inequality (12) can be rewritten as

$$\frac{\partial}{\partial t} Q_{\Psi_1}(\ell) = Q_\Phi(w) - \sum_{k=2}^n \frac{\partial}{\partial x_k} Q_{\Psi_k}(\ell) - Q_\Lambda(\ell).$$

The interpretation of the above equality is described as follows. The change in the stored energy $\frac{\partial}{\partial t} Q_{\Psi_1}(\ell)$ in an infinitesimal volume exactly equals to the difference between the supply rate $Q_\Phi(w)$ into the infinitesimal volume, the energy lost $\sum_{k=2}^n \frac{\partial}{\partial x_k} Q_{\Psi_k}(\ell)$ by the volume, which is called flux, and the dissipation $Q_\Lambda(\ell)$ within the volume. Hence, the rate of change of the stored energy does not exceed the power supplied the system due to this dissipation and flux.

In the remainder of this section, we explain how the dissipativity can be equivalently described in the frequency domain.

Suppose that (5) is an image representation of $\mathcal{B}$. Consider the frequency domain inequality (FDI) expressed as

$$M(j\omega)\varPhi(j\omega)M(j\omega) \geq 0, \quad \forall \omega \in \mathbb{R}^n. \tag{13}$$

The FDI (13) is an interpretation of dissipativity of $\mathcal{B}$ in entire frequency domain.

Proposition 2 [19] Suppose that $\mathcal{B}$ is represented by an image representation (5). Let $\Phi \in \mathbb{R}^{m \times m}$ in (8) be given. Then, the following statements (i) and (ii) are equivalent.

(i) The behavior $\mathcal{B}$ is dissipative with respect to the supply rate $Q_\Phi(w)$.

(ii) The FDI (13) holds for all $\omega \in \mathbb{R}^n$.

The above proposition shows that (13) is an inequality which interprets dissipativity in the frequency domain.

3. Problem Formulation

We characterize finite frequency properties for a linear time-invariant system $\Sigma = (\mathbb{R}^n, \mathcal{C}^n, \mathcal{B})$ using QDFs. We give the problem formulation in this section for this purpose.

We consider the behavior $\mathcal{B}$ typically represented by the kernel representation (3), where $w \in \mathbb{C}^m(\mathbb{R}^n, \mathcal{C}^n)$ is the manifest variable and $R \in \mathbb{C}^{m \times m}(\mathcal{C}^n)$ is the polynomial matrix. Then, $\mathcal{B}$ is given by (4). We set the following assumption on $\mathcal{B}$ throughout this paper.

Assumption 1

(i) The behavior $\mathcal{B}$ in (4) is controllable.

(ii) The kernel representation (3) is minimal.

(iii) An image representation of $\mathcal{B}$ is described by (5), which is possibly unobservable.

Let $\Phi \in \mathbb{R}^{m \times m}(\mathcal{C}^n, \mathcal{C}^n)$ in (8) be given. Suppose that this $\Phi(\zeta, \eta)$ induces the supply rate for $\mathcal{B}$. Let $\omega \in \mathbb{R}^n$ be the frequency vector given by $\omega := (\omega_1, \ldots, \omega_k)$. Define the frequency domain $\Omega \subset \mathbb{R}^n$ as a product of finite intervals by

$$\Omega := \prod_{k=1}^n \Omega_k = \Omega_1 \times \cdots \times \Omega_n, \tag{14}$$

$$\Omega_k := \{ \omega_k \in \mathbb{R} | \tau_k(\omega_k - \varepsilon_{k1})(\omega_k - \varepsilon_{k2}) \leq 0 \} \quad (k = 1, \cdots, n),$$

where $\varepsilon_{k1}, \varepsilon_{k2} \in \mathbb{R}$, $\varepsilon_{k1} \leq \varepsilon_{k2}$ are given and $\tau_k \in \mathbb{Z}$ is either $+1$ or $-1$.

The domain $\Omega$ can represent the various type of finite frequency domain by the choice of $\tau_k$ and $\varepsilon_{k1}, \varepsilon_{k2} \in \mathbb{R}$. For $\tau_k = +1, \forall k = 1, \cdots, n$, $\Omega$ becomes the middle frequency domain

$$\Omega_m := \prod_{k=1}^n \Omega_{m,k}, \tag{15}$$

$$\Omega_{m,k} := \{ \omega_k \in \mathbb{R} | \varepsilon_{k1} \leq \omega_k \leq \varepsilon_{k2} \} \quad (k = 1, \cdots, n).$$

We can also consider the low frequency domain

$$\Omega_h := \prod_{k=1}^n \Omega_{h,k}, \tag{16}$$

$$\Omega_{h,k} := \{ \omega_k \in \mathbb{R} | |\omega_k| \leq \varepsilon_k \} \quad (k = 1, \cdots, n)$$

by putting $\varepsilon_{k1} = -\varepsilon_k$ and $\varepsilon_{k2} = \varepsilon_k$ for $k = 1, \cdots, n$, where $\varepsilon := (\varepsilon_1, \cdots, \varepsilon_n) \in \mathbb{R}^n$ is a given vector satisfying

$$\varepsilon_k \geq 0, \forall k = 1, \cdots, n.$$

On the other hand, $\Omega_h$ expresses the high frequency domain

$$\Omega_{h,k} := \{ \omega_k \in \mathbb{R} | \omega_k \leq \varepsilon_{k1}, \varepsilon_{k2} \leq \omega_k \} \quad (k = 1, \cdots, n)$$

for $\tau_k = -1, \forall k = 1, \cdots, n$. The domain $\Omega$ also becomes the entire real vectors, i.e. $\Omega = \mathbb{R}^n$, by choosing the parameters $\varepsilon_{k1} = \varepsilon_{k2} = 0$ in addition. Of course, we can represent other frequency domains by choosing the values of $\tau_k$ and $\varepsilon_{k1}, \varepsilon_{k2}$, appropriately.

Consider the finite frequency property described by the following finite frequency domain inequality (FFDI)

$$M'(j\omega)\varPhi(j\omega)M(j\omega) \geq 0, \quad \forall \omega \in \Omega. \tag{17}$$
Our goal is to find a characterization of the above FFDI using QDFs from the viewpoint of dissipativity introduced in Section 2.3. Especially, we want to give clear answers to the following two questions from the viewpoint of dissipativity under the restriction of the frequency domain to the product of finite intervals, which are formulated mathematically in this section.

Questions

(i) What a power function newly appears in the dissipation inequality (11), or equivalently the dissipation equality (12), for compensating the restriction of the frequency domain? Specifically, what is the different point comparing with the finite frequency characterization for one-dimensional behaviors [20].

(ii) What additional property of \( \mathcal{B} \) to the dissipativity is equivalent to the FFDI (17)?

An interpretation of the FFDI (17) from the behavioral approach is the following. Consider the QDF \( Q_\Phi(w) \) induced by \( \Phi \in \mathbb{H}^{im}[\zeta, \eta] \) in (8). Fourier transform of \( Q_\Phi(w) \) is computed as

\[
\hat{w}(j\omega)^* \partial \Phi(j\omega) \hat{w}(j\omega) = \hat{\ell}(j\omega)^* M(j\omega)^* \partial \Phi(j\omega) M(j\omega) \hat{\ell}(j\omega),
\]

where \( \hat{w} \in L_2(\mathbb{C}^n, \mathbb{C}^n) \) and \( \hat{\ell} \in L_2(\mathbb{C}^m, \mathbb{C}^m) \) are Fourier transforms of \( w \in \mathcal{B} \cap \mathbb{D}^w(\mathbb{R}^n, \mathbb{C}^n) \) and \( \ell \in \mathbb{D}^w(\mathbb{R}^m, \mathbb{C}^m) \), respectively. Since \( \ell \) can be taken an arbitrarily trajectory in \( \mathbb{D}^w(\mathbb{C}^m, \mathbb{C}^m) \), the inequality

\[
\hat{w}(j\omega)^* \partial \Phi(j\omega) \hat{w}(j\omega) \geq 0, \text{ } \forall \omega \in \mathcal{B} \cap \mathbb{D}^w(\mathbb{R}^n, \mathbb{C}^n), \text{ } \omega \in \Omega
\]

is equivalent to the FFDI (17). We can regard the above inequality imposes a weighted frequency constraint on \( w \in \mathcal{B} \) over the restricted frequency domain \( \Omega \). Hence, it expresses the weighted rate limitation on the trajectories contained in \( \mathcal{B} \), although the FFDI (17) is described by using \( M(\xi) \).

Remark 3 Chakrabarti et.al. [5] considered the two-dimensional frequency domain \( \Omega \subset \mathbb{R}^2 \) which is a compact subset of \([0, \infty) \times [0, \infty)\) containing \( 0 \) in the design of a two-dimensional low pass filter. For example, the set is expressed by a linear combination of \( \pi_1 \) and \( \pi_2 \) as

\[
\Omega = \left\{ \omega \in \mathbb{R}^2 \mid \omega_1 \geq 0, \text{ } \omega_2 \geq 0 \text{ and } \omega_1 + \omega_2 = \pi_0 \right\},
\]

where \( \pi_0 \in \mathbb{R} \) is a nonnegative constant. This set cannot be represented in the form of (14), since the domain is described by the sum of the frequency variables. It remains as a future work to characterize the finite frequency properties for such frequency domains.

4. Characterization of Finite Frequency Properties

This section derives a characterization of the finite frequency properties using QDFs for \( n \)-dimensional behaviors as a main result. We give a finite frequency property characterization in Section 4.1. A physical interpretation of the characterization is provided in Section 4.2. Finally, we give a characterization of the property in terms of \( \mathcal{B} \)-canonical polynomial matrices [22] in Section 4.3.

4.1 Main Theorem

In this subsection, we derive a characterization of the FFDI (17) using QDFs as a main result.

We first point out what issues should be solved in this paper before we provide our main result. In order to generalize the previous characterizations [19], [20] to the \( n \)-dimensional and finite frequency case, we should examine the following two points from a theoretical view point, which are also illustrated in Fig. 1.

- We cannot obtain a spectral factorization of the polynomial matrix \( \partial \Phi(\xi) \) constructed by \( \Phi \in \mathbb{H}^{im}[\zeta, \eta] \) in (8), since we restrict the frequency domain to \( \Omega \) in (14). The factorization played an important role to construct a dissipation rate in the characterization of [19] \(^1\).

- It is not clear how a compensation rate can be expressed in the \( n \)-dimensional case. In [20], the characterization was derived by using a property that a compensation rate is induced by a polynomial matrix which is nonnegative definite in the \( (n) \) frequency domain.

We can resolve the first point on the spectral factorization by naturally generalizing the idea of [20] to the \( n \)-dimensional case. See the proof of Lemma 2.1(i) \( \Rightarrow \) (ii) in the technical report [23] for the detail. Thus, it can be the main focus to tackle the second point on a compensation rate. We explain it in detail after we show the main result (Theorem 1) of this paper. This relates to an answer to the latter part of Question (i).

For the purpose stated in the above paragraph, we introduce some notions to construct a compensation rate. Define \( \sigma_{\xi k-}, \sigma_{\xi k+} : \mathbb{R} \to \mathbb{R} \) \((k = 1, \ldots, n)\) by

\[
\sigma_{\xi k-} := \frac{\sigma_{\xi k, 2} - \sigma_{\xi k, 1}}{2} \text{ and } \sigma_{\xi k+} := \frac{\sigma_{\xi k, 1} + \sigma_{\xi k, 2}}{2},
\]

and the set \( \mathcal{G} \subset \mathbb{H}^{im}[\zeta, \eta] \) by

\[
\mathcal{G} := \left\{ \Gamma \in \mathbb{H}^{im}[\zeta, \eta] : \begin{align*}
\Gamma(\zeta, \eta) := \sum_{k=1}^{n} \chi_k(\zeta, \eta) \Gamma_k(\zeta, \eta) \\
\text{for some } \Gamma_k \in \mathbb{H}^{im}[\zeta, \eta] \quad (k = 1, \ldots, n) \text{ such that (21)}
\end{align*} \right\},
\]

where \( \chi_k(\zeta, \eta) = \left[ \begin{array}{c}
1 \\
\Gamma_{k, 1} - j \sigma_{\xi k, 1} \sigma_{\xi k, 2} - j \sigma_{\xi k, 3} \\
\sigma_{\xi k, 4} - 1 \\
\eta
\end{array} \right] \),

\[
\tau_k \Omega_{\Gamma_k}(t) \geq 0, \forall t \in \mathbb{C}^w(\mathbb{R}^n, \mathbb{C}^m),
\]

where \( \tau_k \) is equal to either \( +1 \) or \( -1 \) for \( k = 1, \ldots, n \). We see that \( \Gamma \in \mathcal{G} \) satisfies the inequality

\[
\partial \Gamma(j\omega) = - \sum_{k=1}^{n} \tau_k (\omega_k - \sigma_{\xi k, 1}) (\omega_k - \sigma_{\xi k, 2}) \cdot \tau_k \partial \Gamma_k(j\omega) \\
\geq 0, \forall \omega \in \Omega.
\]

\(^1\) In \( n \)-dimensional behaviors, there exists a problem that a spectral factor \( F(\xi) \) does not always becomes a polynomial matrix in the spectral factorization \( M(\xi) \otimes \partial \Phi(\xi) \otimes M(\xi) = F(\xi) \cdot F(\xi)^\dagger \), i.e. \( F(\xi) \) can be a rational function matrix. Pillai and Willems [19] have solved the problem by developing a constructive proof for the existence of a polynomial spectral factor. However, since we avoid a spectral factorization based on [20] in this paper, we need not to deal with the problem.
We have seen from Proposition 1 that the FDI (13) is equivalent to the dissipation inequality (11). Since we consider the case where the FDI (13) is restricted to the domain Ω, we can imagine that an analogous inequality to (11) holds from Proposition 1. This is explained as follows.

Assume that there exist 2n-variable polynomial matrices
\[ \Psi(\zeta, \eta) := \begin{bmatrix} \Psi_1(\zeta, \eta), & \cdots, & \Psi_n(\zeta, \eta) \end{bmatrix}, \]
\[ \Psi_k \in \mathbb{P}_{\text{max}}[\zeta, \eta] \quad (k = 1, \cdots, n) \] (23)
and \( \Gamma \in \mathcal{G} \) satisfying the inequality
\[ \text{div} Q_\Gamma(\ell) \leq Q_\Phi(\omega) - Q_\ell(\ell) \] (24)
for all \( \omega \in \mathcal{B} \) with the image representation (5). The above inequality corresponds to the dissipation inequality (11) in the finite frequency case. This is equivalent to the existence of \( \Delta \in \mathbb{P}_{\text{max}}[\zeta, \eta] \) satisfying the 2n-variable polynomial matrix equation
\[ \nabla \Psi(\zeta, \eta) = M(\zeta) \Phi(\zeta, \eta) M(\eta) - \Gamma(\zeta, \eta) - \Delta(\zeta, \eta) \] (25)
and \( Q_\Delta(\ell) \geq 0 \), \( \forall \ell \in \mathbb{C}_{\text{max}}^n \). Substituting \( \zeta = -j\omega \) and \( \eta = j\omega \) into (25), we obtain the FFDI
\[ M(j\omega)^* \Phi(j\omega) M(j\omega) = \partial T(j\omega) + \partial \Delta(j\omega) \geq 0, \quad \forall \omega \in \Omega \] from (22). The above inequality guarantees that the FFDI (17) holds.

The inequality (24) also gives a necessary condition for the finite frequency property. Thus, we obtain the following main result which equivalently characterizes the property in terms of finite frequency property. This theorem gives answers to Questions (i) and (ii) in Section 3.

**Theorem 1** Let \( \mathcal{B} \in (4) \) and \( \Phi \in \mathbb{P}_{\text{max}}[\zeta, \eta] \) in (8) be given. Suppose that Assumption 1 holds. Define \( \Omega \) by (14) and \( \mathcal{G} \) by (19). Then, the following statements (i), (ii), and (iii) are equivalent.

(i) The FFDI (17) holds for all \( \omega \in \Omega \).

(ii) There exist 2n-variable polynomial matrices \( \Psi \in \mathbb{C}_{\text{max}}[\zeta, \eta] \) in (23) and \( \Gamma \in \mathcal{G} \) satisfying the inequality (24) with the image representation (5).

(iii) The inequality
\[ \int_{\Omega} Q_\Phi(\omega) d\omega \geq 0 \] (26)
holds for all \( \omega \in \mathcal{B} \) with the image representation (5) and \( \ell \in \mathbb{D}^n(\mathbb{R}, \mathbb{C}^m) \) satisfying
\[ \tau_k \int_{\mathbb{R}^n} \text{He} \left[ \frac{\partial z_k}{\partial \zeta_k} - j\sigma z_k \right] \left[ \frac{\partial z_k}{\partial \zeta_k} - j\sigma z_k \right] d\omega \leq 0 \] (27)
for \( k = 1, \cdots, n \), where \( z_k \in \mathbb{D}^n(\mathbb{R}, \mathbb{C}^m) \) is defined by
\[ z_k := Z_{N_k} \left( \frac{d}{dx} \ell, \right) \] (28)
\[ Z_{N_k}(\ell) := \text{col} \{ I_{m_1}, \xi I_{m_2}, \cdots, \xi^{N_k} I_m \} \in \mathbb{R}^{[\Pi_{N_k}^{(1)}(N_k+1)]|m|} \] for some multi-index \( N_k := (N_{k,1}, \cdots, N_{k,n}) \in \mathbb{Z}^n \).

**Proof** See Appendix B.1 in [23] for the proof. \( \square \)

We describe the answers to Questions in Section 3 corresponding to their statements.

**Answer 1**

(i) In the inequality (24), the QDF \( Q_\Gamma(\ell) \) is called a compensation rate for \( \mathcal{B} \) with respect to the supply rate \( Q_\Phi(\omega) \) and the frequency domain \( \Omega \). This QDF is the new function which appears in the dissipation inequality (11). Since \( \mathcal{B} \) is not dissipative with respect to the supply rate \( Q_\Phi(\omega) \), \( Q_\Gamma(\ell) \) guarantees dissipativity of some rate constrained subbehavior related to \( \mathcal{B} \) and \( \Omega \). This claim gives an answer to the former part of Question (i). See also Answer 2 (i) in Section 4.2 for the further property of this function. It clarifies this point using a dissipation rate for the subbehavior.

In the following, we give an answer to the latter part of Question (i). The authors [20] proved that a compensation rate is induced by a polynomial which is nonnegative definite in the frequency domain. Although we can use the idea due to [20], the straightforward extension of [20] is not clear. For the difficulty, this paper clarifies that a compensation rate \( Q_\Gamma(\ell) \) is induced by a 2n-variable polynomial matrix \( \Gamma(\zeta, \eta) \) expressed as a summation of 2n-variable polynomial matrices which are nonnegative definite on each frequency domain. This is described in (19) and (22).

The above resolution is completed by proving that a compensation rate satisfies the inequality
\[ v^* \partial T(j\omega)v \]
\[ = -\sum_{k=1}^n \chi_k(j\omega_k)v_k^* \partial T_k(j\omega_k)v_k \]
\[ = \text{tr} \left[ \begin{array}{cccc}
v v^* & 0 & \cdots & 0 \\
0 & v v^* & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & v v^* \\
\end{array} \right] \]
\[ \cdot \begin{bmatrix}
-\partial \chi_1(j\omega_1) \partial T_1(j\omega_1) & 0 & \cdots & 0 \\
0 & -\partial \chi_2(j\omega_2) \partial T_2(j\omega_2) & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & -\partial \chi_n(j\omega_n) \partial T_n(j\omega_n) \\
\end{bmatrix} \]
\[ \geq 0, \quad \forall \omega \in \Omega \]
for some \( v \neq 0 \) and, if there exists an \( \chi_k(\zeta, \eta) \) which does not satisfy the inequality
\[ -\partial \chi_1(j\omega_1) \partial T_1(j\omega_1) \geq 0, \quad \forall \omega \in \Omega, \]
the nonnegativity of the compensation rate is violated in the frequency domain. We omit the detail description due to a space limitation. See the proof of Lemma B.1 (i)\( \Rightarrow \) (ii) and (ii)\( \Rightarrow \) (iii) in the technical report [23] for the detail.

(ii) The statement (iii) in Theorem 1 gives an answer to Question (ii). We see that the matrix integral quadratic constraint (27) is opposed to the inequality (9) as the additional constraint. See Answer 2 (ii) in Section 4.2 for the further description. This point is explained exactly in terms of dissipativity of some rate constrained subbehavior of \( \mathcal{B} \).

**Remark 4** In n-dimensional behaviors, there does not always exist an observable image representation [16]. Hence, the
Remark 5 We should remark that \(\chi_k(\xi, \eta)\) \((k = 1, \ldots, n)\) in \((20)\) is a real coefficient polynomial if \(Q\) is symmetric about the origin, e.g. the low frequency domain \(\Omega\) in \((15)\). If \(M(\xi)\) and \(\Phi(\xi, \eta)\) are all real polynomial matrices, we can restrict \(\Psi(\xi, \eta)\) and \(\Gamma(\xi, \eta)\) in Theorem 1 to real symmetric \(2n\)-variable polynomial matrices without loss of generality.

Remark 6 Similarly to Remark 2, if we regard the variable \(x_1 = t\) and \(x_2, \ldots, x_n\) as the time and space variables, respectively, then the inequality \((24)\) is rewritten as

\[
\frac{\partial}{\partial t} Q_{\Psi}(t) \leq Q_{\Phi}(w) \left\{ \frac{\partial}{\partial x_2} Q_{\Psi}(t) + \cdots + \frac{\partial}{\partial x_n} Q_{\Psi}(t) \right\} - Q_{\Gamma}(t).
\]

This inequality can be interpreted as follows from the viewpoint of dissipativity. The rate of change of the stored energy \(Q\) does not exceed the power \(Q_{\Phi}(w)\) supplied to the system with energy lost due to flux \(\sum_{k=2}^{n} \frac{\partial}{\partial x_k} Q_{\Psi}(t)\) and with the compensating power \(Q_{\Gamma}(t)\).

### 4.2 Physical Interpretation

In this subsection, we clarify a physical interpretation of Theorem 1 from the view point of the dissipation theory. Define the subbehavior \(\mathcal{B}_\Omega \subset \mathcal{B}\) by

\[
\mathcal{B}_\Omega := \left\{ w \in D^\infty(R^n, C^n) \left| w = M \begin{bmatrix} \xi \\ \eta \end{bmatrix}, \forall \xi, \eta \in D^\infty(R^n, C^n) \ s.t. \ (27) \right. \right\}.
\]

The trajectories of \(\mathcal{B}_\Omega\) vary in the frequency \(\Omega\). This implies that \(\mathcal{B}_\Omega\) is the rate constrained subbehavior of \(\mathcal{B}\). Then, we have the following corollary.

**Corollary 1** Let \(\mathcal{B}\) in \((4)\) and \(\Phi \in H^\infty_\text{sym}[\xi, \eta]\) in \((8)\) be given. Suppose that Assumption 1 holds. Define \(\mathcal{G}\) by \((19)\) and define \(\mathcal{B}_\Omega\) by \((29)\) for \(\Omega\) in \((14)\). Then, the following statements (i), (ii) and (iii) are equivalent.

(i) The FFDI \((17)\) holds for all \(w \in \Omega\).

(ii) There exist \(2n\)-variable polynomial matrices \(\Psi \in C^\text{sym}_{\text{max}}[\xi, \eta]\) in \((23)\), \(\Delta \in H^\infty_\text{sym}[\xi, \eta]\) and \(\Gamma \in \mathcal{G}\) satisfying

\[
\begin{align*}
\text{div} Q_{\Psi}(t) &= Q_{\Phi}(w) - Q_{\Delta+1}(t), \\
Q_{\Delta+1}(t) &\geq 0
\end{align*}
\]

for all \(w \in \mathcal{B}_\Omega\).

(iii) The behavior \(\mathcal{B}_\Omega\) in \((29)\) is dissipative with respect to the supply rate \(Q_{\Phi}(w)\).

**Proof** See Appendix B.2 in [23] for the proof.

Corollary 1 provides a physical interpretation of Theorem 1 from the view point of dissipativity. This gives a clear answer to the latter part of Question 3 in Section 3.

### Answer 2

(i) Answer 1 (i) states that the compensation rate \(Q_{\Gamma}(t)\) is the new function which newly appears in the dissipation inequality \((11)\). We further clarify the role of the function in this answer.

From Corollary 1 (ii), if we concentrate ourselves to the subbehavior \(\mathcal{B}_\Omega\), the QDF \(Q_{\Delta+1}(t)\) becomes the dissipation rate for \(\mathcal{B}_\Omega\) with respect to the supply rate \(Q_{\Phi}(w)\). This can be verified as follows.

Since \((31)\) and

\[
\int_{R^n} Q_{\Delta+1}(\ell) d\ell = \int_{R^n} Q_{\Phi}(w) d\ell,
\]

\(\forall w \in \mathcal{B}_\Omega \cap D^\infty(R^n, C^n) \ s.t. \ (5)\)

hold, we observe that the QDF \(Q_{\Delta+1}(t)\) in \((30)\) and \((31)\) becomes the dissipation rate for \(\mathcal{B}_\Omega\) with respect to the supply rate \(Q_{\Phi}(w)\). Since the QDF \(Q_{\Gamma}(t)\) guarantees the dissipativity, \(Q_{\Gamma}(t)\) can be considered as a compensating power. This shows that the compensation rate \(Q_{\Gamma}(t)\) plays a role which guarantees dissipativity of \(\mathcal{B}_\Omega\). This is the reason why we call \(Q_{\Gamma}(t)\) as the compensation rate for \(\mathcal{B}\) with respect to the supply rate \(Q_{\Phi}(w)\).

(ii) It is not difficult to see that \(\mathcal{B}\) is not necessarily dissipative with respect to the supply rate \(Q_{\Phi}(w)\) from Proposition 1. However, Corollary 1 (iii) states that, if we concentrate ourselves to the subbehavior \(\mathcal{B}_\Omega\), then \(\mathcal{B}_\Omega\) becomes dissipative with respect to the supply rate \(Q_{\Phi}(w)\). This corresponds to an answer to the latter part of Question 2.

### 4.3 Characterization Using \(\mathcal{B}\)-canonical Polynomial Matrices

In Theorem 1, the degree of \(\Psi(\xi, \eta)\) and \(\Gamma(\xi, \eta)\) in the statement (ii) are not specified explicitly. However, thanks to \(\mathcal{B}\)-canonical polynomial matrices [22], we can determine the bounds by the degree of the polynomial matrix which induces a kernel representation of \(\mathcal{B}\). See A. 2 and the reference [22] for the definition and basic properties of \(\mathcal{B}\)-canonical polynomial matrices.

We set some assumptions to characterize the upper bound of the degree of \(\Psi(\xi, \eta)\) and \(\Gamma(\xi, \eta)\).  

**Assumption 2**

(i) The polynomial matrix \(R \in C^\infty_\text{sym}[\xi]\) in \((2)\) is row reduced [22],[27].

(ii) The \(2n\)-variable polynomial matrix \(\Phi \in H^\infty_\text{sym}[\xi, \eta]\) in \((8)\) is \(\mathcal{B}\)-canonical.

(iii) The behavior \(\mathcal{B}\) in \((4)\) is represented by an observable image representation \((5)\).

Assumption 2 (i) does not lose any generality, because there always exists a unimodular polynomial matrix \(U \in C^\infty_\text{sym}[\xi]\) satisfying \(R_{\text{red}}(\xi) = U(\xi)R(\xi)\), where \(R_{\text{red}} \in C^\infty_\text{sym}[\xi]\) is row reduced. It should be noted that \(R_{\text{red}}(\xi)\) may be obtained by using the package Singular [24]. Assumption 2 (ii) implies that the following degree constraint holds.

\[
\deg R_k \geq \deg_0 \Phi - 1 = \deg_0 \Phi - 1, \ \forall k = 1, \ldots, n
\]

This assumption does not lose the generality. If \((32)\) does not hold, i.e. \(\deg R_k < \deg_0 \Phi - 1 = \deg_0 \Phi - 1\) for some
Proposition 3

Let \( \mathcal{B} \) in (4) and let \( \Phi \in \mathbb{H}^q_{\text{opp}}[\zeta, \eta] \) in (8) be given. Suppose that Assumptions 1 and 2 hold. Define \( \Omega \) by (14) and \( \mathcal{G}' \) by (33). Then, the following statements (i), (ii) and (iii) are equivalent.

(i) The FFDI (17) holds for all \( \omega \in \Omega \).

(ii) There exist unique 2\( n \)-variable polynomial matrices

\[ \Psi'(\zeta, \eta) = (\Psi'_1(\zeta, \eta), \ldots, \Psi'_n(\zeta, \eta)) \in \mathbb{H}^q_{\text{opp}}[\zeta, \eta] \]

and \( \Gamma' \in \mathcal{G}' \) with \( \mathcal{B} \)-canonical \( \Psi'_k, \Gamma'_k \in \mathbb{H}^q_{\text{opp}}[\zeta, \eta] \) (\( k = 1, \ldots, n \)) satisfying

\[ \text{div} Q_{\psi}(w) \leq Q_{\phi}(w) - Q_{\Gamma}(w). \] (35)

(iii) The inequality (26) holds for all \( w \in \mathcal{B} \) satisfying

\[ \tau_k \int_{\mathbb{R}^2} \text{He} \left[ \frac{\partial \zeta'_k}{\partial x_k} - j \omega \zeta'_k \right] dx \leq 0 \] (36)

for \( k = 1, \ldots, n \), where \( \zeta'_k \in \mathbb{D}^0(\mathbb{R}, \mathbb{C}) \) is defined by \( \zeta'_k = Z_{N'_k} \frac{d}{dx} w \) for some multi-index \( N'_k \in \mathbb{Z}^n \), \( N'_{2k} \leq \text{deg}_R R - 1 \) (\( l = 1, \ldots, n \)).

Proof

See Appendix B.3 in [23] for the proof. \( \square \)

Proposition 3 shows that the upper bounds of the degree of \( \Psi(\zeta, \eta) \) and \( \Gamma(\zeta, \eta) \) are determined by that of \( R(\xi) \).

5. A Numerical Example

In this section, we apply Theorem 1 and Corollary 1 to a numerical example.

Consider a two-dimensional behavior \( \mathcal{B} \subset \mathbb{C}^m(\mathbb{R}^2, \mathbb{R}^2) \) whose kernel representation is described by

\[ R \left( \frac{d}{dx} \right) w = 0, \quad R(\xi) := [-1, \xi_1, \xi^2_2 + 1], \]

where \( w := \text{col}(w_1, w_2) \) is the manifest variable. Define the frequency domain and the symmetric matrix by

\[ \Omega := \{ \omega \in \mathbb{R}^2 \mid |\omega_1|, |\omega_2| \leq 1 \} \quad \text{and} \quad \Phi := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \]

respectively. The domain \( \Omega \) is restricted to the low frequency with respect to \( \omega_2 \), however it represents an entire interval with respect to \( \omega_1 \).

Since \( \mathcal{B} \) is controllable, \( \mathcal{B} \) can be represented by the image representation

\[ w = M \left( \frac{d}{dx} \right) \xi, \quad M(\xi) := \begin{bmatrix} \xi_1 & \xi^2_2 + 1 \\ 1 & 1 \end{bmatrix}, \]

which is an observable image representation. Then, we have

\[ M^*(\xi) \Phi M(\eta) = \zeta_1 + \zeta^2_2 + \eta_1 + \eta^2_2 + 3 \]

From the above equation, we obtain

\[ M^*(\omega) \partial \Phi(\omega) M(\omega) = 3 - 2\omega^2, \]

which implies that the FFDI (17) holds for all \( \omega \in \Omega \). We observe that \( M^*(\xi) \Phi(\zeta, \eta) M(\eta) \) can be decomposed to

\[ M^*(\xi) \Phi(\zeta, \eta) M(\eta) = [\zeta_1 + \eta_1, \zeta_1 + \eta_2] \Psi(\zeta, \eta) + \Gamma(\zeta, \eta) + \Delta(\zeta, \eta). \]

where \( \Psi \in \mathbb{R}^{2\times 1}[\zeta, \eta], \Gamma \in \mathbb{R}[\zeta, \eta] \) and \( \Delta \in \mathbb{R}[\zeta, \eta] \) are given by

\[ \Psi(\zeta, \eta) := \begin{bmatrix} 1 \\ -\zeta_2 \end{bmatrix}, \quad \Gamma(\zeta, \eta) := 2(1 - \eta_2) \quad \text{and} \quad \Delta(\zeta, \eta) := 1, \]

respectively. Then, we have the inequality

\[ \text{div} Q_{\psi}(\ell) = Q_{\phi}(w) - Q_{\Gamma}(\ell) \leq \text{div} Q_{\phi}(w), \quad \ell \in \mathbb{C}^m(\mathbb{R}^2, \mathbb{R}^2), \] (37)

which satisfies Theorem 1 (ii). The inequality (37) shows that \( \mathcal{B} \) dissipates a power with the compensating power \( Q_{\Gamma}(w) \).

From Corollary 1, (37) is equivalently rewritten by the dissipation inequality

\[ \text{div} Q_{\psi}(\ell) \leq Q_{\phi}(w), \quad \forall \ell \in \mathcal{B}_{\Omega}, \]

where \( \mathcal{B}_{\Omega} \subset \mathbb{D}^m(\mathbb{R}^2, \mathbb{R}^2) \) is the subbehavior of \( \mathcal{B} \) defined by

\[ \mathcal{B}_{\Omega} := \{ w \in \mathbb{D}^m(\mathbb{R}^2, \mathbb{R}^2) \mid w = M \left( \frac{d}{dx} \right) \xi, \quad \ell \in \mathbb{D}^m(\mathbb{R}^2, \mathbb{R}^2) \text{ s.t. (38) and (39)} \}, \]

\[ \int_{\mathbb{R}^2} \frac{\partial \ell}{\partial x_1} \frac{\partial \ell}{\partial x_1} dx \geq 0, \]

\[ \int_{\mathbb{R}^2} \left( \frac{\partial \ell}{\partial x_1} \frac{\partial \ell}{\partial x_2} - z \right) dx \leq 0, \quad z := \left[ \ell \frac{\partial}{\partial x} \right]. \] (39)

This shows that \( \mathcal{B}_{\Omega} \) is dissipative with respect to the supply rate \( Q_{\phi}(w) \).

6. Conclusions

In this paper, the authors have characterized the finite frequency properties using some inequality with the compensation rate and an inequality of an integral of the supply rate with a matrix integral quadratic constraint based on QDFs as a main result. The authors have resolved the problem of an expression of the compensation rate in \( n \)-dimensional behaviors. The characterization has led to a physical interpretation in terms of the
dissipation inequality, equivalently dissipativity, for the sub-
behavior with some rate constraints. These results can be regarded
as a generalization of the previous one-dimensional results [20]
to the n-dimensional behaviors. Such an interpretation has not
been clarified by the previous studies of finite frequency prop-
erties. The aforementioned characterization also yields a char-
acterization in terms of B-canonical polynomial matrices.

As a future direction, an LMI characterization should be de-
derived, which is a tractable condition for a numerical check-
ing of the finite frequency properties. For this problem,
Yang et al. [25] derived the generalized KYP lemma to the two-
dimensional discrete-time Roesser state-space system as a su-
ficient characterization. It is desired to derive a necessary and
sufficient characterization which should be tackled in our future
work.

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Appendix Background Materials

In this appendix, we collect the background materials which are
used in the proofs.

A.1 Coefficient Matrices

We define the coefficient matrix of a 2n-variable polynomial
matrix. For this purpose, we first introduce an ordering on the
multi-index i = (i1, ··· , in) using the multi-index notation [19].
Of course, many orderings are possible. We choose the ordering
based on anti-lexicographic ordering [26].

The ordering is defined as follows. For given multi-indices
i := (i1, ··· , in), j := (j1, ··· , jn) ∈ Zn (i0 ≥ 0; k = 1, ··· , n),
we define the ordering i < j if and only if the rightmost nonzero
entry of (i1 − j1, i2 − j2, ··· , in − jn) is negative.

We give the definition of the coefficient matrix of the 2n-
variable polynomial matrices based on the ordering of the
multi-index defined in the above paragraph. With every
Φ ∈ E[R][ζ, η] in (8), we define its coefficient matrix Φ ∈
E[Ω][ζ, η] by

[Φ]
where the $(i,j)$th block matrix $\Phi_{i,j}$ $(i,j = 0, 1, \cdots, K)$ are aligned based on the ordering of multi-indices and $K = (K_1, \cdots, K_q)$. See pp. 1116-1118 of [21] for the more detailed construction of $\Phi$. Then, $\Phi(\zeta, \eta)$ is expressed as $\Phi(\zeta, \eta) = Z_k(\zeta)^t Z_k(\eta)$.

The nonnegativity of a QDF is characterized by the nonnegativity of its coefficient matrix as seen in the following lemma.

**Lemma A.1** [21] Let $\Phi \in \mathbb{H}^{\rho \sigma}[\zeta, \eta]$ in (8) be given. Define $\widetilde{\Phi} \in \mathbb{H}[\Pi_{0,1}(K_{1r1})]_{\rho \sigma}[\Pi_{0,1}(K_{1r1})]_{\rho \sigma}$ by (A.1). Then, we have $Q_0(\ell) \geq 0$ for all $\ell \in C^\rho(\mathbb{R}, \mathbb{C}^\sigma)$ if and only if $\widetilde{\Phi} \geq 0$ holds.

**A.2 $\mathcal{B}$-canonical Polynomial Matrices**

We introduce $\mathcal{B}$-canonicity of polynomial matrices in this appendix, which are taken from the references [22],[27].

We assume that $R \in C^{\rho \sigma}[\zeta]$ in (3) is row reduced [22],[27] in this section. The assumption does not lose the generality as we have explained in Section 4.1.

**Definition A.1** [22] Let $\mathcal{B}$ be represented by a kernel representation (3) for $R \in C^{\rho \sigma}[\zeta]$. Assume that $R(\xi)$ is row reduced. Let $D(\xi) \in C^{\rho \sigma}[\xi]$ be given. Let $d_i \in C^{\rho \sigma}[\xi]$ $(i = 1, \cdots, p)$ denote the $i$th row of $D(\xi)$ and $D(\xi)$, respectively. A polynomial matrix $D(\xi)$ is called $\mathcal{B}$-canonical if $\text{deg} d_i \leq \text{deg} d_j - 1$, $\forall i = 1, \cdots, p$ holds.

The next lemma ensures the uniqueness of an $R$-canonical polynomial matrix up to $\mathcal{B}$-equivalence.

**Lemma A.2** [22] Let $\mathcal{B}$ be represented by a kernel representation (3) for $R \in C^{\rho \sigma}[\zeta]$. Assume that $R(\xi)$ is row reduced. For any $D \in C^{\rho \sigma}[\zeta]$, there exists a unique $\mathcal{B}$-canonical $D' \in C^{\rho \sigma}[\zeta]$ satisfying $D\left(\frac{\partial}{\partial \xi}\right)w = D'\left(\frac{\partial}{\partial \xi}\right)w$, $\forall w \in \mathcal{B}$.

For $\Phi \in \mathbb{H}^{\rho \sigma}[\zeta, \eta]$ in (8), there exist $\tilde{F} \in C^{\text{rank}\Phi}(\Pi_{0,1}(K_{1r1}))_{\rho \sigma}$ satisfying $\Phi = \tilde{F} \Sigma_k \tilde{F}$, where $\Sigma_k \in C^{\text{rank}\Phi \times \text{rank}\Phi}$, $\tilde{F}$ is of full row rank, and $\det \Sigma_k \neq 0$. In this case, we get $\text{rank}\Sigma_k = \text{rank}\Phi$. With such a factorization of $\Phi$, we obtain a canonical factorization of $\Phi(\zeta, \eta)$ as

$$\Phi(\zeta, \eta) = F' \Sigma_k F(\eta), \quad (A.2)$$

where $F \in C^{\text{rank}\Phi}[\xi]$ is defined by $F(\xi) := \tilde{F} Z_k(\xi)$.

**Definition A.2** [22] Let $\mathcal{B}$ be represented by a kernel representation (3) for $R \in C^{\rho \sigma}[\zeta]$. Assume that $R(\xi)$ is row reduced. Let $\Phi \in \mathbb{H}^{\rho \sigma}[\zeta, \eta]$ be given by (8). Let $F \in C^{\text{rank}\Phi}[\xi]$ be defined by the canonical factorization (A.2). Then, $\Phi(\zeta, \eta)$ is called $\mathcal{B}$-canonical if $F(\xi)$ is $\mathcal{B}$-canonical.

The following result is an immediate consequence of the uniqueness of the canonical factorization of $\Phi(\zeta, \eta)$ and of Lemma A.2.

**Lemma A.3** [22] Let $\mathcal{B}$ be represented by a kernel representation (3) for $R \in C^{\rho \sigma}[\zeta]$. Assume that $R(\xi)$ is row reduced. Let $\Phi \in \mathbb{H}^{\rho \sigma}[\zeta, \eta]$ be given by (8). Then, for any $\Phi(\zeta, \eta)$, there exists a unique $\mathcal{B}$-canonical $\Phi' \in \mathbb{H}^{\rho \sigma}[\zeta, \eta]$ satisfying $Q_0(w) = Q_0(w)$, $\forall w \in \mathcal{B}$.