Algebraic Controllability of Nonlinear Mechanical Control Systems

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Abstract: This paper shows that the concept of algebraic controllability is equivalent to the concept of accessibility. Moreover, the paper gives a reduction condition for checking whether or not a given nonlinear mechanical control system is algebraically controllable. Through a quadrotor unmanned aerial vehicle example, it is demonstrated that the condition reduces a computational complexity for checking whether or not the system is algebraically controllable.

Key Words: algebraic controllability, accessibility, nonlinear mechanical control systems.

1. Introduction

The paper studies algebraic controllability of nonlinear mechanical control systems. Since the algebraic controllability is equivalent to accessibility which is defined as absence of autonomous variable [1], if a given nonlinear mechanical control system is algebraically controllable, the system is also accessible. Hence, to study an accessible property, it is valuable to give a convenient way for checking whether or not a given nonlinear mechanical control system is algebraically controllable.

Algebraic controllability of a nonlinear mechanical control system is defined by using hyper-regularity [2] of a polynomial matrix, whose each element is composed of a differential operator ring, derived from the system. If the differential operator ring is a non-commutative Euclidean domain, hyper-regularity of the polynomial matrix can be examined by repeating elementary matrix operations. Furthermore, computer algebra systems [3]–[6] can be applied to study hyper-regularity of the polynomial matrix. Moreover, algebraic controllability of a mechanical control system which can be transformed into affine first order differential equations relates to strong accessibility which is defined as absence of autonomy [7]. Thus for many mechanical control systems, instead of checking accessibility by the Lie rank condition, strong accessibility, which is characterized by the Lie rank condition, can be examined by checking hyper-regularity of the polynomial matrix. Hence, an extremely long calculation time might be required for checking whether or not a polynomial matrix is hyper-regular.

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2. Algebraic Controllability and Accessibility

This section relates algebraic controllability and accessibility of the following system

\[
\dot{x} = f(x, u),
\]

where \(x \in \mathbb{R}^n\) and \(u \in \mathbb{R}^m\) denote state and input variables, respectively, and \(f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n\) is meromorphic. In the paper, meromorphic functions are defined as the elements of the quotient field of the ring of analytic functions [1].

2.1 Algebraic Controllability

In order to define algebraic controllability, first, we give some preliminaries. Let \(M_{n,(a)}\) denote the field of all meromorphic functions depending on a finite number of variables \(\{x_i, u_i^0 \mid 1 \leq i \leq n, 1 \leq k \leq m, l \geq 0\}\). The field \(M_{n,(a)}\) can be endowed with a differential structure determined by Eq. (1) as follows:

\[
\phi(x, u, \ldots) := \frac{\partial \phi}{\partial x} f(x, u) + \sum_{i=0}^{l} \frac{\partial \phi}{\partial u^i} d_i^{(l)},
\]

where \(\phi(x, u, \ldots) \in M_{n,(a)}\). A vector space \(E_{n,(a)}\) of differential one-forms spanned over \(M_{n,(a)}\) is defined as \(E_{n,(a)} := \text{span}_{M_{n,(a)}} \{dx_i, du_i^0 \mid 1 \leq i \leq n, 1 \leq k \leq m, l \geq 0\}\). Then for any \(\phi \in M_{n,(a)}\), differential \(d : M_{n,(a)} \to E_{n,(a)}\) is defined as

\[
d\phi := \frac{\partial \phi}{\partial x} dx + \sum_{i=0}^{l} \frac{\partial \phi}{\partial u^i} du_i^{(l)},
\]

(2)

Let \(D_{n,(a)} := M_{n,(a)} \left\{ \frac{\partial}{\partial x} \right\}\). If we take \(\alpha = \sum_{i=0}^{m} \alpha_i \frac{\partial}{\partial u^i} \in D_{n,(a)}\), where \(\alpha_i \in M_{n,(a)}\), then \(\frac{d}{dt}\alpha\) is defined as

\[
\frac{d}{dt}\alpha := \sum_{i=0}^{m} (\alpha_i \frac{d}{dt} u_i^0 + \alpha_i \frac{d}{dt} u_i^0).
\]

Hence \(D_{n,(a)}\) is a left skew polynomial ring, and thus elements of \(D_{n,(a)}\) can act on the vector space \(E_{n,(a)}\), that is, the vector space \(E_{n,(a)}\) can be endowed with a differential structure by defining a derivative operator \(\frac{d}{dt}\) as follows:
\[ \frac{d}{dt} \left( \sum_{i=1}^{n} a_i dx_i + \sum_{l=0}^{m} c_{l,k} dt_k^{(l)} \right) \]
\[ := \sum_{i=1}^{n} \left( \dot{a}_i dx_i + a_i \sum_{j=1}^{n} \frac{\partial f_j}{\partial x_j} dx_i \right) + \sum_{l=0}^{m} \sum_{k=1}^{n} (c_{l,k} dt_k^{(l)} + c_{l,k} dt_k^{(l+1)}), \]

where \( \sum_{i=1}^{n} a_i dx_i + \sum_{l=0}^{m} c_{l,k} dt_k^{(l)} \in E_{(x,a)}. \) More generally, see [8],[9]. Moreover, \( D_{(x,a)} \) is simple and a non-commutative left and right Euclidean domain [10]. Hence \( D_{(x,a)} \) has the left and right Ore property [10]. Thus, \( D_{(x,a)} \) admits a skew field \( K_{(x,a)} \) of fractions containing elements of the form \( k = r^{-1}n \) or \( k = nr^{-1} \), where \( 0 \neq r \in D_{(x,a)} \) and \( n \in D_{(x,a)} \). Hence, the rank of a matrix \( R_{(x,a)} \in D_{(x,a)}^{\times m} \) is well defined as \( \dim (K_{(x,a)}^* R_{(x,a)}) = \dim (R_{(x,a)} K_{(x,a)}^*) \). A matrix \( U \in D_{(x,a)}^{\times m} \) is called unimodular if there exists a matrix \( U^{-1} \in D_{(x,a)}^{\times m} \) with \( UU^{-1} = U^{-1}U = I_n \). The following proposition can be found in [10].

**Proposition 1.** Suppose that \( R_{(x,a)} \in D_{(x,a)}^{\times m} \). Then there exist unimodular matrices \( U_{(x,a)} \in D_{(x,a)}^{\times n} \) and \( V_{(x,a)} \in D_{(x,a)}^{\times n} \) such that

\[ U_{(x,a)} R_{(x,a)} V_{(x,a)} = \begin{pmatrix} \Delta & 0 \\ 0 & 0 \end{pmatrix}, \]

where \( \Delta := \text{diag}(1, \ldots, 1, \alpha) \in D_{(x,a)}^{\times n} \), \( 0 \neq \alpha \in D_{(x,a)} \), and where \( \text{deg} \alpha \) is constant for any unimodular matrices \( U_{(x,a)} \) and \( V_{(x,a)} \) satisfying (3).

**Remark 1.** The above normal form is called the Jacobson form [10]. Since \( D_{(x,a)} \) is Euclidean [10], the matrices \( U_{(x,a)} \) and \( V_{(x,a)} \) can be obtained by repeating elementary row and column operations for the matrix \( R_{(x,a)} \). Here, elementary row (column) operations are defined as follows:

1. Interchange row (column) \( i \) and row (column) \( j \).
2. To row (column) \( i \) add \( d \in D_{(x,a)} \) times row (column) \( j \), \( i \neq j \).
3. Multiply row (column) \( i \) by a non-zero element in \( M_{(x,a)} \).

Each elementary row (column) operation on a matrix corresponds to the left (right) multiplication of the matrix by an appropriate unimodular matrix.

First, differentiating both sides of Eq. (1), we have

\[ P_{(x,a)} \left( \frac{dx}{du} \right) = 0, \]

where

\[ P_{(x,a)} := \left( \frac{d}{du} I - \frac{\partial f}{\partial u} (x, u) - \frac{\partial f}{\partial u} (x, u) \right). \]

Since \( f \) is meromorphic with respect to each variable, coefficients of polynomials of each element of \( P_{(x,a)} \) are meromorphic functions. Thus \( P_{(x,a)} \in 2^{(x,a) \times m} \). To define algebraic controllability, we define hyper-regularity [2] of \( P_{(x,a)} \).

**Definition 1.** The matrix \( P_{(x,a)} \) defined by (5) is called hyper-regular if there exist unimodular matrices \( U_{(x,a)} \in D_{(x,a)}^{\times n} \) and \( V_{(x,a)} \in D_{(x,a)}^{\times m} \) such that

\[ U_{(x,a)} P_{(x,a)} V_{(x,a)} = \begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix}. \]

**Definition 2.** System (1) is called algebraically controllable if \( P_{(x,a)} \) defined by (5) is hyper-regular.

We note that as mentioned in Remark 1, unimodular matrices \( U_{(x,a)} \) and \( V_{(x,a)} \) satisfying (6) can be obtained by repeating elementary row and column operations for the matrix \( P_{(x,a)} \).

**Remark 2.** Let us consider the following linear system

\[ \dot{x} = Ax + Bu. \]

Then differentiating both sides of (7), we have

\[ \frac{dx}{P_{\text{linear}}} = 0. \]

In the behavioral theory [11], it is known that linear system (7) is controllable in the usual sense if and only if there exist unimodular matrices \( U \in \left( R \left[ \frac{d}{du} \right] \right)^{\times n} \) and \( V \in \left( R \left[ \frac{d}{du} \right] \right)^{\times m} \) such that

\[ UP_{\text{linear}} V = \begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix}. \]

Therefore, if a given system (1) is linear, algebraic controllability is equivalent to the concept of controllability in the usual sense.

**2.2 Accessibility**

This subsection shows that nonlinear system (1) is algebraically controllable if and only if the system is accessible. Accessibility is defined by using a concept of autonomous variable [1]. We define the subspace of \( E_{(x,a)} \) as

\[ X := \text{span}_{M_{(x,a)}} \{ dx_i, i = 1, \ldots, n \}. \]

**Definition 3.** A one-form \( \omega \in X \) is called an autonomous variable of system (1) if there exists \( \alpha \in D_{(x,a)}, \text{deg} \alpha \geq 1 \) such that

\[ \alpha \omega = 0. \]

**Definition 4.** System (1) is called accessible if there does not exist any non-zero autonomous variable in \( X \).

It is known [1] that a one-form \( \omega \in X \) is an autonomous variable if and only if it has an infinite relative degree. Here, if \( d\phi \in X \), \( \phi \in M_{(x,a)} \), has infinite relative degree, it means that \( \phi^{(k)}, k \geq 0 \), is not influenced by a control input \( u \). Moreover, it is known [1],[12] that when system (8) has an affine form

\[ \dot{x} = f_0(x) + \sum_{i=1}^{m} f_i(x) u_i, \]

the concept of accessibility defined in Definition 4 is locally equivalent to strong accessibility defined in [7]. Thus it is valuable to give a convenient way for examining accessibility. As
mentioned in Remark 1, algebraic controllability can be examined by elementary matrix operations for the polynomial matrix \( P_{(\tau,\alpha)} \) derived from a given system (1). Therefore if accessibility and algebraic controllability are equivalent, accessibility of system (1) can be also investigated by elementary matrix operations on the polynomial matrix \( P_{(\tau,\alpha)} \). We want to relate algebraic controllability and accessibility of system (1). To this end, we give some lemmas.

**Lemma 1.** Let \( A \in \mathcal{D}_{(\tau,\alpha)}^{n(n)} \). If there exists a matrix \( B \in \mathcal{D}_{(\tau,\alpha)}^{n(n)} \) such that

\[
AB = I_m,
\]

then \( A \) is unimodular.

**Proof.** See Appendix. \( \square \)

Using Lemma 1, we can relate hyper-regularity and left primeness of the matrix \( P_{(\tau,\alpha)} \) defined by (5). Here, the polynomial matrix \( P_{(\tau,\alpha)} \) is called left prime if there exist matrices \( L_{(\tau,\alpha)} \in \mathcal{D}_{(\tau,\alpha)}^{n(n)} \) and \( P'_{(\tau,\alpha)} \in \mathcal{D}_{(\tau,\alpha)}^{n(n+m)} \) such that \( P_{(\tau,\alpha)} = L_{(\tau,\alpha)}P'_{(\tau,\alpha)} \), where \( L_{(\tau,\alpha)} \) is unimodular.

**Lemma 2.** The matrix \( P_{(\tau,\alpha)} \) defined by (5) is left prime if and only if \( P_{(\tau,\alpha)} \) is hyper-regular.

**Proof.** See Appendix. \( \square \)

By a similar way of the proof of Theorem 14 in [13], we have the following proposition.

**Proposition 2.** System (1) is accessible if and only if \( P_{(\tau,\alpha)} \) defined by (5) is left prime.

By Lemma 2 and Proposition 2, we can relate algebraic controllability and accessibility.

**Theorem 1.** System (1) is algebraically controllable if and only if the system is accessible.

Note that there are some previous studies of algorithms for the computation of the Jacobson form of a polynomial matrix over a non-commutative Euclidean domain [3]–[6]. For the computation of the Jacobson form, we can apply the library called “Jacobson.lib” of the computer algebra system SINGULAR:PLURAL. However, \( \mathcal{D}_{(\tau,\alpha)} \) does not coincide with the skew polynomial ring studied in references [3],[4],[6]. Thus for the computation of the Jacobson form of \( P_{(\tau,\alpha)} \) defined by (5), we need to improve the existing algorithm. Furthermore even if we could develop a new algorithm for the computation of the Jacobson form, an extremely long calculation time might be required due to large matrix size. In order to resolve the problem, the paper gives a reduction condition for the computation of the Jacobson form of (5). To study a reduction condition, it is convenient to give a special structure for general system (5). Hence from the next section, we study nonlinear mechanical control systems.

3. Algebraic Controllability of Mechanical Control Systems

This section restricts our attention to the following mechanical control system

\[
M(q)\ddot{q} = C(q, \dot{q}) + B(q)u, \quad \text{for} \quad \text{system (10)}
\]

where \( q \in \mathbb{R}^n \) and \( u \in \mathbb{R}^m \) denote configuration and input variables, respectively. \( M(q) \in \mathbb{R}^{n \times n} \) is invertible at all \( q \in \mathbb{R}^n \). Here, each entry of \( M(q) \) and \( g(q, \dot{q}, u) \) is meromorphic with respect to each variable. Although we can transform (11) into the form (9), if each element of \( M(q) \) is a complicated function with respect to \( q \), the calculation of the inverse matrix of \( M(q) \) may be very hard. Hence we define algebraic controllability of mechanical control systems (11) without using the relation \( \ddot{q} = M(q)^{-1}g(q, \dot{q}, u) \). To this end, we need some mathematical preliminaries.

Let \( \tilde{M}_{(\tau,\alpha)} \) denote the field of all meromorphic functions depending on a finite number of variables of \( \{q_i^{(0)}, u_i^{(0)} \mid 1 \leq i \leq n, 1 \leq j \leq m, l \geq 0 \} \). Here we do not use the relation \( \ddot{q} = M^{-1}(q)g(q, \dot{q}, u) \) for the above mentioned reason. For any \( \phi(q, \dot{q}, \ldots, u, \dot{u}, \ldots) \in \tilde{M}_{(\tau,\alpha)} \) we define

\[
\dot{\phi}(q, \dot{q}, \ldots, u, \dot{u}, \ldots) := \sum_{l=0} \left( \frac{\partial\phi}{\partial q} q^{(l+1)} + \frac{\partial\phi}{\partial u} u^{(l+1)} \right).
\]

A vector space \( \tilde{E}_{(\tau,\alpha)} \) of differential one forms spanned over \( \tilde{M}_{(\tau,\alpha)} \) is defined as

\[
\tilde{E}_{(\tau,\alpha)} := \text{span}_{\tilde{M}_{(\tau,\alpha)}} \left\{ dq_i^{(l)}, du_i^{(l)} \right\} \quad 1 \leq i \leq n, 1 \leq j \leq m, l \geq 0.
\]

Then for any \( \phi \in \tilde{M}_{(\tau,\alpha)}, \) differential \( \delta : \tilde{M}_{(\tau,\alpha)} \rightarrow \tilde{E}_{(\tau,\alpha)} \) is defined as

\[
\delta \phi := \sum_{l=0} \left( \frac{\partial\phi}{\partial q} dq^{(l)} + \frac{\partial\phi}{\partial u} du^{(l)} \right).
\]

Let \( \tilde{D}_{(\tau,\alpha)} := \tilde{M}_{(\tau,\alpha)} \left[ \frac{d}{d\tau} \right] \). The product of elements of \( \tilde{D}_{(\tau,\alpha)} \) is defined by the same manner in the case of \( D_{(\tau,\alpha)} \). Hence \( \tilde{D}_{(\tau,\alpha)} \) is also simple and a non-commutative Euclidean domain. The vector space \( \tilde{E}_{(\tau,\alpha)} \) can be endowed with a differential structure by defining a derivative operator \( \frac{d}{d\tau} \) in the same way in the case of \( E_{(\tau,\alpha)} \).

Now, differentiating both sides of system (11), we have

\[
\tilde{P}_{(\tau,\alpha)} \left[ \frac{dq}{du} \right] = 0,
\]

where

\[
\tilde{P}_{(\tau,\alpha)} := \left( M(q) \frac{d}{d\tau} - \frac{\partial M}{\partial q} \ddot{q} - \cdots - \frac{\partial M}{\partial \dot{q}} \dot{q} \right).
\]

(12)

Since each entry of \( M(q) \) and \( g(q, \dot{q}, u) \) are meromorphic with respect to each variable, \( \tilde{P}_{(\tau,\alpha)} \in \tilde{D}_{(\tau,\alpha)}^{n(n+m)} \). Since we have a similar proposition with Proposition 1, we can define algebraic controllability of system (11) as follows.

**Definition 5.** System (11) is called algebraically controllable if \( \tilde{P}_{(\tau,\alpha)} \) defined by (12) is hyper-regular, that is, there exist unimodular matrices \( \tilde{U}_{(\tau,\alpha)} \in \tilde{D}_{(\tau,\alpha)}^{n(n)} \) and \( \tilde{V}_{(\tau,\alpha)} \in \tilde{D}_{(\tau,\alpha)}^{n(n+m)} \) such that

\[
\tilde{U}_{(\tau,\alpha)} \tilde{P}_{(\tau,\alpha)} \tilde{V}_{(\tau,\alpha)} = \begin{pmatrix} I_n & 0 \end{pmatrix}.
\]
From now on, we show that if system (11) is algebraically controllable, the transformed system expressed by the form (9) is also algebraically controllable. This means that if we want to check accessibility of a given nonlinear mechanical control system (11), we can examine it by checking algebraic controllability of the system without transforming into the form (9). First, we show that even if we multiply \( M^{-1}(q) \) from the left of both sides, algebraic controllability is invariant.

**Lemma 3.** System (11) is algebraically controllable if and only if system (13) is algebraically controllable.

\[
\ddot{q} = M^{-1}(q)g(q, \dot{q}, u)
\]

is algebraically controllable.

**Proof.** See Appendix. \( \square \)

Next, we want to show that algebraic controllability of system (13) is equivalent to that of system

\[
\begin{cases}
\dot{q} = v, \\
\dot{v} = M^{-1}(q)g(q, v, u)
\end{cases}
\]

To this end, we prepare the following lemma.

**Lemma 4.** System

\[
\ddot{q} = h(q, \dot{q}, u)
\]

is algebraically controllable if and only if system

\[
\begin{cases}
\ddot{q} = v, \\
\dot{v} = h(q, v, u)
\end{cases}
\]

is algebraically controllable, where \( h : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \) is meromorphic with respect to each variable.

**Proof.** See Appendix. \( \square \)

Lemmas 3 and 4 yield the following theorem.

**Theorem 2.** System (11) is algebraically controllable if and only if system (14) is algebraically controllable.

By Theorem 2, if each entry of the matrix \( M(q) \) is very complicated function, algebraic controllability of (14) can be examined by checking algebraic controllability of (11) without calculating \( M^{-1}(q) \). However, a long calculation time might be needed to directly check algebraic controllability of a given system (11). To reduce a computational complexity, in the next section, we show a reduction condition for checking algebraic controllability of system (11).

4. Reduction Condition for Algebraic Controllability

The section gives a reduction condition for checking algebraic controllability of mechanical control systems (11). To this end, we define

\[
A(q, \dot{q}, \ddot{q}) := M(q)\dot{q} - C(q, \dot{q}).
\]

Then system (11) is equivalent to

\[
E := A(q, \dot{q}, \ddot{q}) - B(q)u = 0.
\]

For analysis of system (17), we split \( q = \begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix}, \quad \ddot{q} = \begin{pmatrix} q_{n-m+1} \\ \vdots \\ q_n \end{pmatrix} \)

\[
A(q, \dot{q}, \ddot{q}) = \begin{pmatrix} A^1(q, \dot{q}, \ddot{q}) \\ A^2(q, \dot{q}, \ddot{q}) \end{pmatrix}
\]

\[
B(q) = \begin{pmatrix} B^1(q) \\ B^2(q) \end{pmatrix}
\]

where

\[
q^1 := \begin{pmatrix} q_1 \\ \vdots \\ q_{n-m} \end{pmatrix}, \quad q^2 := \begin{pmatrix} q_{n-m+1} \\ \vdots \\ q_n \end{pmatrix},
\]

\[
A^1(q, \dot{q}, \ddot{q}) := \begin{pmatrix} A_1(q, \dot{q}, \ddot{q}) \\ \vdots \\ A_{n-m}(q, \dot{q}, \ddot{q}) \end{pmatrix},
\]

\[
A^2(q, \dot{q}, \ddot{q}) := \begin{pmatrix} A_{n-m+1}(q, \dot{q}, \ddot{q}) \\ \vdots \\ A_n(q, \dot{q}, \ddot{q}) \end{pmatrix},
\]

\[
B^1(q) := \begin{pmatrix} B_{1,1}(q) & \cdots & B_{1,n}(q) \\ \vdots & \ddots & \vdots \\ B_{n-m,1}(q) & \cdots & B_{n-m,n}(q) \end{pmatrix},
\]

\[
B^2(q) := \begin{pmatrix} B_{n-m+1,1}(q) & \cdots & B_{n-m+1,n}(q) \\ \vdots & \ddots & \vdots \\ B_{n,1}(q) & \cdots & B_{n,n}(q) \end{pmatrix},
\]

Then, differentiating \( E = 0 \), we get

\[
d\bar{E} = \begin{pmatrix} P_1^1 & P_1^2 \\ P_2^1 & P_2^2 \end{pmatrix} \begin{pmatrix} dq_1^1 \\ dq_1^2 \\ dq_2^1 \\ dq_2^2 \end{pmatrix},
\]

where

\[
P_j := \frac{\partial A^j}{\partial q_1} + \frac{\partial A^j}{\partial q_2} \frac{dq_2}{dt} + \frac{\partial A^j}{\partial q_2} \frac{dq_2}{dt} + \sum_{k=1}^m u_k \frac{\partial B_k^j}{\partial q_1},
\]

and \( B_k^j(q) \) represents \( k \)-th column vector of \( B^j(q) \).

To get a sufficient condition for algebraic controllability, we put the following assumption.

**Assumption 1.** \( \det B^2(q) \neq 0 \).

**Remark 3.** If Assumption 1 holds, \( (B^2(q))^{-1} \in (D_{(q,u)})^{m \times m} \).

We note that for many practical systems, Assumption 1 is satisfied because this assumption means that the number of independent control inputs equals \( m \). Let us suppose that Assumption 1 holds. Then,

\[
PV = \begin{pmatrix} p_1^1 & p_1^2 \\ p_2^1 & p_2^2 \end{pmatrix} B^1(B^2)^{-1} I_m,
\]

where

\[
V := \begin{pmatrix} I_{n-m} \\ I_m \end{pmatrix},
\]

In addition,

\[
\tilde{P} := U_1 PV_1 V_2 V_3 \begin{pmatrix} p_1^1 - B^1(B^2)^{-1} p_1^2 \\ p_2^1 - B^1(B^2)^{-1} p_2^2 \\ 0 \\ 0 \\ I_m \end{pmatrix}.
\]
where
\[ U_1 := \begin{pmatrix} I_{n-m} & -B^1(B^2)^{-1} \\ 0 & I_m \end{pmatrix}, \]
\[ V_2 := \begin{pmatrix} I_{n-m} \\ -P_2^1 I_m \end{pmatrix}, \]
\[ V_3 := \begin{pmatrix} I_{n-m} \\ I_m \\ -P_2^1 I_m \end{pmatrix}. \]

We can conclude that if Assumption 1 and the following assumption hold, then system (11) is algebraically controllable.

**Assumption 2.** The matrix \( P_1^1 - B^1(B^2)^{-1}P_2^2 \in \mathcal{D}_{(q,u)}^{(n-m)(n-m)} \) is unimodular or the matrix \( P_1^2 - B^1(B^2)^{-1}P_2^2 \in \mathcal{D}_{(q,u)}^{(n-m)(n-m)} \) is hyper-regular.

From now on, we prove the above mentioned fact.

### 4.1 \( P_1^1 - B^1(B^2)^{-1}P_2^1 \) is Unimodular

Since \( P_1^1 - B^1(B^2)^{-1}P_2^1 \in \mathcal{D}_{(q,u)}^{(n-m)(n-m)} \) is unimodular, there exists unimodular matrix \( R_1 \in \mathcal{D}_{(q,u)}^{(n-m)(n-m)} \) such that

\[ (P_1^1 - B^1(B^2)^{-1}P_2^1)R_1 = I_{n-m}. \]

Correspondingly, we have

\[ \tilde{P}V_4 = \begin{pmatrix} I_{n-m} \\ 0 \\ 0 \\ I_m \end{pmatrix}. \]

where
\[ V_4 := \begin{pmatrix} R_1 \\ I_m \\ I_n \end{pmatrix}. \]

Hence

\[ U_2 \tilde{P}V_4 V_5 V_6 = (I_n, 0), \]

where
\[ V_5 := \begin{pmatrix} I_{n-m} \\ -(P_2^1 - B^1(B^2)^{-1}P_2^2) \\ I_m \end{pmatrix}, \]
\[ V_6 := \begin{pmatrix} I_{n-m} \\ I_n \end{pmatrix}. \]

From (18), if Assumption 1 holds and \( P_1^1 - B^1(B^2)^{-1}P_2^1 \) is unimodular, system (11) is algebraically controllable.

### 4.2 \( P_1^1 - B^1(B^2)^{-1}P_2^1 \) is Hyper-Regular

Since \( P_2^1 - B^1(B^2)^{-1}P_2^2 \in \mathcal{D}_{(q,u)}^{(n-m)(n-m)} \) is hyper-regular, \( n-m \leq m \) and there exist unimodular matrices \( L_1 \in \mathcal{D}_{(q,u)}^{(n-m)(n-m)} \) and \( R_2 \in \mathcal{D}_{(q,u)}^{(n-m)(n-m)} \) such that

\[ L_1(P_2^1 - B^1(B^2)^{-1}P_2)R_2 = (I_{n-m} 0). \]

Correspondingly, we have

\[ U_2 \tilde{P}V_7 = \begin{pmatrix} L_1(P_1^1 - B^1(B^2)^{-1}P_2^2) \\ I_{n-m} \\ 0 \\ 0 \\ I_m \end{pmatrix}. \]

where
\[ U_2 := \begin{pmatrix} L_1 \\ I_m \end{pmatrix}, \]
\[ V_7 := \begin{pmatrix} I_{n-m} \\ R_2 \\ I_m \end{pmatrix}. \]

Hence

\[ U_2 \tilde{P}(V_7 \cdots V_{10}) = (I_n, 0), \] (19)

From (19), if Assumption 1 holds and \( P_1^1 - B^1(B^2)^{-1}P_2^2 \) is hyper-regular, system (11) is algebraically controllable.

### 5. Quadrotor Unmanned Aerial Vehicle

Using a quadrotor unmanned aerial vehicle example [14], which is a mechanical control system with six degrees of freedom and four control inputs, we demonstrate that Assumptions 1 and 2 reduce a computational complexity for checking whether or not a given system (11) is algebraically controllable, that is, accessible.

We regard the quadrotor UAV as a rigid body, whose configuration space is \( \mathbb{R}^3 \times SO(3) \) [15]. Since \( \mathbb{R}^3 \times SO(3) \) is a six dimensional manifold, we can locally consider \( \mathbb{R}^3 \times SO(3) \) as \( \mathbb{R}^6 \). Let \( (x, y, z, \phi, \theta, \psi) \) be local coordinates of \( \mathbb{R}^3 \times SO(3) \), where \( (x, y, z) \) denotes the position of the center of gravity of the quadrotor UAV, and \( \phi, \theta, \psi \) denote the roll, pitch, and yaw angles of UAV in an inertial frame, respectively. The Lagrangian of this system \( L : T(\mathbb{R}^3 \times SO(3)) \rightarrow \mathbb{R} \) is given by

\[ L := \frac{1}{2}m(x^2 + y^2 + z^2) + \frac{1}{2} \omega^T J \omega - mgz, \] (20)

where \( m \) denotes the mass of the vehicle and \( J := \begin{pmatrix} J_1 & 0 & 0 \\ 0 & J_2 & 0 \\ 0 & 0 & J_3 \end{pmatrix} \) is the inertia matrix of UAV in the body frame, and \( g \) is the gravitational acceleration. Further, \( \omega \) in (20) denotes the angular velocity of the vehicle in the body frame [15], and is expressed as \( \omega = \begin{pmatrix} 1 & 0 & -\sin \theta \\ 0 & \cos \phi \sin \theta & \sin \phi \cos \theta \\ 0 & -\sin \phi \cos \theta & \cos \phi \cos \theta \end{pmatrix}. \)

In terms of the local coordinates \( q := (x, y, z, \phi, \theta, \psi) \), the Lagrangian control system of the quadrotor UAV is then subject to the equations of motion
where
\[ B(q) := \begin{bmatrix} b_1 & 0 & 0 & 0 \\ b_2 & 0 & 0 & 0 \\ \cos \phi \cos \theta & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos \phi & \sin \phi \\ 0 & -\sin \theta & \sin \phi \cos \theta & \cos \phi \cos \theta \end{bmatrix}. \]

Differentiating \( E \), we get
\[ dE = P \left( \frac{dq}{du} \right)^2, \]
where
\[ P := \begin{pmatrix} P_1^1 & P_1^2 \\ P_2^1 & -P_2^2 \end{pmatrix}, \]
and
\[ q^1 := (x, y), \quad q^2 := (z, \phi, \theta, \psi), \]
and
\[ P_1^1 := \begin{pmatrix} m \frac{d^2}{dt^2} & 0 \\ 0 & m \frac{d^2}{dt^2} \end{pmatrix}, \quad P_1^2 := \begin{pmatrix} 0 & a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 & 0 \end{pmatrix}, \]
\[ P_2^1 := \begin{pmatrix} 0 & * & * & * \\ 0 & * & * & * \end{pmatrix}, \quad P_2^2 := \begin{pmatrix} \cos \phi \sin \theta & 0 & 0 \\ \cos \phi \sin \psi & \cos \phi \cos \psi & 0 \end{pmatrix}. \]

If we want to directly check the strong accessibility rank condition defined in [7], we have to transform (22) into the form (9). However, since the transformed first order differential equations are very complicated, it is not easy to examine the strong accessibility rank condition. Thus we want to check algebraic controllability which is locally equivalent to strong accessibility defined in [7].

**References**

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Therefore system (22) is algebraically controllable. Hence by Theorem 2, even if system (22) is transformed into the form (9), the transformed system described by a unimodular, so that Assumption 2 holds. From the discussion of Subsection 4.2, there exist unimodular matrices $U$ and $V$ such that

$$U P V = \begin{pmatrix} I_6 & 0 \end{pmatrix}.$$ 

6. Conclusion

We have shown that the concepts of algebraic controllability and accessibility are equivalent. Moreover, we have provided a reduction condition for checking whether or not a given nonlinear mechanical control system is algebraically controllable. By applying the condition to a quadrotor unmanned aerial vehicle, we have demonstrated that it reduces a computational complexity for checking algebraic controllability.

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References


Appendix

Proof of Lemma 1:

Proof. By definition of rank, we have

$$m = \text{rank} \ I_m = \text{rank} \ (AB) \leq \text{rank} \ A \leq m.$$ 

(A. 1)

Hence rank $A = m$. Eq. (A. 1) implies $ABA = A$, so that $A(ABA - I_m) = 0$. Since rank $A = m$,

$$BA = I_m.$$ 

Therefore $A$ is unimodular. 

Proof of Lemma 2:

Proof. Suppose that the matrix $P_{(\alpha)}$ is left prime. By Proposition 1, there exist unimodular matrices $U_{(\alpha,a)} \in D_{(\alpha,a)}^{\rho a}$ and $V_{(\alpha,a)} \in D_{(\alpha,a)}^{\alpha (n+m)}$ such that

$$U_{(\alpha,a)} P_{(\alpha)} V_{(\alpha,a)} = \begin{pmatrix} \Delta & 0 \end{pmatrix},$$

where $\Delta := \text{diag} (1, \cdots, 1, \alpha) \in D_{(\alpha,a)}^{\alpha \rho a}$, $0 \neq \alpha \in D_{(\alpha,a)}$. Then if we denote $V^{-1}_{(\alpha,a)}$ as $\begin{pmatrix} V^{-1}_1 & V^{-1}_2 \end{pmatrix}$, and $P_{(\alpha,a)} = \begin{pmatrix} (U^{-1}_{(\alpha,a)}) & V^{-1}_1 \end{pmatrix}$.

Since $P_{(\alpha,a)}$ is left prime, $U^{-1}_{(\alpha,a)} \Delta$ is unimodular. Then since $U^{-1}_{(\alpha,a)}$ is unimodular, $\Delta$ is unimodular. If $\text{deg} \ \alpha \geq 1$, $\Delta$ is not unimodular, so that $\text{deg} \ \alpha = 0$.

Conversely, suppose that the matrix $P_{(\alpha,a)}$ is hyper-regular, that is, there exist unimodular matrices $U_{(\alpha,a)} \in D_{(\alpha,a)}^{\rho a}$ and $V_{(\alpha,a)} \in D_{(\alpha,a)}^{\alpha (n+m)}$ satisfying

$$U_{(\alpha,a)} P_{(\alpha)} V_{(\alpha,a)} = \begin{pmatrix} I_n & 0 \end{pmatrix}.$$
Then by a straightforward calculation, $\left( P_{(x,u)} \right)_{(T_{(x,u)})} = \left( U_{(x,u)}^{-1} 0 \begin{pmatrix} 0 & V_{(x,u)}^{-1} \end{pmatrix}, where $T_{(x,u)} := \begin{pmatrix} 0 & I_m \end{pmatrix} V_{(x,u)}^{-1}$. Hence $\left( P_{(x,u)} \right)_{(T_{(x,u)})}$ is unimodular, so that there exists $\begin{pmatrix} P_{(x,u)} & \bar{T}_{(x,u)} \end{pmatrix} \in D_{(x,u)}(n+m)_{(x,u)}$ such that $\left( P_{(x,u)} \right)_{(T_{(x,u)})} = \begin{pmatrix} I_n & \bar{T}_{(x,u)} \end{pmatrix}$. Therefore there exists $P_{(x,u)} \in D_{(x,u)}(n+m)$ such that $P_{(x,u)} \bar{T}_{(x,u)} = I_n$. Thus if there exist matrices $L_{(x,u)} \in D_{(x,u)}(n+m)$ and $P'_{(x,u)} \in D_{(x,u)}(n+m)$ such that $P_{(x,u)} = L_{(x,u)} P'_{(x,u)}$, we have $L_{(x,u)} P'_{(x,u)} = I_n$.

Then by Lemma 1, $L_{(x,u)}$ is unimodular. □

**Proof of Lemma 3:**

**Proof.** From (13), we define $F := \tilde{q} - M^{-1}(q)g(q, \tilde{q}, u) = 0$.

Then $dF = \tilde{P}_{(q,u)} \frac{dq}{du}$, where

$$
\tilde{P}_{(q,u)} := \left( \frac{\partial}{\partial q} I_n - M^{-1}(q) \frac{\partial}{\partial \dot{q}} M^{-1} \right) \frac{\partial}{\partial \ddot{q}},
$$

Since

$$
\frac{\partial M^{-1}}{\partial q} = -M^{-1} \frac{\partial M}{\partial q} M^{-1}, \quad i = 1, \ldots, n,
$$

we have

$$
\frac{\partial}{\partial q} \left( M^{-1}(q)g(q, \tilde{q}, u) \right) = -M^{-1}(q) \left( \frac{\partial}{\partial \dot{q}} \tilde{q} \right) \cdots \frac{\partial}{\partial \ddot{q}} \tilde{q}
$$

$$
+ M^{-1}(q) \frac{\partial}{\partial \ddot{q}}.
$$

Thus, we obtain

$$\tilde{P}_{(q,u)} = M^{-1}(q) \tilde{P}_{(q,u)}.$$

Hence if system (11) is algebraically controllable, there exist unimodular matrices $U_{(q,u)}, \tilde{V}_{(q,u)}$ such that

$$
U_{(q,u)} \tilde{P}_{(q,u)} \tilde{V}_{(q,u)} = \begin{pmatrix} I_n & 0 \end{pmatrix},
$$

$$
\Leftrightarrow \left( U_{(q,u)} M(q) \right) M^{-1}(q) \tilde{P}_{(q,u)} \tilde{V}_{(q,u)} = \begin{pmatrix} I_n & 0 \end{pmatrix},
$$

$$
\Leftrightarrow \left( U_{(q,u)} M(q) \right) \tilde{P}_{(q,u)} \tilde{V}_{(q,u)} = \begin{pmatrix} I_n & 0 \end{pmatrix}.
$$

Therefore system (13) is also algebraically controllable.

Conversely, if system (13) is algebraically controllable, there exist unimodular matrices $\hat{U}_{(q,u)}, \hat{V}_{(q,u)}$ such that

$$
\hat{U}_{(q,u)} \hat{P}_{(q,u)} \hat{V}_{(q,u)} = \begin{pmatrix} I_n & 0 \end{pmatrix},
$$

$$
\Leftrightarrow \left( \hat{U}_{(q,u)} M^{-1}(q) \right) M(q) \hat{P}_{(q,u)} \hat{V}_{(q,u)} = \begin{pmatrix} I_n & 0 \end{pmatrix},
$$

$$
\Leftrightarrow \left( \hat{U}_{(q,u)} M^{-1}(q) \right) \hat{P}_{(q,u)} \hat{V}_{(q,u)} = \begin{pmatrix} I_n & 0 \end{pmatrix}.
$$

Therefore system (11) is also algebraically controllable. □

**Proof of Lemma 4:**

**Proof.** From (15), we define $F_1 := \tilde{q} - h(q, \tilde{q}, u) = 0$. Then $dF_1 = P_1 \frac{dq}{du}$, where

$$
P_1 := \left( \frac{\partial}{\partial \tilde{q}} I_n + \frac{\partial}{\partial \dot{q}} \frac{\partial}{\partial \ddot{q}} - \frac{\partial}{\partial \dot{q}} \frac{\partial}{\partial \ddot{q}} \right).
$$

In addition, from (16), we define $F_2 := \begin{pmatrix} \tilde{q} - v \\ \dot{v} - h(q, v, u) \end{pmatrix} = 0$.

Then $dF_2 = P_2 \frac{dv}{du}$, where

$$
P_2 := \left( I_n \frac{\partial}{\partial \tilde{v}} I_n - \frac{\partial}{\partial \dot{v}} - \frac{\partial}{\partial \ddot{v}} \right).
$$

Then by a straightforward calculation,

$$
P_3 := U_1 P_2 V_1 = \begin{pmatrix} P_1 & 0 \\ 0 & I_n \end{pmatrix}, (A.2)
$$

where $U_1 := \left( \frac{\partial}{\partial \tilde{v}} I_n \right)$, $V_1 := \begin{pmatrix} I_n & 0 \end{pmatrix}$. If system (15) is algebraically controllable, there exist unimodular matrices $U, V$ such that $UP_V = \begin{pmatrix} I_n & 0 \end{pmatrix}$. Hence

$$
\begin{pmatrix} U & 0 \\ 0 & I_n \end{pmatrix} P_3 \begin{pmatrix} V & 0 \\ 0 & I_n \end{pmatrix} = \begin{pmatrix} UP_V & 0 \\ 0 & I_n \end{pmatrix} = \begin{pmatrix} I_n & 0 \\ 0 & I_n \end{pmatrix}.
$$

Thus system (16) is algebraically controllable.

Conversely, suppose that system (16) is algebraically controllable. Since (A.2) is satisfied, if $P_1$ is hyper-regular, $P_1$ is also hyper-regular. Thus system (15) is algebraically controllable. □

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