Finite-Dimensional Adaptive $H_{\infty}$ Consensus Control for Infinite-Dimensional Systems

Yoshihiko MIYASATO *

Abstract: Design methodologies of finite-dimenional adaptive $H_{\infty}$ consensus control of multi-agent systems composed of a class of infinite-dimensional systems are provided in this paper. The proposed control schemes are derived as solutions of certain $H_{\infty}$ control problems, where the effects of neglected infinite-dimensional modes and the imperfect knowledge of the leader are regarded as external disturbances to the process. It is shown that the resulting control systems are robust to uncertain system parameters and that the desirable consensus tracking is achieved approximately via finite-dimensional adaptive control schemes.

Key Words: adaptive control, consensus control, infinite-dimensional system, $H_{\infty}$ control.

1. Introduction

In recent years, much attention has been paid to cooperative control problems of multi-agent systems, and many control strategies have been developed such as formation control, task assignment, traffic control, and scheduling et al. (for example, [1]–[7]). Among those, distributed consensus tracking with limited communication networks has been a basic and important topic, and various research results have been proposed for various processes and under various conditions [8]–[11]. In those research works, adaptive control or sliding mode control methodologies were also applied in order to deal with uncertainties of agents, and stability of control systems was assured via Lyapunov function analysis. Furthermore, robustness properties of the control schemes were also discussed. However, those results are restricted to simple and low-order multi-agent systems, and those approaches do not have been applied to the control of complicated and high-order systems via low-order compensators.

On the contrary, there have been several researches of adaptive control for infinite-dimensional systems [12]–[21]. In the author’s previous work [20], the author has developed design methods of adaptive control for distributed parameter systems via finite-dimensional controllers, where stabilizing control signals are added to regulate the effects of infinite-dimensional modes, and those signals are derived as a solution of a certain $H_{\infty}$ control problem. The purpose of the present paper is to apply our previous result [20] to the consensus tracking problems, and present design methods of adaptive consensus control of multi-agent systems composed of a class of infinite-dimensional systems (distributed parameter systems of hyperbolic type). The proposed control strategy is composed of finite dimensional compensators, and is derived as a solution of the certain $H_{\infty}$ control problem where the effects of neglected infinite-dimensional modes of the process and the imperfect knowledge are regarded as external disturbances to the processes. It is shown that the resulting control systems are robust to uncertain system parameters and neglected infinite-dimensional modes, and that the desirable consensus tracking is achieved approximately via finite-dimensional adaptive control schemes. The present work would provide a useful strategy to deal with the coordinate control of certain flexible structures. The formation control version with the same policy, was also discussed in [22].

2. Multi-Agent System and Information Network

First, mathematical preliminaries on information network graphs of multi-agent systems are summarized [10],[11]. We consider a weighted undirected graph $G = (\mathcal{V}, \mathcal{E}, A)$ as a model of interactions among agents. $\mathcal{V} = \{1, \cdots, n\}$ is a node set, which corresponds to a set of agents, and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is an edge set. An edge $(i, j) \in \mathcal{E}$ indicates that the agent $i$ and $j$ can communicate with each other. Associated with the edge set $\mathcal{E}$, a weighted adjacency matrix is introduced such as $A = [a_{ij}] \in \mathbb{R}^{n \times n}$, and the entry $a_{ij}$ of it is defined by

\[
a_{ij} = a_{ji} = 0 \iff (i, j) \in \mathcal{E},
a_{ij} = a_{ji} > 0 \iff \text{otherwise}.
\]

A path is a sequence of edges in the form $(i_1, i_2), (i_2, i_3), \cdots$, where $i_j \in \mathcal{V}$, and the undirected graph is called connected, if there is always an undirected path between every pair of distinct nodes. For the adjacency matrix $A = [a_{ij}]$, the Laplacian matrix $L = [l_{ij}] \in \mathbb{R}^{n \times n}$ is defined by

\[
l_{ii} = \sum_{j=1}^{n} a_{ij},
\]

\[
l_{ij} = -a_{ij}, \quad (i \neq j).
\]

It is known that the Laplacian matrix is symmetric and positive-semidefinite. Especially, Laplacian matrix has a simple 0 eigenvalue with the associated eigenvector $\mathbf{1} = [1 \cdots 1]^T$, and all other eigenvalues are positive, if the corresponding undirected graph is connected.

In this manuscript, we consider a consensus control problem of leader-follower type on an undirected network graph $G$. Let
\( y_0 \) be a leader which each agent \( i \in \mathcal{V} \) should follow (\( i \) is called a follower). Then, \( a_{ij} \) is defined such as
\[
 a_{ij} = \begin{cases} 
 0 & \text{leader’s information is available to follower } i, \\
 > 0 & \text{otherwise},
\end{cases}
\] (1)
and from \( a_{ij} \) and \( L \), the matrix \( M \in \mathbb{R}^{m \times n} \) is defined by
\[
 M = L + \text{diag}(a_{10}, \ldots, a_{0n}).
\] (2)

\( M \) is symmetric and positive definite, if \( 1 \) at least one \( a_{ij} (1 \leq i \leq n) \) is positive, and 2. the graph \( \mathcal{G} \) is connected [11]. Hereafter, we assume those two assumptions.

### 3. Problem Statement

We consider a multi-agent system composed of distributed parameter systems of hyperbolic type as a class of infinite-dimensional systems. Let \( \Omega \) be a bounded open domain in a finite dimensional Euclidian space, and \( L^2(\Omega) \) is defined as the Hilbert space of all square integrable functions with the inner product
\[
(u, v) = \int_{\Omega} u(x) v^*(x) dx,
\] (3)
where \( v^* \) is a complex conjugate of \( v \). We consider a single-input, single-output distributed parameter system of hyperbolic type in \( L^2(\Omega) \) [23] described by
\[
\frac{d^2}{dt^2} u_i(t) + 2\alpha_i A_i \frac{d}{dt} u_i(t) + A_i u_i(t) = b_i f_i(t),
\] (4)
where \( u_i(t) (\in L^2(\Omega)) \) is a state, \( f_i(t) \) (an input) and \( y_i(t) \) (an output) are scalar functions on \( t \in [0, \infty) \), \( b_i (\in L^2(\Omega)) \) is an input influence function, and \( c_i (\in L^2(\Omega)) \) is a sensor influence function. \( \alpha_i \) is a small damping constant \( (0 < \alpha_i << 1) \). It is assumed that the operator \( A_i \) is self-adjoint, positive definite, and unbounded operator with compact resolvent whose eigenvalues \( \lambda_{ij} \)
\[
0 < \lambda_1 < \lambda_2 < \cdots \lambda_{ij} < \cdots, \quad (\lim_{j \to \infty} \lambda_{ij} = \infty),
\] (6)
are simple. The normalized eigenfunctions of \( A_i \) are denoted by \( \phi_{ij} \) which satisfy the following relations.
\[
A_i \phi_{ij} = \lambda_{ij} \phi_{ij}, \quad (j = 1, 2, \cdots).
\] (7)
The set \( \phi_{ij} \) forms a complete orthonormal system in \( L^2(\Omega) \). For each controlled process (4), (5), only the input \( f_i(t) \) and the output \( y_i(t) \), and \( \frac{d}{dt} y_i(t) \) are available for measurement, but the state \( u_i(t) \) and the systems parameters in \( A_i, b_i, c_i, \) and \( \alpha_i \) are unknown.

The control objective is to design an adaptive consensus control system for a swarm of infinite-dimensional systems (4), (5) in which consensus tracking is achieved via finite-dimensional adaptation schemes (8), (9), and it corresponds to coordinate control of certain flexible structures.
\[
y_i \rightarrow y_j, \quad \dot{y}_i \rightarrow \dot{y}_j, \quad (i, j = 1, \cdots, n),
\] (8)
\[
y_i \rightarrow y_0, \quad \dot{y}_i \rightarrow \dot{y}_0, \quad (i = 1, \cdots, n).
\] (9)

Hereafter in this paper, the index \( i \) corresponds to each agent, and \( i = 1, \cdots, n \).

### 4. Mathematical Preliminary

In this section, mathematical preliminaries on the solution of the process (4) are summarized. The next assumption is introduced.

Assumption 1. \( \alpha_i \) and \( \lambda_{ij} \) satisfy the following conditions.
\[
\begin{cases}
\alpha_i^2 \lambda_{ij}^2 - \lambda_{ij} \neq 0, & (j \geq 1), \\
\alpha_i \lambda_{ij} < \frac{1}{\lambda_{ij}},
\end{cases}
\] (10)
The second inequality of (10) indicates that \( \alpha_i \) is sufficiently small. Based on Assumption 1, \( g(A_i) \) is defined by
\[
\begin{aligned}
g_1(\lambda_{ij}) & \equiv \left( \alpha_i^2 \lambda_{ij}^2 - \lambda_{ij} \right)^{1/2}, \\
g_1(A_i) & \equiv \sum_{j=1}^{n} g_1(\lambda_{ij}) \phi_{ij} \phi_{ij}^*.
\end{aligned}
\] (11)

Since \( g_1(\lambda_{ij}) \sim \alpha_i \lambda_{ij} \) as \( j \to \infty \), \( g_1(A_i) \) is an unbounded operator and \( D(g_1(A_i)) = D(A_i) \), where \( D(\cdot) \) is a domain of (\( \cdot \)). Furthermore, (10) shows that \( g_1(A_i)^{-1} \) is a bounded operator. By utilizing \( g_1(A_i) \), the solution of the process (4) is given by

**Lemma 1.** [23] The next evolution equations in \( L^2(\Omega) \) are considered,
\[
\begin{aligned}
\frac{d}{dt} \xi_i(t) &= A_i^* \xi_i(t) + g_1(A_i)^{-1} b_i f_i(t), \\
\frac{d}{dt} \eta_i(t) &= A_i^* \eta_i(t) - g_1(A_i)^{-1} b_i f_i(t),
\end{aligned}
\] (12)
\[
A_i^* \eta_i(t) = -\alpha_i \lambda_{ij} g_1(\lambda_{ij}), \\
\lim_{j \to \infty} \mu_{ij} = -\alpha_i \lambda_{ij} g_1(\lambda_{ij}).
\] (13)

Then, the unique solution \( u_i(t) \) of (4) is described as follows:
\[
u_i(t) = \frac{\xi_i(t) + \eta_i(t)}{2},
\] (14)
where initial conditions \( \xi_i(0), \eta_i(0) \) of (12) are determined uniquely from initial conditions \( u_i(0), \frac{d}{dt} u_i(0) \) of (4)

The operator \( A_i^* \) have eigenfunctions \( \phi_{ij} \) and corresponding eigenvalues \( \mu_{ij} \) as follows:
\[
A_i^* \phi_{ij} = \mu_{ij} \phi_{ij},
\] (15)
\[
\begin{aligned}
\mu_{ij} &= -\alpha_i \lambda_{ij} - g_1(\lambda_{ij}), \\
\lim_{j \to \infty} \mu_{ij} &= -\alpha_i \lambda_{ij} \lim_{j \to \infty} \mu_{ij} = -\infty.
\end{aligned}
\] (16)

Then, \( \xi_i(t), \eta_i(t) \) and \( u_i(t) \) are rewritten into the following eigenfunction expansion forms:
\[
\begin{aligned}
\xi_i(t) &= \sum_{j=1}^{\infty} \xi_{ij}(t) \phi_{ij}, \\
\eta_i(t) &= \sum_{j=1}^{\infty} \eta_{ij}(t) \phi_{ij}, \\
\frac{d}{dt} \xi_i(t) &= \mu_{ij} \xi_{ij}(t) + g_1(\lambda_{ij})^{-1} b_i f_i(t), \\
\frac{d}{dt} \eta_i(t) &= \mu_{ij} \eta_{ij}(t) - g_1(\lambda_{ij})^{-1} b_i f_i(t),
\end{aligned}
\] (17)
\[
\begin{aligned}
u_i(t) &= \sum_{j=1}^{\infty} \xi_{ij}(t) + \eta_{ij}(t) \\
\frac{d}{dt} \nu_i(t) &= \frac{\mu_{ij}}{2} \phi_{ij}.
\end{aligned}
\] (19)

\( \xi_{ij}(t), \phi_{ij}, \eta_{ij}(t) = (\eta_i(t), \phi_{ij}), b_{ij} = (b_i, \phi_{ij}) \), respectively.

### 5. System Representation

In the present section, we derive an input-output representation of each process (4), (5) of the multi-agent system. Although the most part of this section is similar to the previous works [20],[22], there are certain differences that the system representation of the form \( \frac{d}{dt} \nu(t) + a_{01} \nu(t) + a_{02} \nu(t) = \cdots \) is to be obtained here, instead of \( \frac{d}{dt} \nu_i(t) + a_{01} \nu_i(t) + a_{02} \nu_i(t) = \cdots \) in [20] or
By utilizing where
\[ C_i N \]

\( \bar{\kappa}_N (t) = \) a stable matrix defined by
\[ C_i N \bar{A}_N + C_i N \bar{K}_N \in \mathbf{C}^{2N \times 2N}. \]

\[ \bar{K}_N \in \mathbf{C}^{2N \times 1} \]

and is observable, there exists \( \bar{K}_N \) satisfying (42) for an arbitrary stable matrix \( F_i \in \mathbf{R}^{2N \times 2N} \). \( \theta_3, \theta_2 \in \mathbf{R}^{2N} \) are vectors satisfying the following relation (43) (since \( \bar{F}_N, \bar{g}_N \) is controllable, there exists \( \theta_1, \theta_2 \in \mathbf{R}^{2N} \) satisfying (43) [24],[25]).

\[ \bar{A}_{NK} = \]

\[ C_i N \bar{A}_{NK} = \]

\[ C_i N \bar{K}_N C_i N \bar{A}_{NK} + \]

\[ \cdot \left\{ \right\} \cdot \bar{K}_N \bar{y}_N(t) \right\}

\[ = \]

\[ \bar{g}_i(\alpha_i) \bar{b}_1 N + \cdots + \bar{g}_i(\alpha_i) \bar{b}_1 N \in \mathbf{C}^{2N}, \]

\[ C_i N = \]

\[ \left[ \right] \cdot \bar{K}_N \bar{y}_N(t) + \bar{e}_i(t). \]

Furthermore, \( \theta_3, \theta_4 \in \mathbf{R} \), \( \delta(t) \) are defined by
\[ \theta_3 = (\bar{C}_i N \bar{A}_{NK} + \bar{C}_i N \bar{K}_N C_i N \bar{A}_{NK}) \cdot \]

\[ \theta_4 = (\bar{C}_i N \bar{K}_N C_i N \bar{A}_{NK}) \cdot \]

\[ \delta(t) = - (\bar{C}_i N \bar{A}_{NK} + \bar{C}_i N \bar{K}_N C_i N \bar{A}_{NK}) \cdot \]

\[ \cdot \left\{ \right\} \cdot \bar{K}_N \bar{y}_N(t) + \bar{e}_i(t). \]

Therefore, the input-output representation of each process of the multi-agent system is deduced such as (40), and is composed of two terms, \( \theta_3^T \bar{y}_N(t) + \theta_4^T \bar{y}_N(t) + \theta_3 f_1(t) + \theta_4 y(t) + \theta_3 f_1(t) \) and \( \delta(t) \). The former term is constructed by finite dimensional systems, and is considered as a primal part for the
controller design. On the contrary, the latter term $\delta(t)$ comes from the infinite dimensional system $[S_\omega]$, and is seen as a residual part for the design of the control systems. That obstacles from the infinite dimensional system $[S_\omega]$ is to be taken into consideration to stabilize the total system.

Next, the residual part $\delta(t)$ is to be evaluated for that purpose. We introduce state variable filters whose dimensions are 1 by utilizing design parameters $\lambda_i$ and $\lambda_i^\prime$. 

$$
\begin{aligned}
\frac{d^2}{dt^2}w_i(t) &= -\lambda_i w_i(t) + |f_i(t)|, \\
\frac{d}{dt}w_i(t) &= -\lambda_i^\prime w_i(t) + w_i(t).
\end{aligned}
$$

We assume that the sensor influence function $c_i$ and the input influence function $b_i$ are smooth in the following sense.

**Assumption 3.** The following inequalities hold.

$$
\sum_{j=1}^\infty |b_j^i |\frac{1}{j} < \infty, \quad \sum_{j=1}^\infty |c_j^i |\frac{1}{j} < \infty, \quad (k = 1, 2).
$$

Then, $\delta(t)$ of (46) is evaluated by Lemma 2.

**Lemma 2.** On Assumption 3, $\delta(t)$ is evaluated as follows:

$$
\begin{aligned}
|\delta(t)| &\leq g_\omega(t)\frac{d}{dt} \delta_\omega(t) + |\epsilon(t)|, \\
g_\omega &\equiv \left[ |f_i(t)|, \quad w_i(t), \quad w_i(t) \right]^T, \\
d_\omega &\equiv \left[ M_{i1}, \quad M_{i2}, \quad M_{i3} \right]^T, \\
0 < M_{i1} &\sim M_{i2} < \infty, \\
\epsilon(t) &\sim e^{-\lambda_0 t}, \quad e^{-\lambda_1 t}, \quad e^{-\lambda_2 t} \rightarrow 0.
\end{aligned}
$$

**Proof.** See [20].

Hereafter, the input-output representation of each agent is written in the following form.

$$
\begin{aligned}
\frac{d^2}{dt^2}y_i(t) &= \Theta_i^\prime \omega_i(t) + \theta_0f_i(t) + \delta(t) + \epsilon(t), \\
\Theta_i &= \left[ \theta_i^1, \quad \theta_i^2, \quad \theta_i^3 \right]^T, \\
\omega_i(t) &= \left[ v_{iN1}(t)^T, \quad v_{iN2}(t)^T, \quad f_i(t), \quad y_i(t) \right]^T.
\end{aligned}
$$

6. Adaptive $H_\infty$ Consensus Control

6.1 Assumptions

By utilizing the system representations in the previous section, the proposed adaptive $H_\infty$ consensus control schemes are constructed via finite dimensional controllers. The next assumptions are introduced.

**Assumption 4.** $\theta_0 \neq 0$, and sgn $\theta_0$ is known. In the following, it is assumed that $\theta_0 > 0$ without loss of generality.

**Assumption 5.** There exist $M_{f0}$ and $M_{f1}$ such that

$$
|f_i(t)| \leq M_{f0} + M_{f1} \sup_{0 \leq \tau \leq T} \left\{ |v_i(\tau)|, \frac{d}{d\tau} y_i(\tau) \right\},
$$

$$(0 \leq M_{f0} < \infty, \quad 0 < M_{f1} < \infty).
$$

**Remark 1.** Assumption 4 states that the relative degree of each agent is 2, and Assumption 5 asserts that the process has a stable inverse.

6.2 Control Law and Error Equation

The communication structure among agents and a leader is prescribed by the information network graph $G$ with the adjacency matrix $A$, the Laplacian matrix $L$, and the matrix $M$ (2). Associated with $G$, $A$, and $M$, we employ the following control law.

$$
\begin{aligned}
f_i(t) &= \hat{p}_i(t) - \Theta_i(t)^\top \omega_i(t) - \sum_{j=0}^n a_{ij}[y_i(t) - y_j(t)] \\
&\quad - \alpha \sum_{j=0}^n a_{ij}[\hat{y}_i(t) - \hat{y}_j(t)] + n_{0}\tilde{y}_0(t) + v_i(t) \\
&\equiv \hat{p}_i(t)\tilde{y}_0(t) + v_i(t),
\end{aligned}
$$

where $\alpha > 0$ is a design parameter. $\hat{\cdot}$ is denoted as a current estimate of $(\cdot)$, and $p_i$ is defined by

$$
p_i = 1/\theta_0.
$$

Concerned with $a_{00}, n_{0}$ is defined as follows:

$$
n_{0} = \begin{cases} 1 & : a_{00} > 0, \\
0 & : \text{otherwise}. \end{cases}
$$

$v_i$ is a stabilizing signal to be determined later based on $H_\infty$ control criterion. A tracking error between the leader $\tilde{y}_0$ and the follower $y_i$ is defined by

$$
\tilde{y}_i(t) \equiv y_i(t) - \tilde{y}_0(t),
$$

and the substitution of (56) and (59) into (52) yields

$$
\begin{aligned}
\tilde{y}_i(t) &= \Theta_i^\top \omega_i(t) + \theta_0 f_i(t) + \delta(t) + \epsilon(t) - \tilde{y}_0(t) \\
&= |\Theta_i - \hat{\Theta}_i(t)|^\top \omega_i(t) + \theta_0 |\hat{p}_i(t) - p_if_0(t)| \\
&\quad + \theta_0 v_i(t) + \delta(t) + \epsilon(t) \\
&\quad + \left\{ -(l_{ii} + a_{00})\tilde{y}_i(t) - \sum_{j=1}^n l_{ij}\tilde{y}_j(t) \right\} \\
&\quad + \left\{ -(l_{ii} + a_{00})\hat{y}_i(t) - \sum_{j=1}^n l_{ij}\hat{y}_j(t) \right\} \\
&\quad + \left\{ -(l_{ii} - 1)\tilde{y}_0(t) \right\}.
\end{aligned}
$$

Then, the total representation of the multi-agent system is given as follows:

$$
\begin{aligned}
\hat{y}(t) &= \Omega(t)(\Theta - \hat{\Theta}(t)) + F_0(t)\Theta_0(\hat{p}(t) - p) \\
&\quad - M\hat{y}(t) - \alpha M\tilde{y}(t) + \Theta_0v(t) \\
&\quad + \delta(t) + (N_0 - 1)\tilde{y}_0(t) + \epsilon(t),
\end{aligned}
$$

$$
\hat{y} = \left[ \tilde{y}_1, \cdots, \tilde{y}_n \right]^T,
$$

$$
\Theta = \left[ \Theta_1^\top, \cdots, \Theta_n^\top \right]^T,
$$

$$
F_0 = \text{diag}(f_0, \cdots, f_0),
$$

$$
\Theta_0 = \text{diag}(\theta_{01}, \cdots, \theta_{0d}),
$$

$$
|\hat{\Theta}(t)| \leq \Theta_0 |\hat{p}(t) - p|.
$$
\[ p = [p_1, \ldots, p_6]^T, \]
\[ N_0 = [n_{10}, \ldots, n_{66}]^T, \]
\[ \mathbf{1} = [1, \ldots, 1]^T, \]
\[ v = [v_1, \ldots, v_6]^T, \]
\[ \delta = [\delta_1, \ldots, \delta_6]^T, \]
\[ \epsilon = [\epsilon_1, \ldots, \epsilon_6]^T. \]

### 6.3 Adaptive \( H_{\infty} \) Consensus Control

For the matrix \( M \) and the positive constants \( \alpha, \gamma \), the matrices \( P \) and \( Q \) are defined such as
\[
P = \begin{bmatrix}
\frac{1}{2} M^2 & \frac{1}{2} M \\
\frac{1}{2} M & \frac{1}{2} M
\end{bmatrix},
\]
\[
Q = \begin{bmatrix}
\gamma^2 M^2 & \frac{\gamma^2 M^2 - \alpha M^2}{\alpha - \gamma M}
\end{bmatrix}.
\]

It can be shown that \( P \) and \( Q \) are both positive definite, if \( \gamma \) satisfies the next condition [11].
\[
0 < \gamma < \min \left( \frac{\sqrt{L_{\min}^2(M)}}{4 \sqrt{\lambda_{\min}(M)}}, \frac{4 \alpha \lambda_{\min}(M)}{4 + \alpha^2 \lambda_{\min}(M)} \right).
\]

Hereafter, it is assumed that \( \gamma \) satisfies (75). Utilizing the positive definite matrix \( P \), a positive function \( W \) is defined by
\[
W(t) = \tilde{z}(t)^T P \tilde{z}(t)
\]
\[
= \frac{1}{2} \left[ \tilde{\Theta}(t) - \Theta \right]^T \Gamma_1^{-1} \left[ \tilde{\Theta}(t) - \Theta \right]
\]
\[
+ \frac{1}{2} \left[ \tilde{\gamma}(t) - \gamma \right]^T \Gamma_2^{-1} \left[ \tilde{\gamma}(t) - \gamma \right]
\]
\[
+ \frac{1}{2} \left[ \tilde{\theta}(t) - \theta \right]^T \Gamma_3^{-1} \left[ \tilde{\theta}(t) - \theta \right],
\]
\[
\tilde{z}(t) = [\tilde{z}^T, \tilde{z}^T]^T,
\]
\[
(\Gamma_1 = \Gamma_1^T > 0, \quad \Gamma_2 = \Gamma_2^T > 0, \quad \Gamma_3 = \Gamma_3^T > 0),
\]
where \( \Gamma_2 \) is especially chosen as a diagonal positive definite matrix, and \( \theta_0 \) is defined as follows:
\[
\theta_0 = [\theta_{00}, \ldots, \theta_{06}]^T.
\]

The tuning laws of \( \tilde{\Theta}, \tilde{\gamma}, \tilde{\theta}, \theta_0 \) are determined by
\[
\dot{\tilde{\Theta}}(t) = \text{Pr} \left[ \Gamma_1 [\tilde{\Theta}(t) - \Theta]^T \tilde{z}(t) \right],
\]
\[
\dot{\tilde{\gamma}}(t) = \text{Pr} \left[ \Gamma_2 [\tilde{\gamma}(t) - \gamma]^T \tilde{z}(t) \right],
\]
\[
\dot{\tilde{\theta}}(t) = \text{Pr} \left[ \Gamma_3 [\tilde{\theta}(t) - \theta]^T \tilde{z}(t) \right],
\]
\[
\tilde{z}(t) = [\tilde{z}^T, \tilde{z}^T]^T + \gamma \tilde{z}(t),
\]
\[
V = \text{diag}(v_1, \ldots, v_6),
\]
where \( \text{Pr} \) are projection operations in which tuning parameters \( \tilde{\Theta}, \tilde{\gamma}, \tilde{\theta} \) and \( \theta_0 \) are constrained to bounded regions deduced from upper-bounds of \([\theta_0]^T\) and upper-bounds and lower-bounds of each element of \( \Theta \) and \( \theta_0 \), respectively [25]. Then, the time derivative of \( W \) along its trajectory is given as follows:
\[
\dot{W}(t) = -\tilde{z}(t)^T Q \tilde{z}(t) + \tilde{z}(t)^T M \tilde{\Theta}(t) \tilde{z}(t)
\]
\[
+ \tilde{z}(t)^T M \left[ (N_0 - I) \tilde{v}(t) + \epsilon(t) \right] + \tilde{z}(t)^T M \delta(t).
\]

From the evaluation of \( \dot{W} \) (82), we introduce the next virtual system:
\[
\dot{\tilde{z}} = f + g_{11} d_1 + g_{12} d_2 + g_2 v,
\]
\[
f = \begin{bmatrix}
0 & I \\
-M & -\alpha M
\end{bmatrix} \tilde{z},
\]
where
\[
g_{11} = \begin{bmatrix} 0 & G_0 \end{bmatrix}, \quad g_{12} = \begin{bmatrix} I \end{bmatrix},
\]
\[
G_0 = \text{block diag} \left( \begin{bmatrix} \gamma_1^T, \ldots, \gamma_6^T \end{bmatrix} \right),
\]
\[
g_2 = \begin{bmatrix} 0 \\
0 \end{bmatrix},
\]
\[
d_1 = \begin{bmatrix} d_{i1} \\
\vdots \\
d_{i6} \end{bmatrix}, \quad d_2 = (N_0 - I) \tilde{v}_0 + \epsilon.
\]

We are to stabilize the virtual system via a control input \( v \) by utilizing \( H_{\infty} \) criterion, where \( d_1, d_2 \) are regarded as external disturbances to the process [26],[27]. For that purpose, we introduce the following Hamilton-Jacobi-Isaacs (HJI) equation and its solution \( W_0 \).
\[
\mathcal{L}_f W_0 + \frac{1}{4} \left( \sum_{i=1}^n \| L_{e_i} W_0 \|^2 \right) = (L_{e_0} W_0) R^{-1} (L_{e_0} W_0)^T
\]
\[
+ q = 0,
\]
\[
W_0 = \tilde{z}^T P \tilde{z},
\]
where \( q \) and \( R \) are a positive function and a positive definite matrix respectively, and those are derived from HJI equation based on inverse optimality [26],[27] for the given solution \( W_0 \) and the positive constants \( \gamma_1, \gamma_2 \). The substitution of the solution \( W_0 \) (88) into HJI equation (87) yields
\[
-\tilde{z}^T Q \tilde{z} + \frac{1}{4} \tilde{z}^T M \left( \Gamma_2 G_0^T \gamma_1 + \frac{I}{\gamma_2} - \Theta_0 R^{-1} \Theta_0^T \right) M \tilde{z}
\]
\[
+ q = 0.
\]

Then, \( R \) and \( q \) are obtained such as
\[
R = \left( \Theta_0^{-1} G_0 \gamma_1^T \Theta_0^{-T} + \Theta_0^{-1} \Theta_0^{-T} + K \right)^{-1},
\]
\[
q = \tilde{z}^T Q \tilde{z} + \frac{1}{4} \tilde{z}^T M \Theta_0 K \Theta_0^T M \tilde{z},
\]
where \( K \) is a diagonal positive definite matrix (a design parameter). From \( R, q \) is derived as a solution of the corresponding \( H_{\infty} \) control problem as follows:
\[
v = -\frac{1}{2} R^{-1} (L_{e_0} W_0) \tilde{z} = -\frac{1}{2} R^{-1} \Theta_0^T M \tilde{z}
\]
\[
= -\frac{1}{2} \left( \Theta_0^{-1} G_0 \gamma_1^T \Theta_0^{-T} + \Theta_0^{-1} \Theta_0^{-T} + K \right) \Theta_0^T M \tilde{z},
\]
where the entries of \( \Theta_0 \) are constructed from the elements of \( \tilde{\theta}_0 \) (79). Then, the time derivative of \( W \) is evaluated by
\[
\dot{W} \leq \tilde{z}^T M \left[ \delta + (N_0 - I) \tilde{v}_0 + \epsilon \right]
\]
\[
- q - \frac{1}{4} \tilde{z}^T M \left( \Gamma_2 G_0^T \gamma_1 + \frac{I}{\gamma_2} - \Theta_0 R^{-1} \Theta_0^T \right) M \tilde{z}
\]
\[
+ \tilde{z}^T M \Theta_0 R v
\]
\[
= -q - v^T R v
\]
\[
+ \left( v + \frac{1}{2} R^{-1} \Theta_0^T M \tilde{z} \right)^T \left( v + \frac{1}{2} R^{-1} \Theta_0^T M \tilde{z} \right)
\]
\[
+ \gamma_1^T \| d_1 \|^2 - \gamma_1^T \sum_{i=1}^n \| d_{i1} - \frac{\sum_{j=1}^n m_{ij} \tilde{s}_j}{2 \gamma_1} \|^2
\]
\[
+ \gamma_2^T \| d_2 \|^2 - \gamma_2^T \sum_{i=1}^n \| d_{i6} - \frac{\sum_{j=1}^n m_{ij} \tilde{s}_j}{2 \gamma_2} \|^2.
\]
\[ M = [m_{ij}] \in \mathbb{R}^{n \times n}, \]
\[ d_2 = (N_0 - 1) \gamma_0 + \epsilon = [d_{21}, \cdots, d_{2n}]^T, \]
and we obtain the next theorems.

**Theorem 1.** In the proposed adaptive control system (56), (79), (92), the stabilizing signal \(\hat{\gamma} \) is a sub-optimal control input minimizing the upper bound on the cost functional \( J \).

\[
J(t) = \sup_{\hat{d}_1, \hat{d}_2 \in \mathcal{D}_2} \left[ \int_0^t [q + \nu^T R \nu] \, dt + W(t) \right]
- \sum_{i=1}^2 \gamma_i^2 \int_0^t \| d_i \|^2 \, dt.
\]

(96)

Also we have the next inequality.

\[
\int_0^t [q + \nu^T R \nu] \, dt + W(t)
\leq \sum_{i=1}^2 \gamma_i^2 \int_0^t \| d_i \|^2 \, dt + W(0).
\]

(97)

**Theorem 2.** The adaptive control system (56), (92), (79) is uniformly bounded, and if \((N_0 - 1) \gamma_0 = 0\) (that is; \(\gamma_0(t) = 0\) or the information of the leader \(\gamma_0\) is available for all followers), then it follows that

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T \left( \|\hat{\gamma}(t)\|^2 + \|\tilde{\gamma}(t)\|^2 \right) \, dt \leq \text{const} \cdot \gamma_1^2. \]

(98)

Otherwise, if \((N_0 - 1) \gamma_0 \neq 0\) (that is; \(\gamma_0(t) \neq 0\) and the information of \(\gamma_0\) is not available for all followers), then the next relation holds.

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T \left( \|\hat{\gamma}(t)\|^2 + \|\tilde{\gamma}(t)\|^2 \right) \, dt \leq \text{const} \cdot (\gamma_1^2 + \gamma_2^2). \]

(99)

**Remark 2.** Theorem 2 states that the approximate consensus tracking with the ratio of \(\gamma_1\) or \(\sqrt{\gamma_1^2 + \gamma_2^2}\), is achieved according to the availability of \(\gamma_0\) or the value of \(\gamma_0\). Furthermore, it should be noted that the adaptive control scheme is constructed via \(M \hat{s}\) and local informations of each agents, and can be implemented in a distributed fashion.

### 6.4 Adaptive \(H_\infty\) Consensus Control with Velocity Tracking

As a special version of the proposed control scheme, we also consider the case \(\gamma = 0\). Then, although \(P\) remains positive definite, \(Q\) becomes positive semidefinite

\[
P = \begin{bmatrix}
\frac{1}{2}M^2 & 0 \\
0 & \frac{1}{2}M
\end{bmatrix}, \quad Q = \begin{bmatrix}
0 & 0 \\
0 & \alpha M^2
\end{bmatrix},
\]

(100)

and the consensus tracking of velocity is achieved approximately such that

\[
y_i \sim \gamma, \quad (i, j = 1, \cdots, n),
\]

(101)

\[
y_i \sim y_0, \quad (i = 1, \cdots, n).
\]

(102)

The adaptive consensus control system with velocity tracking is easily derived by replacing \(\tilde{s}(t)\) by \(\hat{s}(t)\), and by utilizing newly defined \(P, Q\) (100) and \(q\), and we obtained the following two theorems.

**Theorem 3.** In the adaptive control system (56), (92), (79) with \(\gamma = 0\) (that is, \(s\) is replaced by \(\hat{s}\)), the velocity tracking error \(\hat{y}\) and the tuning parameters \(\Theta, \hat{\beta}_i, \hat{b}_i\) are uniformly bounded, and if \((N_0 - 1) \gamma_0 = 0\) (that is; \(y_0(t) = 0\) or the information of the leader \(y_0\) is available for all followers), then it follows that

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T \|\hat{y}(t)\|^2 \, dt \leq \text{const} \cdot \gamma_1^2.
\]

(103)

Otherwise, if \((N_0 - 1) \gamma_0 \neq 0\) (that is; \(y_0(t) \neq 0\) and the information of \(y_0\) is not available for all followers), then the next relation holds.

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T \|\hat{y}(t)\|^2 \, dt \leq \text{const} \cdot (\gamma_1^2 + \gamma_2^2).
\]

(104)

### 7. Numerical Example

For numerical simulation studies, we consider a multi-agent systems composed of uniform cantilever beams with internal viscous damping of Voigt type \([23]\) as follows:

\[
\frac{\partial^2}{\partial t^2} u_i(t, x_i) + 2\alpha_i \frac{\partial^2}{\partial t \partial x_i} u_i(t, x_i) + \frac{\partial^4}{\partial x_i^4} u_i(t, x_i) = b_i(x_i) f_i(t),
\]

\((x_i \in \Omega_i \equiv (0, 1)), \quad (\alpha_i > 0), \quad (i = 1, 2, 3; n = 3).\)

Each structure is clamped at \(x_i = 0\), and is free at the other end \(x_i = 1\). The boundary conditions are given by

\[
u_i(t, 0) = 0, \quad \frac{\partial}{\partial x_i} u_i(t, 0) = 0,
\]

\[
\frac{\partial}{\partial x_i} u_i(t, 1) + 2\alpha_i \frac{\partial^{k+1}}{\partial x_i^{k+1}} u_i(t, 1) = 0, \quad (k = 2, 3).
\]

The control output (sensor output) is described as

\[
y_i(t) = \int_0^t \phi_i(x_i) u_i(t, x_i) \, dx_i,
\]

and it is assumed that \(\alpha_i, b_i(x_i), c_i(x_i), u_i(t, x_i)\) are unknown. For this process, the operator \(A_i\) is defined by

\[
A_i u_i = \frac{\partial^2}{\partial x_i^4} u_i,
\]

\[
D_i(A_i) = \{ u_i : A_i u_i \in L^2(\Omega_i), \quad \nu_i(0) = 0, \quad \frac{\partial}{\partial x_i} u_i(0) = 0 \},
\]

and corresponding eigenvalues and eigenfunctions are given as follows:

\[
\lambda_i \phi_i(x_i) = \lambda_i \phi_i(x_i), \quad (0 < \lambda_i < 1),
\]

\[
\lambda_i = \beta_i^2 (> 0),
\]

\[
\phi_i(x_i) = [\cosh(\beta_i x_i) - \cos(\beta_i x_i)]
\]

\[
-\gamma_i \sinh(\beta_i x_i) - \sin(\beta_i x_i)],
\]

where \(\beta_i\) are positive solutions of the equation

\[
\cosh(\beta_i) \cos(\beta_i) + 1 = 0, \quad (j = 1, 2, 3, \cdots),
\]
and the damping constant $\beta$ is determined such as $0 < \beta_0 < \beta_1 < \cdots$, and $\gamma$ are defined by

$$
\gamma = \frac{\cosh \beta_j + \cos \beta_j}{\sinh \beta_j + \sin \beta_j}, \quad (j = 1, 2, 3, \cdots).
$$

For the numerical simulation, $b_i(x_i), c_i(x_i), \alpha_i$ are determined such as

$$
b_i(x_i) = c_i(x_i) = 10^3 x_i^3 (1 - x_i)^8 \exp[-(x_i - 0.3)^2],
\theta_0 = C_i b_i = 10.78196,
\alpha_1 = 0.00001, \alpha_2 = 0.00005, \alpha_3 = 0.0001,
$$

and the damping constant $\lambda_{Ni} (> 0)$ and the corresponding integer $N_i$ are chosen as follows:

$$
\lambda_{Ni} = 0.1, \quad N_i = 3.
$$

Then, the state variable filter $\bar{F}_{Ni}, \bar{g}_{Ni}$ is of 6 degrees ($2N_i = 6$), and are determined in the following way:

$$
\bar{F}_{Ni} = \text{diag} \left( -1 \quad -2 \quad -3 \quad -4 \quad -5 \quad -6 \right) \in \mathbb{R}^{6 \times 6},
\bar{g}_{Ni} = \left[ 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \right]^T \in \mathbb{R}^6,
\lambda_{fi} = 1.
$$

Associated with the information network structure (Fig. 1), the adjacency matrix $A = [a_{ij}]$ and $a_{i0}$ are chosen such that

$$
A = \begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 1 \\
0 & 1 & 0
\end{bmatrix}, \quad a_{10} = 1, \quad a_{20} = a_{30} = 0.
$$

The virtual leader $y_0$ is determined such as

$$
y_0 + 2y_0(t) + y_0(t) = \sin t.
$$

The design parameters are chosen as follows:

$$
\Gamma = I, \quad K = I, \quad \alpha = 1, \quad \gamma = 0.4, \quad \gamma = 0.1.
$$

The eigenfunction expansion method of order 20 is used for the numerical computation. Figure 2 shows the results where the proposed method is applied. For comparison, Fig. 3 show the result where the stabilizing signal $v$ is not added at all. Those results show the effectiveness of the stabilizing signals $v$ for stabilization of infinite-dimensional systems and achievement of the desired consensus tracking. Further numerical experiments for more complex network graphs are left in the future study.

8. Concluding Remarks

Design methodologies of finite-dimensional adaptive $H_\infty$ consensus control of multi-agent systems composed of a class of infinite-dimensional systems (distributed parameter systems of hyperbolic type) have been provided in the present paper. The proposed control strategy is composed of finite dimensional compensators, and is derived as a solution of certain $H_\infty$ control problem, where the effects of neglected infinite-dimensional modes of the processes and the imperfect knowledge of the leader are regarded as external disturbances to the processes. It is shown that the resulting control systems are robust to uncertain system parameters and neglected infinite-dimensional modes, and that the desirable consensus tracking is achieved approximately via finite-dimensional adaptive control schemes. Effectiveness of the proposed design schemes was also confirmed in the simulation studies. The proposed method would provide a basic and useful strategy to deal with the coordinate control of certain flexible structures.

References


[24] K.S. Narendra and A.M. Annaswamy: *Stable Adaptive Sys-