Event-Triggered and Self-Triggered Control for Discrete-Time Average Consensus Problems

Kenta HAMADA *, Naoki HAYASHI *, and Shigemasa TAKAI *

Abstract: This paper considers event-triggered and self-triggered control for discrete-time consensus problems. The event-triggered approach for multi-agent systems has attracted great attention in terms of reducing computation resources. In this paper, we show sufficient conditions to achieve an average consensus for centralized and distributed discrete-time event-triggered protocols. The results are then extended to self-triggered control where each agent computes its next update time based on the measurement errors of its neighbor agents.

Key Words: multi-agent systems, discrete-time consensus problem, event-triggered control, self-triggered control.

1. Introduction
In recent years, multi-agent systems that integrate physical systems with information systems have attracted great attention. Multi-agent systems are distributed systems where each agent cooperatively operates by communicating with neighbor agents. Reaching a consensus is one of the fundamental problems of cooperative control to achieve a group objective of a multi-agent system [1]–[4]. The main objective of consensus problems is to guarantee that states of all agents converge to a common value by local communications.

To achieve a consensus for more complicated systems, a number of consensus algorithms have been reported. However, most of those methods are time-triggered protocols where each agent periodically computes a control input. In many applications, the processor embedded on each agent may have low computational and communicational abilities. Though time-triggered control can simplify design and analysis of consensus problems, it also causes conservative usage of network and computation resources.

To overcome such drawbacks, event-triggered control has been proposed. The event-triggered control does not have to update a control input periodically and the load of processors on each agent can be reduced. Tabuada proposed centralized event-triggered control for continuous-time nonlinear systems [5]. Dimarogonas et al. showed sufficient conditions to achieve an average consensus by continuous-time distributed event-triggered and self-triggered protocols [6].

From an implementation point of view, discrete-time consensus dynamics is an important issue. Chen and Hao reported sufficient conditions to achieve an average consensus for discrete-time event-triggered and self-triggered protocols based on an LMI approach [7]. However, their methods require the global information to obtain the conditions for an average consensus.

This paper considers a fully distributed approach to event-triggered and self-triggered average consensus problems of multi-agent systems. The local communications are assumed to be represented by a static and undirected graph. The sufficient conditions to achieve an average consensus for discrete-time consensus protocols are provided based on the analysis of the nonlinear control theory.

The rest of the paper is organized as follows: Section 2 states some preliminaries. Section 3 considers sufficient conditions for an event-triggered protocol to achieve an average consensus. Section 4 shows sufficient conditions to achieve a consensus for self-triggered control. Section 5 presents a numerical example. Finally, Section 6 concludes this paper.

2. Preliminaries

2.1 Graph Theory
This subsection reviews some fundamental results of graph theory. An undirected graph $G = (V, E)$ consists of a node set $V = \{1, 2, \ldots, n\}$ and an edge set $E$. Each node $i$ in a graph $G$ represents each agent $i$. An undirected edge $(i, j) \in E$ indicates that agents $i$ and $j$ exchange their states with each other. The adjacency matrix $A = A(G) = \{a_{ij}\}$ is an $n \times n$ matrix with $a_{ij} > 0$ if $(i, j) \in E$ and $a_{ij} = 0$ otherwise. If there is an edge $(i, j) \in E$, then $i$, $j$ are called adjacent and $j$ is called a neighbor of node $i$.

The neighbor set of node $i$ is denoted by $N_i = \{j \in V \mid (i, j) \in E\}$. An undirected path in a graph $G$ is a sequence of nodes $i_1, i_2, \ldots, i_k$ such that $(i_j, i_{j+1}) \in E$ for $j = 1, 2, \ldots, k-1$. If there is a path between any two nodes of the graph $G$, then $G$ is called connected. The degree matrix $D = D(G) = \text{diag}(d_1, \ldots, d_n)$ is an $n \times n$ diagonal matrix, where $d_i = \sum_{j=1}^{n} a_{ij}$. The Laplacian $L$ of a graph $G$ is a positive semidefinite matrix such that $L = D - A$. If a graph $G$ is connected, then the Laplacian has a single zero eigenvalue and the corresponding eigenvector is $1$, where $1 = [1, 1, \ldots, 1]^\top \in \mathbb{R}^n$.

2.2 Event-Triggered Protocol
We consider a multi-agent system of $n$ agents in which agent $i$ has the state $x_i \in \mathbb{R}$ and information exchanges are described as a connected undirected graph. The discrete-time protocol of each agent can be described by

$$x_i(k + 1) = x_i(k) + \epsilon u_i(k), \quad (1)$$

where $u_i(k) \in \mathbb{R}$ is a control input of agent $i$ at step $k$ and $\epsilon > 0$. 

1 Division of Electrical, Electronic and Information Engineering, Graduate School of Engineering, Osaka University, Suita, Osaka 565-0871, Japan.
E-mail: n.hayashi@cei.eng.osaka-u.ac.jp
(Received February 3, 2014)
(Revised April 29, 2014)
We consider the following control input $u_i$:

$$u_i(k) = -\sum_{j=1}^{n} a_{ij} (x_i(k_i) - x_j(k_j)), \tag{2}$$

where $k_0, k_1, \ldots$ is the sequence of synchronized trigger times of agent $i$. The control input is updated only at trigger times and held constant between these trigger times.

The measurement error for agent $i$ is defined by

$$e_i(k) = x_i(k) - x(k), \quad \forall k \in [k_i, k_{i+1}], \quad i = 0, 1, \ldots, \tag{3}$$

and we consider a stack vector of agents’ errors $e(k) = [e_1(k), e_2(k), \ldots, e_n(k)]^T$. The measurement error $e_i$ shows the difference between the last updated state and the current state. In Section 3, we derive a trigger condition which decides trigger times using the measurement error.

From (1)–(3), we have

$$x_i(k + 1) = x_i(k) - e \sum_{j=1}^{n} a_{ij} (x_i(k_i) - x_j(k_j))$$

and the closed-loop system can be described as

$$x(k + 1) = x(k) - \epsilon L x(k) - \epsilon E e(k), \tag{4}$$

where $x(k) = [x_1(k), x_2(k), \ldots, x_n(k)]^T$ is a stack vector of agents' states.

The following lemma shows that if a multi-agent system achieves a consensus, the consensus value is the average of the initial values of agents.

**Lemma 1** Consider the discrete-time protocol (1) with a connected undirected graph, using the event-triggered protocol (2), if a multi-agent system achieves a consensus, then there holds

$$\lim_{k \to \infty} x_i(k) = \frac{1}{n} \sum_{j=1}^{n} x_j(0), \quad \forall i \in V.$$  

**Proof** Since the graph $G$ is a connected undirected graph, the Laplacian of $G$ is a symmetric matrix, that is, $L = L^\top$. Then, for all $k = 0, 1, \ldots$, we have

$$1^\top x(k + 1) = 1^\top x(k) - \epsilon 1^\top L e(k)$$

$$= 1^\top x(k) - \epsilon (L E e(k))$$

$$= 1^\top x(k) - \epsilon 1^\top x(0). \tag{5}$$

Therefore, if the states of agents converge to a common point $x^\ast$, then we have

$$\lim_{k \to \infty} \sum_{j=1}^{n} x_j(k) = n x^\ast = \sum_{j=1}^{n} x_j(0).$$

\hfill \Box

**Remark 1** The proposed method requires undirected communications to achieve an average consensus. If a communication network is described as a directed graph, the corresponding graph Laplacian is asymmetric ($L \neq L^\top$) except for the special case where the communication graph is balanced. Therefore, in general, (5) does not hold and an average consensus cannot be achieved. For the case of a directed graph, we need a compensation term for the consensus protocol (4) to keep the sum of states constant as proposed in [8]. The extension to the directed graph case is our future work.

3. **Event-Triggered Protocol**

In this section, we consider event-triggered control to achieve an average consensus. We show sufficient conditions for centralized and distributed event-triggered protocols.

3.1 **Centralized Approach**

In this subsection, we show sufficient conditions for a centralized event-triggered protocol. In the centralized event-triggered control, states of all agents are monitored by a leader agent. The leader agent decides trigger times at which all agents synchronously update their control inputs.

The procedure of the centralized event-triggered control for an average consensus is summarized as follows: Firstly, agents report their states to a leader agent at every step $k$. The leader agent estimates the states at the next step $k + 1$ based on the assumption that a trigger is not executed at the current step $k$:

$$\hat{x}_i(k + 1) = x_i(k) - \epsilon \sum_{j \in N_i} a_{ij} (x_i(k_i) - x_j(k_j)). \tag{6}$$

Then the leader agent computes the measurement error $\hat{e}(k + 1) = x(k) - \hat{x}(k + 1)$ and $f(k + 1) = \|\hat{e}(k + 1)\| - \alpha \|L \hat{x}(k + 1)\|$, where $\hat{x}(k) = [\hat{x}_1(k), \hat{x}_2(k), \ldots, \hat{x}_n(k)]^T$ and $\alpha$ is a positive constant.

Note that the leader agent can obtain $f(k + 1)$ at step $k$ since $\hat{x}_i(k + 1)$ can be estimated by (6) with the current state and the last updated state. If $f(k + 1) \geq 0$, a trigger is synchronously executed at step $k$ by the leader agent and each agent $i$ updates its state as follows:

$$x_i(k + 1) = \left\{ \begin{array}{ll}
  x_i(k) - \epsilon \sum_{j \in N_i} a_{ij} (x_i(k_i) - x_j(k)) , & \text{if } f(k + 1) \geq 0, \\
  \hat{x}_i(k + 1), & \text{otherwise}.
\end{array} \right.$$

Theorem 1 guarantees that an average consensus is achieved by the above procedure of the centralized event-triggered control. It shows that an update of a control input is not required if the measurement error $e$ is small enough.

**Theorem 1** Suppose that the graph $G$ is connected and $0 < \epsilon < 2/(2||L||)$. If a trigger is executed at step $k$ when $f(k + 1) \geq 0$, then the multi-agent system with (2) achieves an average consensus, where

$$f(k) = ||\hat{e}(k)|| - \alpha ||L \hat{x}(k)||,$$

$$0 < \alpha < \frac{-(1 + \epsilon ||L||) + \sqrt{1 + 4\epsilon ||L||}}{\epsilon ||L||^2}.$$  

**Proof** We consider a candidate ISS Lyapunov function:

$$V(k) = x^\top(k) L x(k).$$

We have

$$V(k + 1) - V(k) = x^\top(k + 1) L x(k + 1) - x^\top(k) L x(k)$$

$$= -2\epsilon x^\top(k) L L x(k) - 2\epsilon x^\top(k) L L \hat{x}(k)$$

$$+ \epsilon^2 x^\top(k) L L \hat{x}(k) + \epsilon^2 \epsilon^\top(k) L L \hat{x}(k)$$

$$+ 2\epsilon x^\top(k) L L \hat{x}(k)$$

$$\leq -2\epsilon ||L||^2 \epsilon \|L x||^2 \epsilon ||L|| ||L x||^2$$

$$+ 2\epsilon^3 ||L||^2 ||L x||^2 \epsilon ||L x||^2$$

$$+ 2\epsilon^2 ||L||^2 ||L x||^2 \epsilon ||L x||^2.$$

If $||e|| \leq \alpha ||L x||$, then we have...
Therefore, in this section, we consider the following consensus:

\[ V(k + 1) - V(k) \leq \epsilon \left( -2 + 2||L||\alpha + \epsilon||L|| + \epsilon||L||^2 + 2\epsilon||L||^2 \alpha^2 \right) \|Lx||^2, \]

where \( \alpha > 0 \). Now we define \( D(\alpha) \) by

\[ D(\alpha) = \epsilon ||L||^2 \alpha + 2||L||\epsilon||L|| + 1\alpha \epsilon \|L\| - 2 \]

\[ = \epsilon ||L||^2 (\alpha - \alpha_1) (\alpha - \alpha_2), \]

where

\[ \alpha_1 = -(1 + \epsilon||L||) + \sqrt{1 + 4\epsilon ||L||}, \]

\[ \alpha_2 = -(1 + \epsilon||L||) - \sqrt{1 + 4\epsilon ||L||}. \]

Note that if \( 0 < \epsilon < (2/||L||) \), \( \alpha_1 > 0 \). Then, if \( \alpha < \alpha_1 \), \( D(\alpha) < 0 \) and we have

\[ V(k + 1) - V(k) \leq \epsilon D(\alpha) \|Lx||^2 \leq 0. \]

Because \( V(k) \geq 0 \), \( V(k + 1) - V(k) \leq 0 \) implies that \( V(k) \) has a finite limit and \( V(k + 1) - V(k) \to 0 \) as \( k \to \infty \). Thus we have

\[ 0 = \lim_{k \to \infty} (V(k + 1) - V(k)) \leq \epsilon D(\alpha) \|Lx||^2 \leq 0. \]

Since \( \epsilon > 0 \) and \( D(\alpha) < 0 \), \( \epsilon D(\alpha) < 0 \). Therefore we have \( \|Lx|| \to 0 \) as \( k \to \infty \).

3.2 Distributed Approach

The centralized event-triggered control stated in Section 3.1 is suitable when a global communication network and a leader agent with enough computational ability are available since a trigger condition is efficiently solved by the leader agent based on the global information of all agents. However, it is not the case for many practical situations due to limited capacity of network bandwidth and lack of robustness for a failure of the leader agent. In this section, we consider a distributed approach for an event-triggered average consensus protocol.

Note that the parameter \( \epsilon \) in (1) is the global information of a multi-agent system since \( \epsilon \) is a common parameter of agents. Therefore, in this section, we consider the following consensus protocol instead of (1).

\[ x_i(k + 1) = x_i(k) - \sum_{j = 1}^{n} a_{ij} \left( x_j(k) - x_j(k) \right). \]  

(7)

Each agent \( i \) estimates the states of its own and the neighbor agents \( j \in N_i \) based on the assumption that a trigger is not executed at step \( k \) as follows:

\[ \hat{x}_i(k + 1) = x_i(k) - \sum_{j \in N_i} a_{ij} \left( x_j(k) - x_j(k) \right). \]  

(8)

\[ \hat{x}_j(k + 1) = x_j(k) - \sum_{i \in N_j} a_{ji} \left( x_i(k) - x_i(k) \right), \quad j \in N_i. \]  

(9)

Then agent \( i \) computes the differences of states \( z_i \) and errors \( r_i \) as follows:

\[ z_i(k + 1) = \sum_{j \in N_i} a_{ij} \left( \hat{x}_j(k + 1) - \hat{x}_j(k + 1) \right), \]

\[ r_i(k + 1) = \sum_{j \in N_i} a_{ij} \left( \hat{x}_j(k + 1) - \hat{x}_j(k + 1) \right), \]

where \( \hat{x}_j(k + 1) = x_j(k) - \hat{x}_j(k + 1) \). To check a trigger condition, each agent also computes \( g_i(k + 1) \) by

\[ g_i(k + 1) = r_i^2 (k + 1) + \sigma_i \eta_i z_i^2 (k + 1), \]

where

\[ \eta_i = -2 + y_i + b_i \left( 1 + \frac{\gamma_i}{y_i} + \frac{1}{2\gamma_i} \right), \]

\[ 0 < \eta_i < 1, \] and \( b_i \) and \( y_i \) are positive constants. Note that each agent can obtain \( g_i(k + 1) \) at step \( k \) since \( z_i(k + 1) \) and \( r_i(k + 1) \) are computed by \( \hat{x}_j(k + 1) \) and \( \hat{x}_j(k + 1) \) which are estimated at step \( k \). If \( g_i(k + 1) \geq 0 \), agent \( i \) sends a request signal to other agents. Then agent \( i \) updates its state as follows:

\[ x_i(k + 1) = \begin{cases} x_i(k) - \sum_{j \in N_i} a_{ij} \left( x_j(k) - x_j(k) \right), \quad \text{if agent } i \text{ received a request signal,} \\ x_i(k + 1), \quad \text{otherwise.} \end{cases} \]

**Remark 2** In this paper, the trigger times are synchronized even in distributed event-triggered control. If an agent needs to execute a trigger at step \( k \), the agent sends a request signal to its neighbor agent. By repeating this procedure until all agents receive the request signal, agents can synchronize trigger times. For example, each agent \( i \) sets a variable \( \xi_i \) as \( \xi_i(0) = 0 \) if agent \( i \) requires a trigger, and \( \xi_i(0) = 1 \) otherwise. Then agents run the following distributed algorithm:

\[ \xi_i(s + 1) = \min_{j \in N_i} \xi_j(s). \]

This algorithm converges to \( \min_{i \in V} \xi_i(0) \) after at most \( k = \text{diam}(G) \) iterations, where \( \text{diam}(G) \) is a diameter of a communication network graph \( G \) [9]. Thus agent \( i \) executes a trigger at step \( k \) if \( \xi_i(k) = 0 \). However, this scheme requires some extra communications and the sampling period of the consensus protocol (7) should be long enough to share the request signal among all agents. Therefore the proposed distributed control method may not be applicable for the system with relatively high dynamics. To relax the constraint on the synchronized trigger times is our future work.

Next, we consider a sufficient condition for a distributed event-triggered protocol. Suppose that \( Lx \pm z = [z_1, z_2, \ldots, z_n]^\top \in \mathbb{R}^n \) and \( Le \pm \nu = [\nu_1, \nu_2, \ldots, \nu_n]^\top \in \mathbb{R}^n \), where

\[ z_i(k) = \sum_{j = 1}^{n} a_{ij} (x_j(k) - x_j(k)), \quad i \in V, \]

\[ r_i(k) = \sum_{j = 1}^{n} a_{ij} (e_i(k) - e_i(k)), \quad i \in V. \]

Theorem 2 shows that an average consensus is achieved by the proposed procedure of the distributed event-triggered control. It implies that agents update their control inputs only when the difference of the measurement error \( r_i \) of some agent is larger than the weighted difference of states \( z_i \).

**Theorem 2** Suppose that the graph \( G \) is connected and triggers are synchronized. Suppose also that \( 0 < a_{ij} \leq \frac{\Delta x_i}{\max_{j \in N_i} \{a_{ij}\}} \) and \( \frac{2 - b_i - \sqrt{\gamma_i}}{5b_i + 2} < \gamma_i < \frac{2 - b_i + \sqrt{\gamma_i}}{5b_i + 2} \), where \( |N_i| \) is the number of elements of the neighbor set \( N_i \). Then, \( g_i(k + 1) \geq 0 \) for \( i \in V \), then the multi-agent system with (2) achieves an average consensus, where
\[ g_i(k) = r_i^2(k) + \sigma_i \left( -2 + \gamma_i + b_i \left( 1 + \frac{5}{2} \gamma_i + \frac{1}{2y_i} \right) \right) \tag{10} \]
and \(0 < \sigma_i < 1 \).

**Proof** Consider again the candidate Lyapunov function \( V(k) = x^T(k)Lx(k) \). For simplicity of notation, we omit the time step \( k \).

Then we have
\[
V(k + 1) - V(k) = x^T(k + 1)Lx(k + 1) - x^T(k)Lx(k)
= \left\{ x - (z + r)^T L \left( x - (z + r) \right) - x^T Lx \right\}
= -2x^T Lz + r^T (z + r)^T Lz + r^T Lr
\leq -2x^T z + 2z^T r + z^T Lz + 2r^T Lr + r^T Lr.
\tag{11} \]
From the inequality of arithmetic and geometric means [6], we have
\[
|\xi_j| \leq \frac{\gamma_j}{2} z_j^2 + \frac{1}{2y_j} r_j^2, \quad \forall \xi, z \in \mathbb{R}, \forall y_i > 0.
\tag{12} \]
Since the graph \( G \) is undirected, we have
\[
\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \xi_j^2 \leq \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \xi_i^2 \leq \sum_{i=1}^{n} a_{sup[N]} \xi_i^2,
\tag{13} \]
where \( a_{sup} = \sup_{i,j \in V} a_{ij} \). Using the inequalities (12) and (13), we have
\[
|z^T r| \leq \sum_{i=1}^{n} |z_i r_i| \leq \sum_{i=1}^{n} \left( \frac{\gamma_j}{2} z_j^2 + \frac{1}{2y_j} r_j^2 \right), \tag{14} \]
\[
z^T Lz = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} z_j (z_j - z_i)
\leq \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} z_j^2 + \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} |z_i z_j|
\leq \sum_{i=1}^{n} a_{sup[N]} z_j^2 + \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \left( \frac{\gamma_j}{2} z_j^2 + \frac{1}{2y_j} z_j^2 \right)
\leq \sum_{i=1}^{n} a_{sup[N]} \left( 1 + \frac{y_j}{2} + \frac{1}{2y_j} \right) \xi_i^2.
\tag{15} \]
Similarly, we have
\[
z^T Lr \leq \sum_{i=1}^{n} a_{sup[N]} |y_i r_i^2| + \sum_{i=1}^{n} a_{sup[N]} \frac{1}{y_i} r_i^2,
\tag{16} \]
\[
r^T Lr \leq \sum_{i=1}^{n} a_{sup[N]} \left( 1 + \frac{y_j}{2} + \frac{1}{2y_j} \right) r_j^2.
\tag{17} \]
From (11) and (14)–(17), we have
\[
V(k + 1) - V(k)
\leq \sum_{i=1}^{n} \left( -2 + \gamma_i + a_{sup[N]} \left( 1 + \frac{5}{2} \gamma_i + \frac{1}{2y_i} \right) \right) \xi_i^2
+ \sum_{i=1}^{n} \left( \frac{1}{y_i} + a_{sup[N]} \left( 1 + \frac{5}{2} \gamma_i + \frac{5}{2y_i} \right) \right) r_i^2.
\]
Since \( \gamma_i > 0, a_{sup} > 0 \) and \( |N_i| > 0 \), we get
\[
\frac{1}{y_i} + a_{sup[N]} \left( 1 + \frac{5}{2} \gamma_i + \frac{5}{2y_i} \right) > 0.
\]

Now we consider
\[
f_i(\gamma_i) = -2 + \gamma_i + b_i \left( 1 + \frac{5}{2} \gamma_i + \frac{1}{2y_i} \right)
= \left( 5b_i + 2 \right) \gamma_i^2 + 2(b_i - 2) \gamma_i + b_i \leq D_i(\gamma_i),
\]
By solving \( D_i(\gamma_i) = 0 \) for \( \gamma_i \), we obtain
\[
\gamma_i = \frac{-b_i - 2 \pm \sqrt{b_i^2 + 2}}{5b_i + 2},
\Delta_i = -4b_i^2 - 6b_i + 4.
\]
If \( 0 < b_i \leq 1 \), we have \( 2 - b_i > \Delta_i \geq 0 \). Thus, if \( a_{ij} \leq 1 \), we have \( \Delta_i \geq 0 \). Moreover, if
\[
0 < \frac{2 - b_i - \sqrt{b_i^2 + 2}}{5b_i + 2} < \gamma_i < \frac{2 - b_i + \sqrt{b_i^2 + 2}}{5b_i + 2},
\]
we have
\[
f_i(\gamma_i) = -2 + \gamma_i + b_i \left( 1 + \frac{5}{2} \gamma_i + \frac{1}{2y_i} \right) < 0.
\]
Therefore, if
\[
r_i^2 \leq -\sigma_i, \quad \frac{1}{y_i} + b_i \left( 1 + \frac{5}{2} \gamma_i + \frac{1}{2y_i} \right) \xi_i^2.
\]
where \(0 < \sigma_i < 1 \), we have
\[
V(k + 1) - V(k)
\leq - \sum_{i=1}^{n} \left( -\sigma_i \right) \gamma_i \xi_i^2
= - \sum_{i=1}^{n} \left( -\sigma_i \right) \gamma_i \xi_i^2
\leq 0.
\]
Because \( V(k) \geq 0, V(k + 1) - V(k) \leq 0 \) implies that \( V(k) \) has a finite limit and \( V(k + 1) - V(k) \to 0 \) as \( k \to \infty \). Thus we have
\[
0 = \lim_{k \to \infty} (V(k + 1) - V(k))
\leq - \sum_{i=1}^{n} \left( -\sigma_i \right) \gamma_i \xi_i^2
\leq 0.
\]
Since \(0 < \sigma_i < 1 \) and \( \left| -2 + \gamma_i + b_i \left( 1 + \frac{5}{2} \gamma_i + \frac{1}{2y_i} \right) \right| > 0 \), we have \( z_i(k) \to 0 \) as \( k \to \infty \) for \( i \in V \). \qed

### 4. Self-Triggered Protocol

Although the event-triggered control scheme stated in Section 3 can reduce the number of updates of a control input, agents have to observe the current states of its neighbor agents as well as its own state at every discrete-time step. In this section, we consider a self-triggered control scheme where agents estimate the next trigger time based on the current states to reduce such computational load.

Let \( T \) be a trigger interval between the trigger times \( k_i \) and \( k_{i+1} \). That is, \( T = k_{i+1} - k_i \). Note that a trigger is not executed and agents do not update their control inputs \( u_i \) between the trigger times \( k_i \) and \( k_{i+1} \). Then we have
\[
x_i(k_{i+1} + T) = x_i(k_i) - T \epsilon \sum_{j=1}^{n} a_{ij} \left( x_i(k_i) - x_j(k_i) \right),
\]
\[
e_i(k_{i+1} + T) = e_i(k_i) - x_i(k_i) - x_i(k_{i+1} + T)
= T \epsilon \sum_{j=1}^{n} a_{ij} \left( x_i(k_i) - x_j(k_i) \right).
\]
The closed-loop system is shown as follows:

\[ x(k_t + T) = (I - eTL)x(k_t), \quad (18) \]
\[ e(k_t + T) = eTLx(k_t). \quad (19) \]

In the following sections, we consider sufficient conditions for centralized and distributed self-triggered control protocols.

4.1 Centralized Approach

In centralized self-triggered control, agents report their states to a leader agent at trigger time \( k_t \). Then the leader agent estimates the next trigger time \( k_{t+1} \) based on the current state \( x(k_t) \) as follows:

\[ k_{t+1} = k_t + \frac{-2\alpha^2(Lx(k_t))^\top Lx(k_t) + \sqrt{\Delta}}{2\epsilon(\|Lx(k_t)\|^2 - \alpha^2\|L^2x(k_t)\|^2)}, \quad (20) \]

where

\[ \alpha = \frac{-(1 + e\|L\|) + \sqrt{1 + 3e\|L\|}}{e\|L\|^2}, \]
\[ \Delta = 4\alpha^4(\|Lx(k_t)\|^2Lx(k_t))^2 + 4\alpha^2(\|Lx(k_t)\|^2(\|Lx(k_t)\|^2 - \alpha^2\|L^2x(k_t)\|^2)). \]

Note that the next trigger time \( k_{t+1} \) is computed only at trigger times \( k_0, \ldots, k_{L-1} \) in the self-triggered control while, in the event-triggered control, the leader agent has to check the trigger condition at every discrete-time step. Therefore, the self-triggered control scheme can reduce computational load to check the trigger condition compared to the event-triggered control scheme proposed in Section 3.1. The following theorem shows that a multi-agent system can achieve an average consensus when a trigger is executed at the synchronized trigger times (20).

**Theorem 3** Suppose that the graph \( G \) is connected, \( 0 < \epsilon \leq \frac{\sqrt{5 - \sqrt{21}}}{6\|L\|} \). If a trigger is executed at

\[ k_{t+1} = k_t + \frac{-2\alpha^2(Lx(k_t))^\top Lx(k_t) + \sqrt{\Delta}}{2\epsilon(\|Lx(k_t)\|^2 - \alpha^2\|L^2x(k_t)\|^2)}, \quad l = 0, 1, \ldots, \]

the multi-agent system with (2) achieves an average consensus, where

\[ \alpha = \frac{-(1 + e\|L\|) + \sqrt{1 + 3e\|L\|}}{e\|L\|^2}, \]
\[ \Delta = 4\alpha^4(\|Lx(k_t)\|^2Lx(k_t))^2 + 4\alpha^2(\|Lx(k_t)\|^2(\|Lx(k_t)\|^2 - \alpha^2\|L^2x(k_t)\|^2)). \]

**Proof** We denote the next trigger time by \( k_{t+1} = k_t + T \). Then the trigger condition \( f(k + 1) = 0 \) is written as follows:

\[ \|e(k_t + T)\| = \alpha\|Lx(k_t + T)\|. \quad (21) \]

From (18)–(21), we have

\[ \|eTLx(k_t)\| = \alpha\|LI - eTL\| \cdot x(k_t)\| \]

Then we have

\[ (\|Lx(k_t)\|^2 - \alpha^2\|L^2x(k_t)\|^2)(\epsilon T)^2 + 2\alpha^2(Lx(k_t))^\top Lx(k_t)eT - \alpha^2\|Lx(k_t)\|^2 = 0. \quad (22) \]

We also have the following inequality

\[ \|Lx(k_t)\|^2 - \alpha^2\|L^2x(k_t)\|^2 \geq (1 - \alpha^2\|L\|^2)\|Lx(k_t)\|^2. \]

Since \( \alpha < \alpha_1 \), we have

\[ 1 - \alpha^2\|L\|^2 > 1 - \alpha^2\|L\|^2 = \frac{\epsilon^2\|L\|^2}{\epsilon\|L\|^2} = \frac{2\|Lx(k_t)\|^2}{\|L\|^2}. \]

If \( 0 \leq \epsilon < \sqrt{1 + 3\|L\|} \), then \( \epsilon \) is larger than \( 1 \).

Since \( \|Lx(k_t)\|^2 - \alpha^2\|L^2x(k_t)\|^2 > 0 \), then \( \sqrt{\Delta} > 2\alpha^2\|Lx(k_t)\|Lx(k_t) \). Therefore, we obtain

\[ \epsilon T \geq \frac{2\alpha^2\|L\|^2}{2\alpha^2\|L\|^2 + 2\alpha\|L\|^2} = \frac{\alpha}{\alpha\|L\|^2 + 1}. \]

Now we define \( \alpha \) by

\[ \alpha = \frac{-(1 + e\|L\|) + \sqrt{1 + 3e\|L\|}}{e\|L\|^2} < \alpha_1. \]

Note that if \( 0 < \epsilon < (1/\|L\|) \), \( \alpha > 0 \). We also have

\[ T \geq \frac{1}{e\|L\|^2} - \frac{1}{e\|L\|^2} = \frac{1}{e\|L\|^2} \frac{\|L\|^2}{1 + 3e\|L\|} = \frac{2 - \sqrt{1 + 3e\|L\|}}{3e\|L\|} \]

By solving the inequality \( 2 - \sqrt{1 + 3e\|L\|} \geq 3e\|L\| \) for \( e\|L\| \), we get

\[ e\|L\| \leq \frac{5 - \sqrt{13}}{6} < 1. \]

Therefore, if \( e \leq (5 - \sqrt{13})/(6\|L\|) \), then \( T \) is larger than 1. □
4.2 Distributed Approach

In this subsection, we consider an average consensus problem with a distributed self-triggered control protocol. As stated in Section 3.2, we consider the consensus protocol (7) which does not depend on the global parameter $\epsilon$.

At every trigger time $k_i$, each agent $i$ computes the allowable maximum trigger interval $T_i^*$ as follows:

$$T_i^* = \max \left\{ \frac{\sqrt{\beta_i}}{1 - \sqrt{\beta_i} P_i}, \frac{\sqrt{\beta_i}}{1 + \sqrt{\beta_i} P_i} \right\},$$

where

$$\beta_i = -\sigma_i - 2 + \gamma_i + b_i \left( 1 + \frac{2}{\gamma_i} + \frac{h_i}{\gamma_i} \right),$$

$$\rho_i = \sum_{j=1}^{n} a_{ij} (x_i(k_i) - x_j(k_i)), $$

$$P_i = \sum_{j=1}^{n} a_{ij} \rho_i + \sum_{j=1}^{n} a_{ij} \rho_j.$$

To synchronize the trigger times, agents share their maximum trigger intervals $T_i^*$ and set the next trigger time as $k_{i+1} = k_i + \max \{T_i^* \}$.

**Remark 3** The synchronization of trigger times can be computed by a distributed algorithm as the same way in Remark 2. Each agent $i$ sets a variable $\xi_i$ as $\xi_i(0) = T_i^*$. Then agents run the following distributed algorithm:

$$\xi_i(s+1) = \min_{j \in \mathcal{N}_i} \xi_j(s).$$

This algorithm converges to $\min_{\mathcal{N}_i} \xi_i(0)$ after at most $k = \text{diam}(G)$ iterations [9]. This scheme requires some extra communications and the sampling period should be long enough as stated in Remark 2. The self-triggered control scheme without the synchronized process is our subsequent work.

The following theorem shows that a multi-agent system can achieve an average consensus by the proposed self-triggered control scheme.

**Theorem 4** Suppose that the graph $G$ is connected and triggers are synchronized. Suppose also that $0 < a_{ij} \leq \frac{1}{\max_{i \in \mathcal{N}_i} |N_i|}$, $2 - \frac{2}{\max_{i \in \mathcal{N}_i} |N_i|} < \gamma_i < \frac{1}{\max_{i \in \mathcal{N}_i} |N_i|}$, and $\beta_i < 1$, where $\Delta_i = 4\beta_i P_i^2$ and $\sum_{j=1}^{n} a_{ij} \rho_j$, then the trigger condition $g_i(k+1) \geq 0$ is written as follows:

$$(P_i T)^2 - 2(P_i T + \rho_i)^2. $$

To solve the equation $(P_i T)^2 = \rho_i (P_i T + \rho_i)^2$ for $T$, we consider

$$(1 - 2\rho_i P_i T) - 2\beta_i P_i T - \beta_i^2 T^2 = 0.$$

Then, from the quadratic formula, we have the solution

$$T = \frac{2\beta_i P_i T \pm \sqrt{\Delta_i}}{2(1 - \beta_i) P_i^2},$$

where

$$\Delta_i = 4\beta_i^2 P_i^2 + 4\beta_i P_i^2(1 - \beta_i) P_i^2 = 4\beta_i P_i^2.$$

Hence, if $1 - \beta_i \neq 0$, then we have

$$T = \frac{\sqrt{\beta_i} \pm 1}{(1 - \beta_i) P_i}.$$ 

\hfill $\Box$

5. Simulation

We consider a multi-agent system with 4 agents. The communication network is given by a static and undirected graph as shown in Fig. 1. We define $\alpha$ by

$$\alpha = -\frac{1 + \epsilon |A| + \sqrt{1 + 4\epsilon |A|}}{\epsilon |A|^2}.$$

Note that

$$\alpha < \alpha_1 = -\frac{1 + \epsilon |A| + \sqrt{1 + 4\epsilon |A|}}{\epsilon |A|^2}.$$

The initial states of the agents are $x(0) = [x_1(0), x_2(0), x_3(0), x_4(0)]^T = [1.2, 0.8, -0.4, -1.6]^T$.

Figure 2 illustrates the state trajectories of 4 agents. The
states of all agents converge to the average of their initial states \((1/4) \sum_{i=1}^{4} x_i(0) = 0\) and the multi-agent system achieves an average consensus. The parameters of the system are shown in Table 1.

Figure 3 shows the trigger times and the trigger intervals of the agents. From these figures, we see that the proposed event-triggered and self-triggered control can reduce computation resources compared to the time-triggered control that requires computation at every step.

6. Conclusion

This paper considered both centralized and distributed protocols of event-triggered and self-triggered average consensus problems with discrete-time consensus protocols. We showed sufficient conditions to achieve an average consensus based on the analysis of the nonlinear control theory.

Our future work is to show sufficient conditions for higher-order discrete-time systems to achieve an average consensus without synchronization of trigger times.

Acknowledgments

This research is supported in part by the JSPS Grant-in-Aid for Young Scientists (B) No. 25820180, and Mitutoyo Association for Science and Technology.

References


Kenta Hamada
He received the M.E. degree from Osaka University in 2014. His research interest includes cooperative control.

Naoki Hayashi (Member)
He received the B.E., M.E. and Ph.D. degrees from Osaka University in 2006, 2008, and 2011, respectively. He was a Research Assistant at Kyoto University in 2011. He is currently an Assistant Professor at Osaka University. His research interests include cooperative control and distributed optimization. He is a member of ISCIE, IEICE, and IEEE.

Shigemasa Takai (Member)
He received the B.E. and M.E. degrees from Kobe University in 1989 and 1991, respectively, and the Ph.D. degree from Osaka University in 1995. From 1992 to 1998, he was a Research Associate at Osaka University. He joined Wakayama University as a Lecturer in 1998, and became an Associate Professor in 1999. From 2004 to 2009, he was an Associate Professor at Kyoto Institute of Technology. Since 2009, he has been a Professor at Osaka University. His research interests include supervisory control and fault diagnosis of discrete event systems. He is a member of ISCIE, IEICE, and IEEE.